Electromagnetic Theory
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Module -1
Elements of Vector Calculus
Lecture -3
Divergence and Curl of Vector Fields

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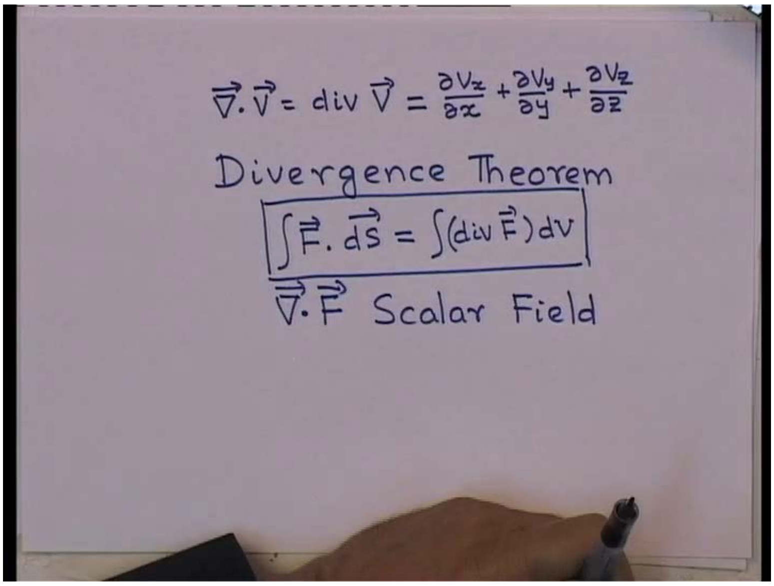

In the last lecture, we had introduced you with the ideas of divergence of a vector field. We had seen that the definition of divergence is given. Divergence is also written as del dot V or div V, if you like. We had seen that this is given by d V x by d x plus d V y by d y plus d V z by d z. This is the Cartesian expression for that. One of the things that we talked about is, what is known as the divergence theorem, which connects the surface integral of a vector field written as, for instance, if F is a vector field, then F dot d S is the same as the volume integral of divergence of F over the volume, which is described by this surface. Now, this is the theorem which has many many applications in subjects such as fluid mechanics and as we will see in electricity magnetism, which we are discussing as well.

What I wish to do today, is to take this concept of divergence a little further and make you more familiar with, how to use the divergence theorem and what is the physical meaning of the word divergence. I would do that and subsequent to this, I will also introduce to you what is known as curl of a vector field. As we can see that the divergence of a vector field is a scalar field, because del dot is there. del dot F is a scalar field.

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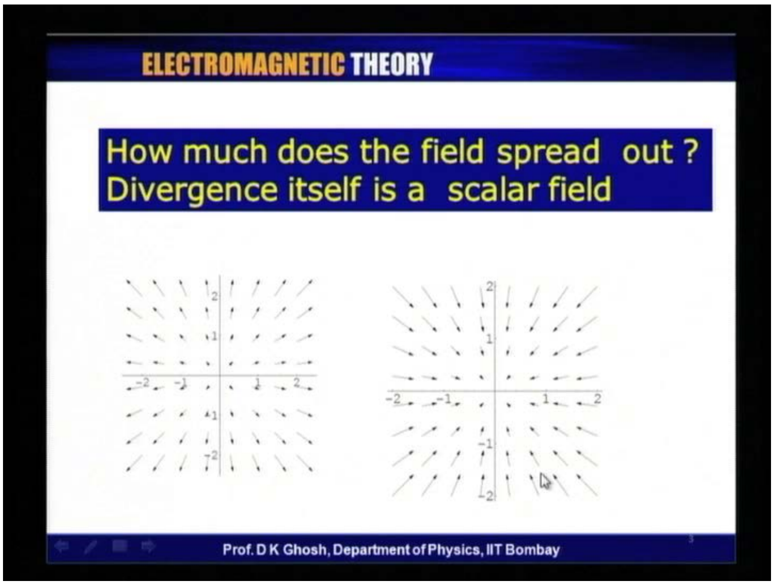

So, let us proceed with that. So, as the name divergence suggests, the divergence of a vector field essentially is a measure of the amount of spread that a vector field has got at a particular point. Now, let us, for instance, if you could see these pictures you will find that, in this case, let us take the origin and you find that the fields are spreading out from the origin. On the other hand, this is the type of vector field that you would expect. For example, the electrostatic field due to a positive charge. Of course, it will not be exactly this, but essentially it will be spreading out. This, on the other hand, you notice that the fields are converging to the center. So, these are examples of positive and the negative divergence.

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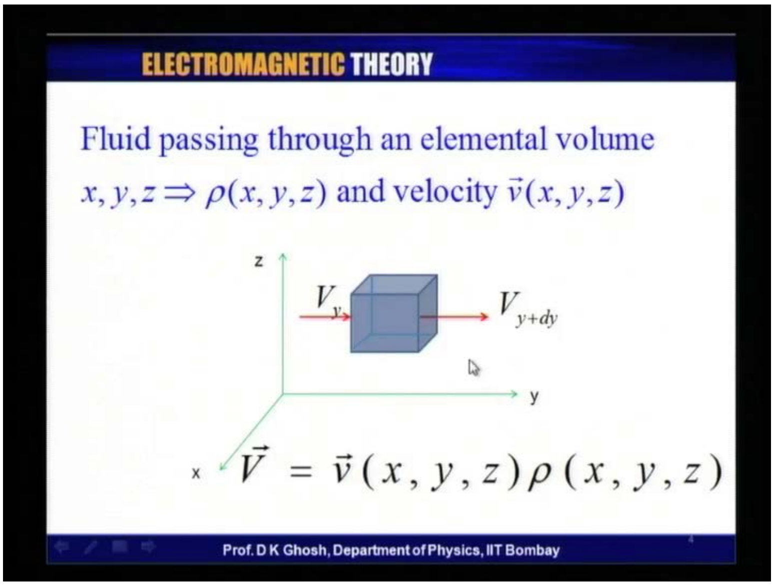

Now, let us look at what it actually means. So, one of the things that I would like to point out is that, the concept of divergence curl etcetera, they all came because of, they were first used in the field of fluid dynamics. So, let me try to illustrate the concept of a vector field using fluid dynamics as example. So, let us look at an elemental volume at the point x y z, having a length, breadth and width dimension of dx, dy and dz. So, that is what we are doing and what we are saying is that, at the point x, y, and z, the density of the fluid is given by rho x y z and the fluid velocity at that point is given by v of x y z.

