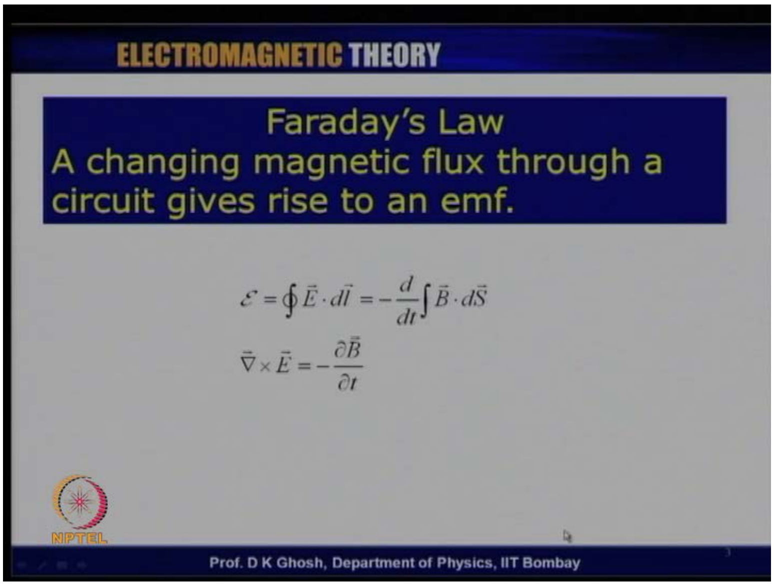
Electromagnetic Theory
Prof. D. K. Ghosh
Department of Physics
Indian Institute of Technology, Bombay

Module -4
Time Varying
Lecture - 29
Faraday's Law and Inductance

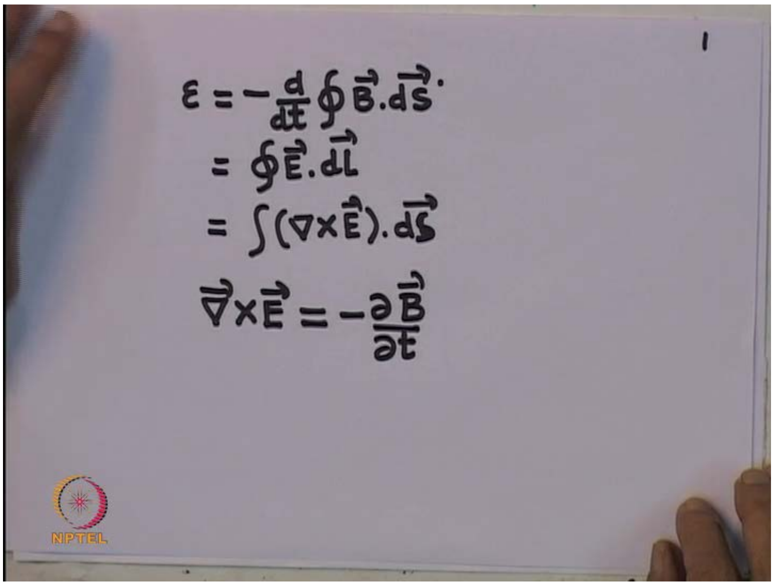
In the previous lecture, we had started discussing the time dependent phenomena, and I had enunciated what is known as Faraday's Law, what faraday had proposed based on certain experiments is that, whenever there is a relative motion between a magnet and a circuit, or if the magnetic fields trying this change in with time that is if any of these effects, results in the flux through a circuit changing. Then it leads to an emf in the circuit and as you had seen emf, we defined emf as line integral of electric field. And so what we had seen is that the time dependent phenomena, lead to a non conservative type of electric field, because del cross of E did not becomes 0, but had a value.

(Refer Slide Time: 01:25)



So, this what we did that the emf, which is the line integral of the electric field is rate of change of flux.

(Refer Slide Time: 01:39)