So, this is, I am simply showing what happens to the y component of the velocity. For convenience, I define a vector capital v at the point x, y, z as the velocity vector at that point, small v x y z multiplied by the density rho at that point. This sort of tells you that this is essentially, this quantity entering an elemental volume dx dy dz and here, from the other phase it is leaving. This is just the y component of that coming in here and this is the y component at the point y plus dy. So, this is the y component is y here, and it is y plus dy.

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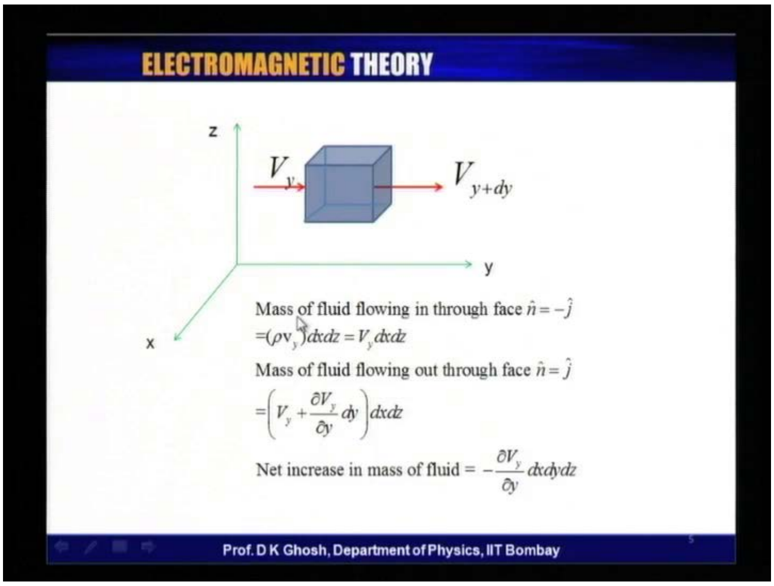

Now, let us let us look at this elemental volume and ask the question, what is the mass of the fluid flowing in. Now notice, this is the phase, the perpendicular to which is along the minus y direction or minus j direction. This is the phase for which the perpendicular is along the plus y direction, but the y coordinate of this is at y and the y coordinate of this phase at y plus d y. So, how much fluid is flowing into this volume though this phase? Now, obviously, we are talking about a mass. So therefore, the rho times V y, that is the distance moved per unit time along this direction and of course, you multiply it with dx dz, which is the perpendicular phase there. So, this is the amount of fluid that is flowing in through the phase n is equal to minus j, namely capital V y, which as I told you is a product of rho multiplied by small V y that is the velocity times dx dz.

Now, so that is the amount of fluid that is getting in and how much is the amount that is getting out. Now, the difference between this phase and this phases that their areas are the same, but this, its y coordinate is y, this as the y coordinate, y plus dy. So, what we do is this. We assume that this elemental volume dx dy dz are small. So that, I need to only retain the first order change in quantities to calculate how much is the mass of the flowing that is flowing out. The amount of fluid that is flowing out is given by y component of the velocity here. What is the y component? That is equal to V y, that is the velocity on this phase plus the rate of change of the velocity with distance, namely d V y by dy times dy, because that is the distance though which it has moved and of course, multiplied by the area. So, that is the amount of mass that is flowing out.

Now, so therefore, if this is the mass that is flowing in and this is the mass that is flowing out, the net amount of mass that is accumulating inside this mass, namely the net increase in the mass of the fluid is this minus this, which is simply minus d V y by dy into dx dy dz. You recall, dx dy dz is the total volume of this element. Now, mind you, this is just the increase only from the flow along the y direction.

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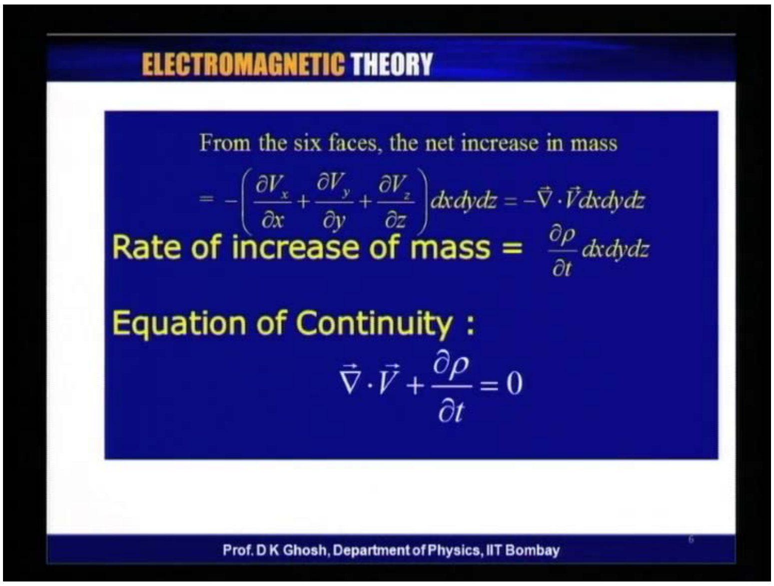

Now, what I am going to do now is to, by symmetry, I can write down an identical expression for the flow from the x direction and the y direction.

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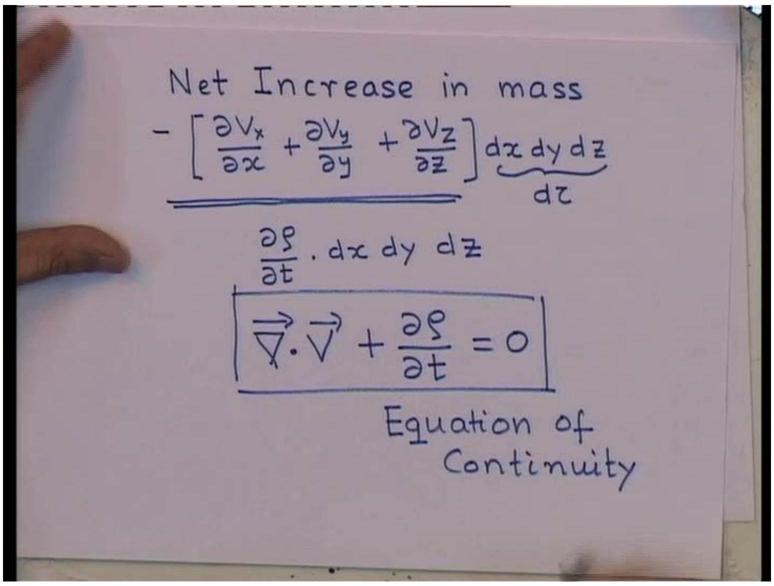

So, since that term was minus d V by dy, so what I get is, the result would be the net flow, net increase in mass If you like, there is a minus sign in front of that, multiplied by d V x by d x plus d V y by d y plus d V z by d z multiplied by the volume, which was d x d y d z. This is of course, your volume of the element, which if you like, I will not write it as d V, so that, you do not get confused with this velocity field V. So, this is, let us say d tau.

So therefore, now there is another way in which I can talk about the rate of increase of mass. So, what I can do is this. I know mass is nothing but the volume times the density. Now obviously, the volume here is fixed. Now, since volume is fixed, the rate of change of the mass is simply given by time rate of change of the density times dx dy dz. These two must be identical. This two must be identical. This is one way of doing it and this is from the definition. That tells me that I have del dot V plus d rho by d t is equal to 0. Now, in fluid dynamics, this is known as the equation of continuity. Now, I have drawn some pictures.

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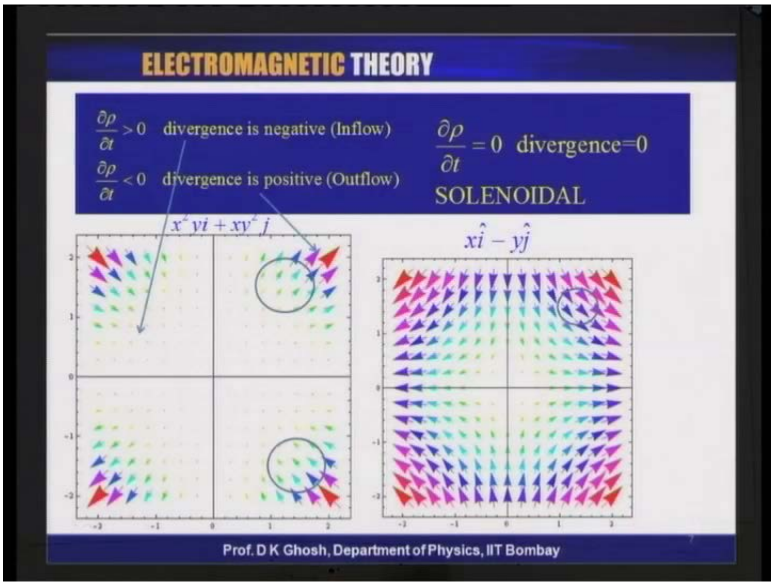

So, let us look at what are these. This will give you an appreciation for the name divergence as well.

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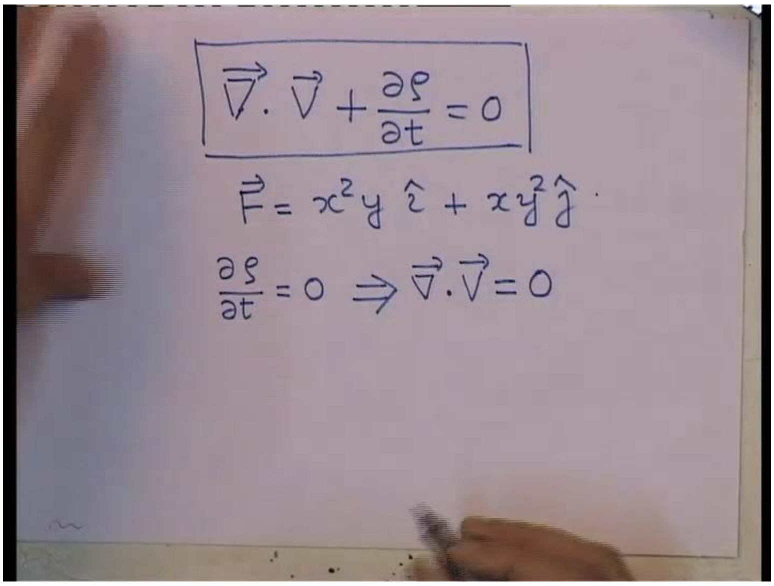

But, let me return by to the equation of continuity. I have got del dot v plus d rho by d t is equal to 0. That is my equation of continuity. I am giving an illustration. Look at this picture. Now, this is the field which is written on the top. So, this field, the left hand side field is given by x s square y times i, the unit vector along the x direction plus x y square times j. So, the field is, let us say, let us called field as F and this is equal to x s square y times the unit vector i plus x y square times unit vector j.

Now, this is the field that has been plotted in this graph. You remember that, we had said that, this we have used mathematic to plot it and the way we plotted is that the vector, the size of the arrows are proportional to the magnitudes of the vectors and the direction is represented by the direction on the arrows. Now, if I look at the first quadrant of this, I am looking at the circular region to set two-dimensional plot. You notice that the, let us suppose this represents a fluid field, the velocity field of a fluid. So, this is, so here you notice that, if you call that these small arrows are the velocity vectors for the fluid, you notice that the fluid velocities which are entering into this circle, they are of smaller magnitude than those which are going out. In other words, from the circular region, more fluid is going out than it is coming in. Thus, more outflow. There is more outflow. Now, obviously, such a thing can happen if the density is decreasing with time. So, if d rho by d t is negative, then the divergence of the field del dot V will be positive and this represents a case of positive divergence. So, this is and the identical statement would be true, if we look at the third quadrant as well. I have not shown it here, but you can sort of check that if you draw a circle here, this same identical argument would be true.

Look on the other hand. In the second quadrant here, now in the second quadrant here or the picture is given on the forth quadrant, identical story would be true of the second quadrant. If you look at the forth quadrant here, you notice that the arrows which are pointing in into this circle are much bigger in magnitude than those which are going out. In other words, this is a case of net inflow. More fluid is coming in than going out. Now, such a thing can happen if there is an increase in the density of the fluid with time.

So, this is the field is diverging and divergence is positive and this field is divergence is negative. Let us look at this in a slightly different field, for which, d rho by d t is 0. It is an incompressible fluid. Now, if d rho by d t is 0, then del dot of V will also be 0. In other words, the velocity field has 0 divergence. Look at this picture here. You notice that, if you take a circle or circular region here, as much fluid is getting in as is going out. Now, in such a situation, where the divergence of the field is 0, for example, is a field which is x i minus y j which we had seen earlier. Such a field for which the divergence of the vector field is 0 is known as Solenoidal vector field.

I have already introduced you with what we call as divergence theorem. So, let us recall, what is the divergence theorem. The divergence theorem, if you recall, connects a surface integral with the volume of the surface, volume of the body bounded by the surface.