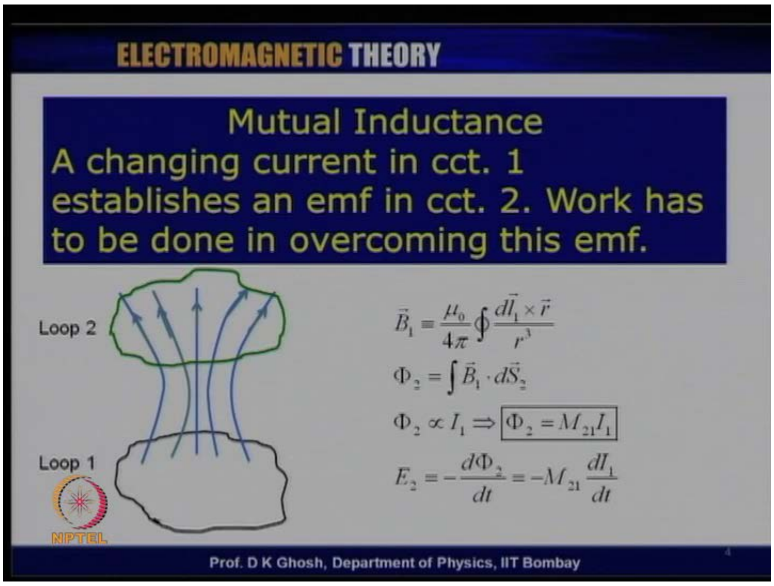


So, emf is given by minus d by d t of B dot d S, and emf by definition is the line integral of the electric field, which by Stokes law we know that we can write it as, a surface integral of the curl of the electric field. And by comparing this expression with this expression, since this is valid for any surface, we can write down a relationship which is the differential form of faraday's law that is del cross of E is equal to minus d B by d t.

So, notice that for the static phenomena, my del cross of E was equal to 0, but now I write del cross of E as time derivative of the magnetic field. So, I repeat again that whatever is the cause for a flux through a circuit changing, the we talked about motional emf which arises when there is a relative motion between the magnetic field and the circuit. For instance either a magnetic is moved towards a circuit, or if a circuit moves towards or away from a magnetic field, which is of course we know by relative motion it is the same effect.

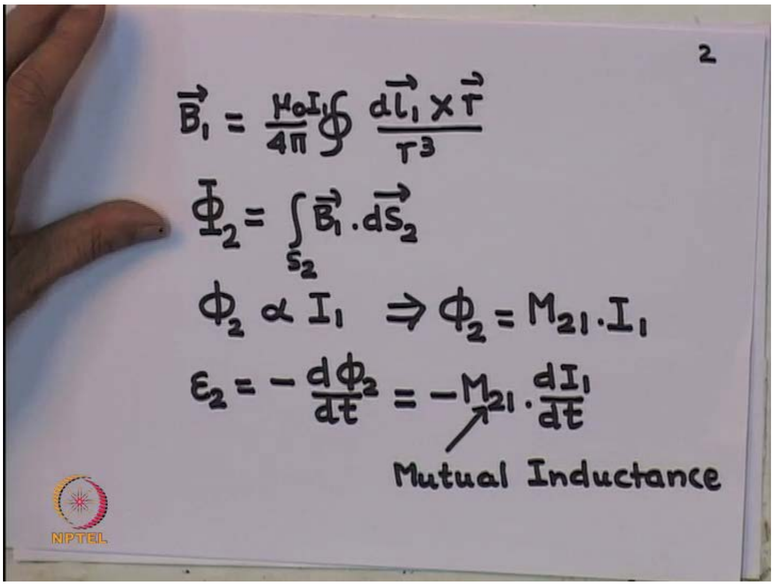
The other problem is that while this motional emf could be explained by invoking the Lawrence force, that is as the circuit moves, the charge on the circuit also experiences a force due to the magnetic field. But, what was surprising is that even if there is no relative motion, if the magnetic fields strength changes with time, even then there is an emf, this is born out by the experiment. So, therefore, we postulated that, whenever there is a changing magnetic field, that is a change flux changes through a circuit, for whatever reason; then there is an induced electric field produced, and that is what is contained in this law that I have written down.

(Refer Slide Time: 04:18)



With this let me introduce the concept of an inductance, so notice that supposing I have two circuits, I have called this loop 1 and this I have called as loop 2. Now, let us suppose that there is a magnetic there is a current which is flowing through loop 1, and this current is changing with time, as a result the magnetic field that it produces changes with time. And the loop 2 is a loop which intersects these magnetic field lines, so therefore, as the magnetic field strength changes, magnetic field strength produced due to the 1st loop, changes it results in a change of flux in the 2nd loop.

(Refer Slide Time: 05:18)