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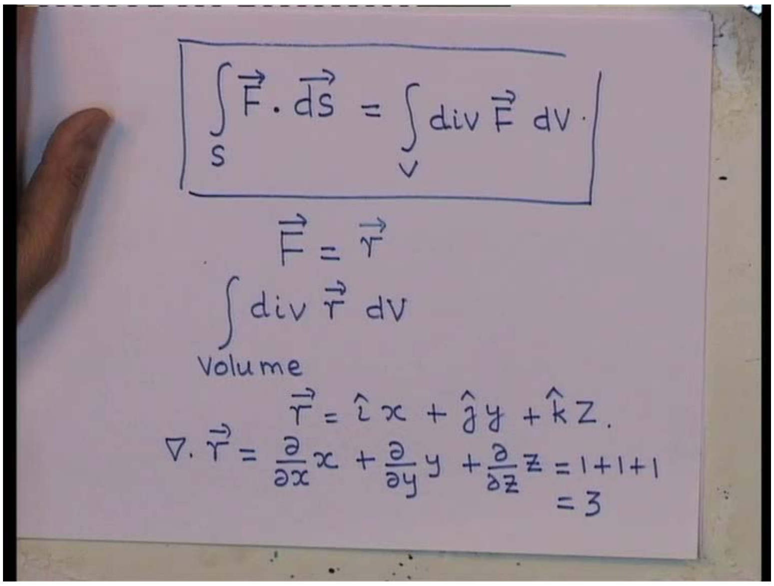

So, suppose I have a vector field, which I represent by F. Then, the surface integral of F, F dot dS over whatever surface you are talking about. Now remember, we had said that, only some special types of surfaces are permitted or rather special type of surfaces like mobia strip are not permitted. This F dot dS, the direction of S is according to convention, the outward normal and this is equal to the volume integral of the divergence of F.

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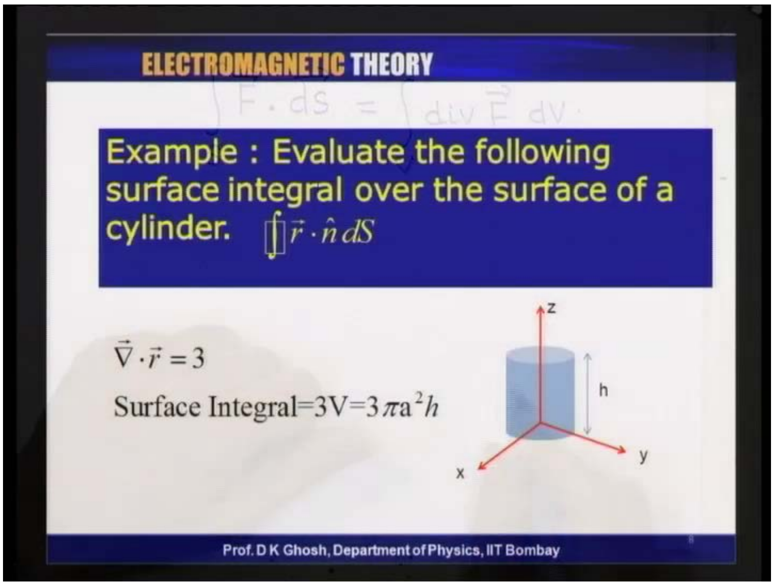

Now, what I am going to do is to illustrate the use of such a thing. So, here what I am doing is that, I am trying to evaluate, this is a cylinder, the base here x y and the height of the cylinder is along the z direction. I want to integrate, find the surface integral of the vector r, r dot n dS. ‘n’, if you recall is along the outward normal to any element of surface. Now first, let me do it the easiest way. The easier one is this; to use the divergence theorem.

So, in this case, the vector field is my position, vector r. Now, I am interested in finding out, what is r dot dS over the surface of the cylinder that has been shown in this picture. So, let us look at this. So, in other words, this surface integral is same as integral of divergence of the vector r over the volume of the cylinder.

So, this is over volume of the cylinder. Incidentally, divergence of the position vector which keeps on coming in various applications is a good thing to remember. It is trivial to calculate because, you recall that vector r is given by i x plus j y plus k z. So therefore, del dot r, which is d by d x of the x component of the vector, namely x, plus d by d y of the y component of the vector, which is y. Similarly, d by d z of the z component of the vector which is z, which is simply 1 plus 1 plus 1 which is equal to 3.

So, divergence of r is 3. So therefore, if I write it here, this is over the volume of the cylinder, 3 which is the number of times d V and how much? Because, there is nothing to integrate, it is integral d V, which is the volume. We all know the volume of the cylinder is pi, a is the radius, so a square times the height h. So, this is the result. So, this surface integral, which we calculated in an indirect fashion, namely calculating through the divergence, works out to 3 pi a square h. Now, what I am going to do now is, to show that this is exactly the result that you would get, if you calculated the surface integral directly.

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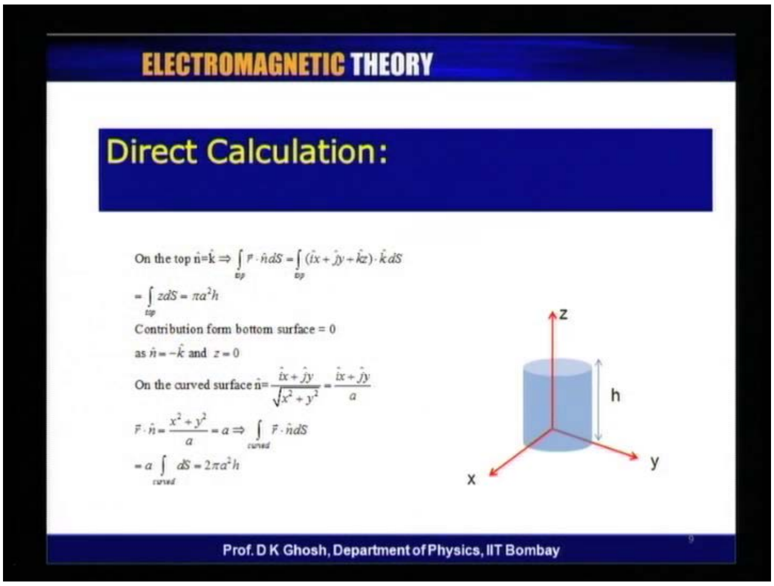

So, let us do that. So, let us look at what, so, let me redraw this picture here.

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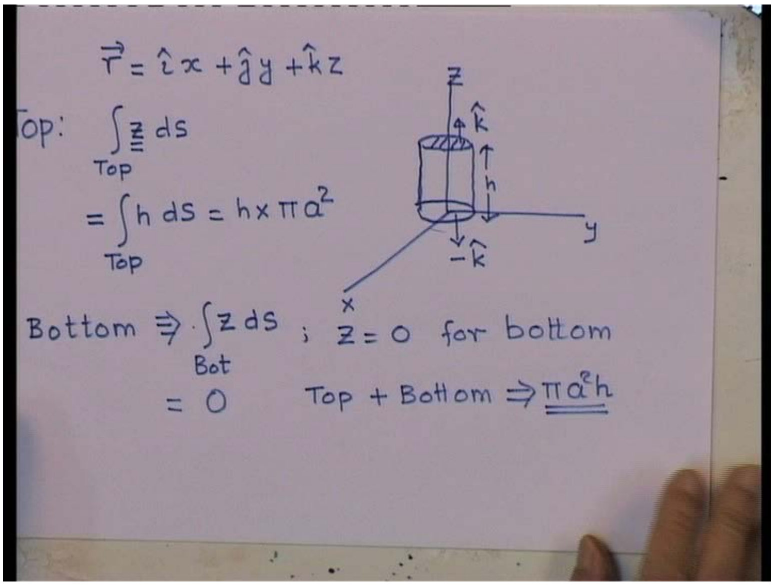

This is x y z and I have a cylinder, whose base is at the origin. First thing to know is this. The cylinder has three surfaces. It is a closed cylinder because, we have said it is over the surface of the cylinder, closed cylinder. It has a top surface, which is this one and the outward normal to the top surface is just the unit vector k. It has the bottom surface, which is, because it is a outward normal, so, it is minus k. We will come back to what happens to this side surfaces. But let us first compute, how much is the contribution to the surface integral from the top and the bottom surface first. Let us recall my field r is i x plus j y plus k z.

I am interested, let us say, first calculating the top surface. So, the top surface is surface integral of r dot k. Now, r dot k d s; d s is an element of the surface. Now, so, i dot k is 0, and j dot k is 0. I am only left with k dot k. So therefore, k dot k is 1. I am left with z and of course, dS, the element of the surface and it is only the top surface. But notice that height of the cylinder is h and this is z equal to 0. So, the z value on the top cap of the cylinder is fixed and is equals to h. So, this is nothing but h times dS on the top surface and h is constant, so it comes out.

So, I am left with simply the area of the top surface, which is pi a square. Now, the calculation of the lower one is equally straight forward. So, let us look at the bottom surface. The only difference now is the unit vector outward normal is along minus k. So, this will not be z dS, but will be minus z dS. So, let us write down, integral bottom surface of z dS. Now, this is actually even simpler because, the value of z for this surface, the lower surface is 0. z is equal to 0 for bottom surface. So therefore, this integral is 0. So, from the top and the bottom, the surface integral gives me pi a square h. I will now calculate this for the curved surface, which is the only surface remaining now.

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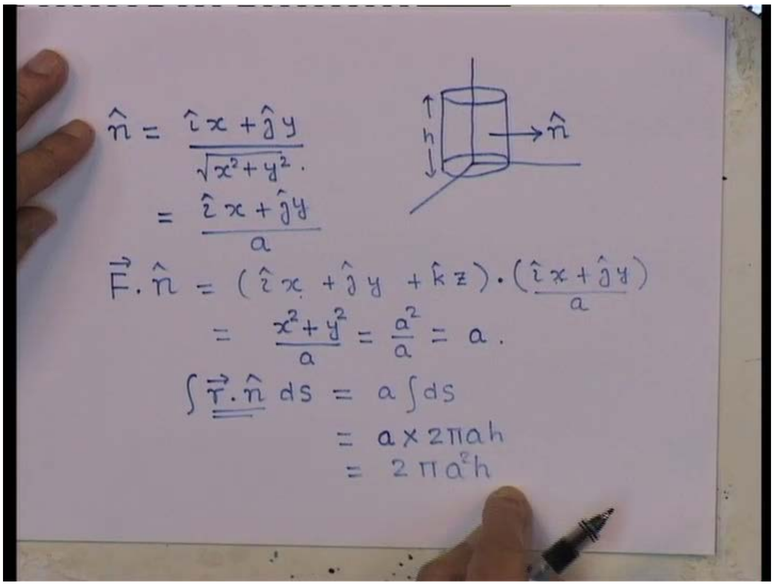

Now, what I have to do now is this. I have to find out what is the unit vector on the curved surface. So, this will be some something like this. Now, let us see what it is. So, notice that this is parallel to the x y plane, a perpendicular to the curved surface is parallel to the x y plane and it is on the surface of the cylinder. So therefore, the unit vector n is nothing but i x plus j y, which is just the radial vector in the x y plane. But I have to find a unit vector.

So therefore, I have to divide it by square root of x square plus y square, where x and y are on the curved surface of the cylinder. But remember that if the radius of the cylinder is a, the square root of x square plus y square is nothing but the radius a itself, because it has to be on that circle. So, this is i x plus j y divide by a. x and y are arbitrary, but because it is on the curved surface, this relationship is there.

So now, let us compute F dot m. F is i x plus j y plus k z dotted with i x plus j y divided by a. So, let us look at what it gives me. So, x into x i dot i is 1 x into x, I get x square. j dot j is 1. I get y square. k dot i and j are both 0. So therefore, this is x square plus y square by a, which is nothing but a square divided by a. So, which is equal to a itself.

So, what do I have? I have here, I have to calculate r dot m dS and r dot n is a times integral of dS. Now, how much is the area of the curved surface? The area of the curved surface of height h is nothing but the circumference of any of these circles multiplied by the height h, which is 2 pi a times the height, which gives me 2 pi a square h. If you recall, from the top and the bottom surface, we had pi a square h. From the curved surface, I had 2 pi a square h. So, the net result I get 3 pi a square h, which is the result I had obtained from the divergence theorem. So, divergence theorem makes it easy to compute certain surface integrals. That is one of the major applications of the divergence theorem.

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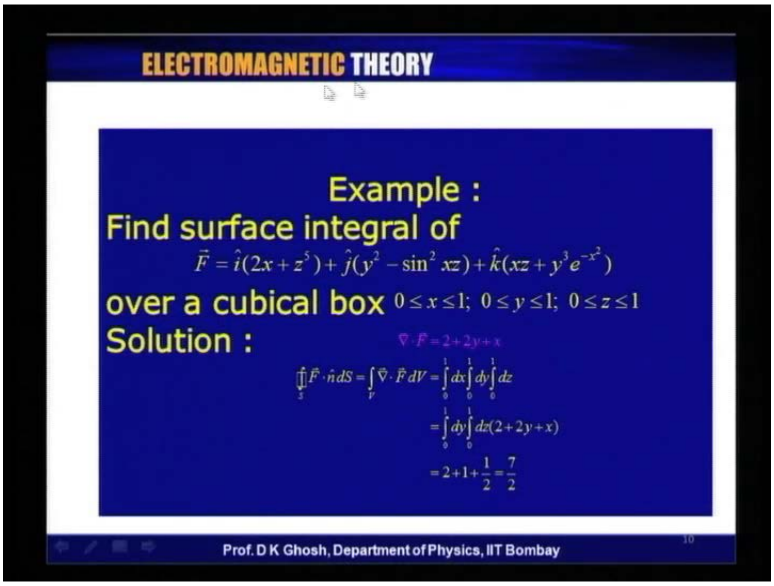

As I further example, let me show you a rather nasty looking field. As you can see it, the field F is i times 2 x plus z to the power 5 j times y square minus sine square k z k times x z plus y cube e to the power minus x square. I want surface integral over a cubical box of, you know, 1 by 1 by 1 cubical box x from 0 to 1, y from 0 to 1 and z from 0 to 1. Now, you realize that if I am trying to attempt to calculate this directly, it is going to be a mess because, I have to worry about how to integrate many of these things. However, this situation is not as bad as it looks.

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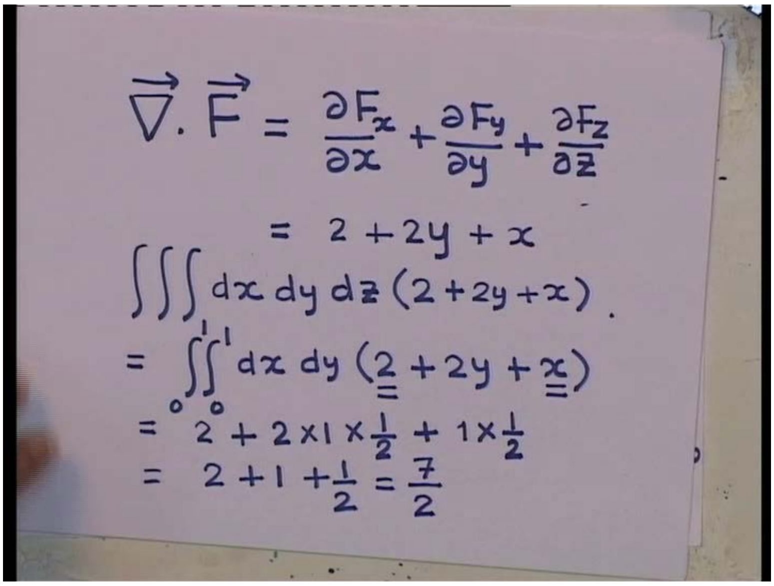

The reason is the following. If you look at what is the divergence of this vector, remember the divergence is d F x. Only x derivative of the x component plus d F y by d y plus d F z by d z. Now, x component? Notice this z to the power 5, which was somewhat nasty, it has its derivative with respect to x is 0. So therefore, my derivative is simply 2. y component again, the sine square x z d by d y is 0. So therefore, I need only d by d y of y square, which is 2 y. Finally, d by d z of F z, again this is a function of x and y. So therefore, x z has to be differentiated with respect to z and I simply get x. So, it is 2 plus 2 y plus x.

Now, I need to integrate this, but fortunately, over the volume of the cylinder. So, I need to calculate, if you like, I will write it explicitly as triple integral d x d y d z of 2 plus 2 y plus x. First thing to notice in this integral is there is no z dependence. Now, since there is no z dependence, I can integrate z out from 0 to 1 and it simply gives me 1. So therefore, I am left with a double integral d x d y of 2 plus 2 y plus x. Each one of them is rather simple to work out. First, so, this is all are from 0 to 1.

First this 2, so, two times integral d x d y. So, it is 2 into 1 into 1, and that is 2 plus 2 times. Now, integral d x over 0 to 1, since there is no x dependence gives me 1 and I have got y square by 2, which is 0 to 1 is 1 by 2 plus; this has only x. So, y integral is done, which gives me 1 into x s square by 2 which when, so x s square by 2 from 0 to 1 is 1 by 2. So, what I do? What do I get? I get 2 plus 1 plus a half, which is 7 by 2. So, this is the result of this surface integral. Now, because the function is so nasty, I will not be attempting direct evaluation on the surface. So much about divergence.

For vector fields, so, divergence of a vector field is a scalar. So, it is a scalar field. Now, for a vector field, it is possible to have an operation, which results in another vector field and this is called the curl of a vector. The curl of a vector came from the word circulation.

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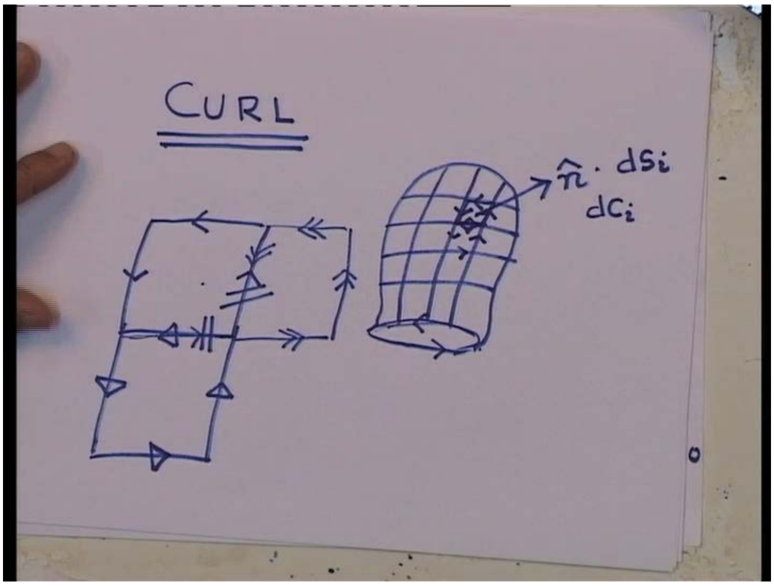

Now, we will see as the name suggests, the meaning of the CURL is associated with how much a vector field is curling about that point.

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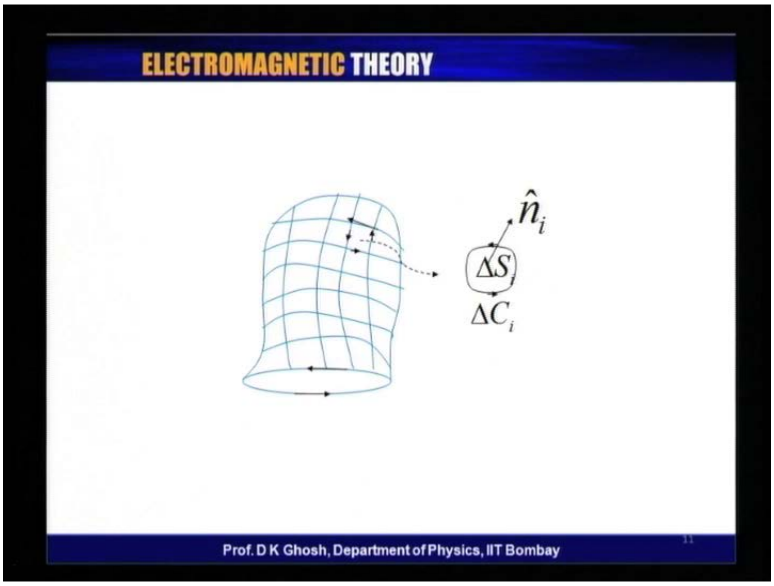

So, let us, so, what I have done here is to draw a picture of a. This is an open surface, something like inverted pot and this open surface has a circular boundary. So, this is a boundary. Now, let us look at how does one calculate the surface integral of a vector field over this surface. Now, let me illustrate that problem a little bit. So, I have this surface. This is an open; I have given it a sense of direction. Now, if I make segments of the surface, so, what I do is this. Just draw these segments as has been shown there. Make elements of segments.

Now, let me try to calculate the surface integral over, for instance, the area bounded by this. Now, let us look at first, instead of going to the area, which I am coming to in a second, let me concentrate on this element of area and let me give it a direction. So, the direction that I will give is this. That is an anticlockwise direction and the surface corresponding to this has an outward normal. So, I will call this element as, supposing this is the i th segment, let me called it n times d S i and this curve, which is the boundary of this d S i, I will called it d C i.

Now, the thing that I want you to notice is this. If I go in each of these segments, if I take the line integral in the same sense all the time, in this case I am taking in the anticlockwise fashion, then you notice from an adjacent circuit or curve, my result will be something like this. This will go like this, this will go like this and this will be exactly in the opposite direction to the previous one. Let me let me illustrate this by making, amplifying these elements. Supposing the same two adjacent elements, I am amplifying. These are two adjacent elements.

So, this is my d S 1. Let us say this is d S 2 and I am going on this, the upper one in an anticlockwise fashion. The line integral will be over this, over this, over this and over that. Now, when I come to number 2 and I still go in the anticlockwise fashion. Just to make it clear, let me give these arrows in a slightly different manner.

Supposing this is the anticlockwise arrow. You notice that this common line for the top one is traversed this way, and for the bottom one is traversed in the exactly opposite direction. So, if I am to now add up, supposing I want to find out how much is the surface integral over this plus that. What remains are only the contribution from the outside boundaries. This will happen that, supposing I am now add up, add another one, it will be like this. You notice again, this has cancelled and this has canceled.

So, what will I be left with? So, if I split up this into such elemental curves on the surface, then I will be left with only the outside boundary, which is nothing but this edge of this object. This is made clear in the next picture.

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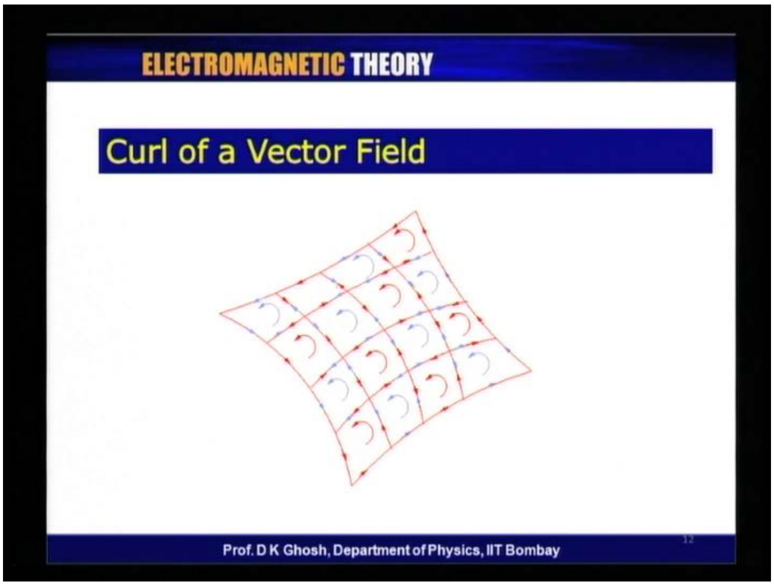

So, you notice here that I have shown a little stretched out thing and everywhere I have gone with the same sense, the anticlockwise fashion. So, if you look at this one and that one, the common areas cancel out and you will be always left with only the outside things.

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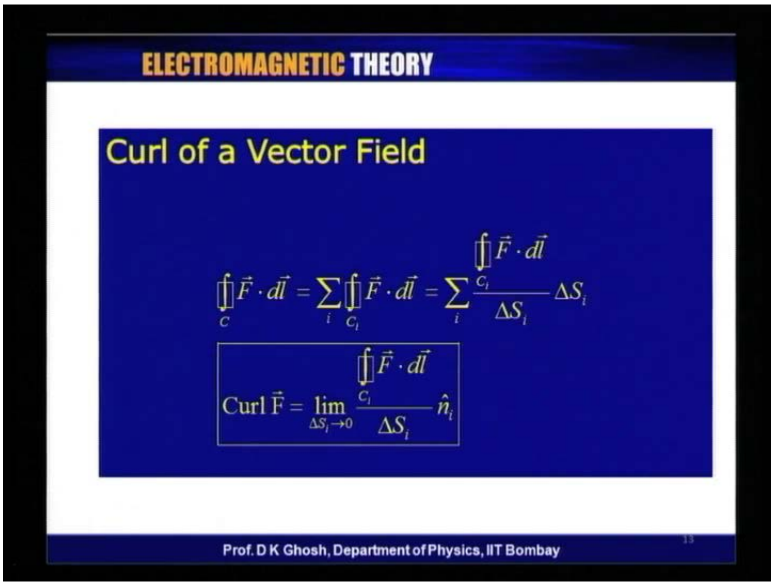

Now, so therefore, I can write down that the net contribution from this, so, let me write it down clearly, is, supposing I am talking about the entire curve F dot d l.

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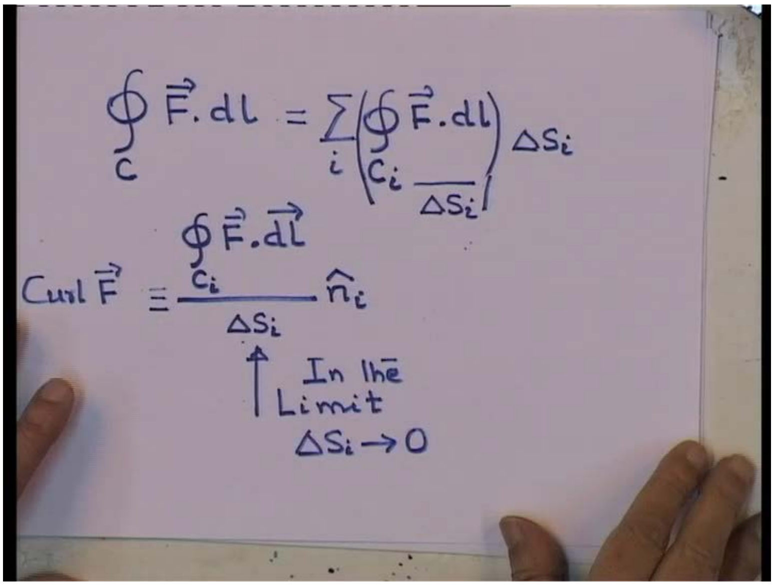

Over all these little close curves that I showed you, then this can be written as sum over i, which is summing over all those little curves and this integral is to be taken over the i th curve of F dot d l. Now, what I do is this. Now, this quantity here is different for different curves. Let me divide this by the area of the surface enclosed by the i th curve and multiply this with the same number. Now, what is this quantity? So, this quantity, which I have written as c i F dot d l over the i th closed curve, divided by delta s i. Now, this is the line integral of the boundary of the i th surface and the area of that i th, enclosed by i th curve is delta s i. The direction associated with this area is the outward normal, that I will call as n i. This quantity is defined as the curl of the vector F at that point i. It is a point relationship because, this relationship is true only in the limit delta s i going to 0. So, if I take the limit of this delta s i going to 0, so, this is a point relationship at the point i.

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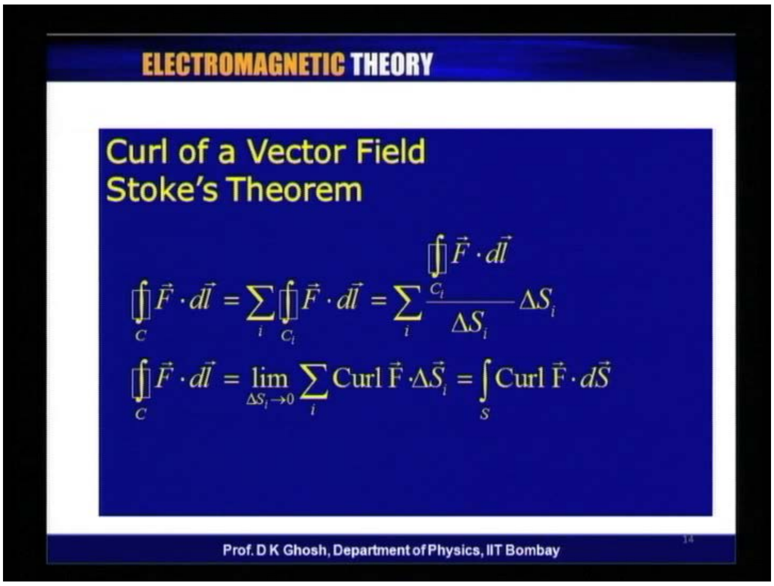

Now, this definition, just as when we define divergence, we obtained a relationship between the surface integral of a vector field with the volume integral of the divergence of that vector field. This definition of the curve in a very similar way gives me or gives us a relationship between line integral of a vector field with the surface integral of the curl of the vector. This relationship is known as the Stoke's theorems.

Let us look at how does this come. So remember, the line integral of this curve, now this is the curve bounding the surface in the picture of the inverted pot that I have showed you. It was the rim, the circular rim that I had showed you. So, this integral is nothing but sum over the integrals; the line integrals of little constituents on the surface. So, sum over i integral over c i. Now, what do is, divided by delta c i multiplied by delta c i and suppose, I take its limit, that is making the circuits smaller and smaller. Then, since this quantity is defined to be the curl of the vector, I get integral F dot d m is nothing but the surface integral of the curl of the vector. That is called Stoke's theorem. It is an extremely important theorem.

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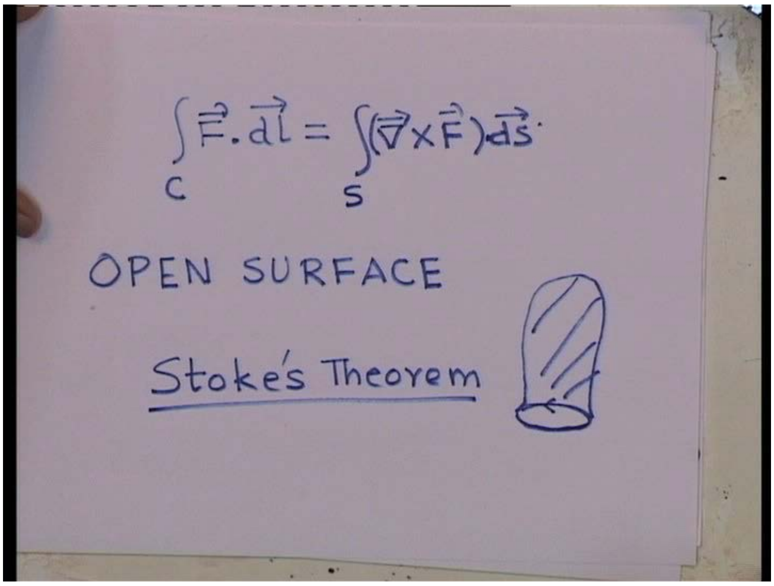

So, what is the c? You take a surface. You take an open surface. It is very important to realize. It is an open surface and not a closed surface, like that pot that I showed you. So, line integral on the boundary of the open surface. So, for example, going back to that picture, if I am interested in surface integral over this surface, I am relating it only to the line integral on this boundary. So, F dot d L is equal to, it is the surface that is bounded by this curve and the curl of that I have to take, so curl of F dotted with d s over the surface, which is described by this. This is called Stoke's theorem. What I will do next time would be to obtain an expression for the curl in the Cartesian coordinate system. This will be, the essential will be following the same technique as we followed for obtaining an expression for the divergence.

To summarize what we have done today is to look at two things. We started with an interpretation of the divergence of a vector field. I repeat, the divergence of a vector field is a scalar field and divergence as the name suggests, is a measure of, it is a point relationship, it is a measure of how much the vector field is diverging or of course, it could be converging at that point. This will be extremely important when we look at the electrostatic phenomenon. Look at for example, the electrostatic field due to the positive charges or negative charges. We will be returning back to the divergence of a vector field. The divergence of a vector field gives us a handle for computing surface integral of a vector in terms of the volume integral of the divergence.

The next thing that we did, which we will take up in greater detail in the next lecture is to define the curl of a vector field. The curl of vector field is itself another vector field. As we will see next time, the curl gives a measure if you like of how much a vector is curling around, as the name suggests, at a given point. What we have done is similar to the divergence theorem. We have obtained a theorem, which relates the line integral of a vector field with the surface integral of the curl of the vector field. In the next lecture, when I have a mathematical expression for the curl of a vector field, we will also give a few examples of the application of Stoke’s theorem.