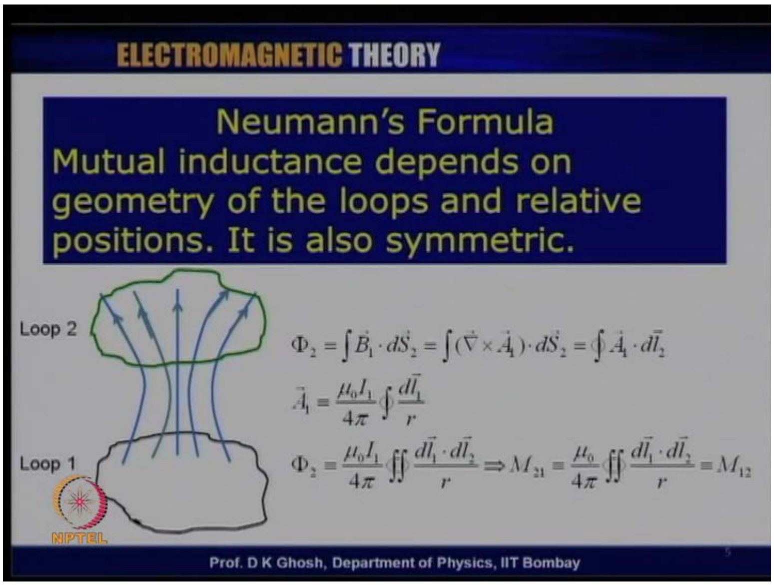


Let us now look at, what it actually means, now if you recall you are Biot-Savart Law it tells me B 1, because I have a loop 1, B 1 is the magnetic field due to the loop 1 that is lower loop in that picture; that is given by mu 0 by 4 pi. Then the integral over the loop of d l 1 my index 1 subscript 1 is suppose to be for the loop 1 cross r divided by r cube. So, this is the magnetic field produced by the current in the 1st loop, so therefore, what I get is this mu 0 I have must put it there, so the flux through the second circuit, this circuit is given by of course, integral B 1 dot d S, if you like I will write it as d S 2.

So, B 1 is the magnetic field produced by 1st loop and d S 2 is because, I am integrating over the surface of the 2nd loop, so you notice since B 1 is proportional to the current, let me write it as a I 1 so phi 2 which is the flux through, the second circuit is proportional to the current in the first circuit. So, this tells me that the flux I can write, the flux to the second circuit, I can write as equal to some constant, let me write it as M 2 1 multiplied by I 1.

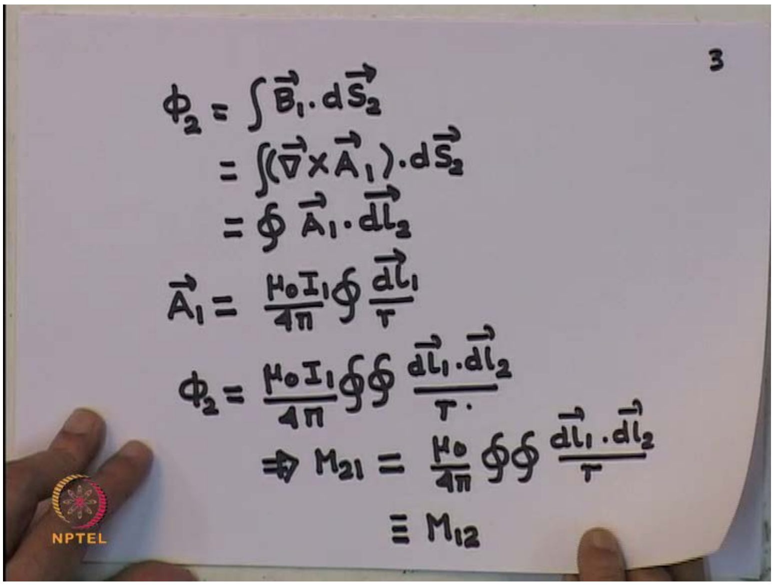
So, therefore, flux in the second circuit is proportional to the current in the first circuit, and that is what it is, and now the emf in the second circuit, which is minus d phi 2 by d t that is simply equal to minus M 2 1 times d I 1 by d t. So, notice that this proportionality constant, which comes when we express the emf in the second circuit, with the rate of change of current in the first circuit, this is what I call as a mutual inductance. The index, the indices have been subscripts have been written so that, this M 2 1 stands for the mutual inductance, when I am talking about the emf in the second circuit due to the cause is in the first circuit, the effect is in the first index.

(Refer Slide Time: 08:08)



Now, we can do something interesting those formula that I will derive is of no great use, but on the other hand, it establishes two things. Firstly, it will establish that the mutual inductance is a property, which depends upon the geometry of the loops, relative position of that loops, and also I will show that M 2 1 is equal to M 1 2 that is whether, I am changing in the changing current in the plus circuit, and looking at the effect in the second circuit, that is major the flux in the second circuit. This gives me the same proportionality constant, if I do the reverse namely, if I change the current in the second circuit, and look for the flux in the second circuit.

(Refer Slide Time: 09:07)



So, therefore, let us look at what we did, so for example, we said that phi 2 is integral B 1 dot d S 2, B 1 because the magnetic field is produced by the first loop. Now, let me express this, in terms of the vector potential produced by the current in the first circuit. So, I will write this as del cross A, I will write as A 1 again all the quantity dot d S 2, and my Stoke's law this would become line integral of A 1 dot d l 2 notice that, the vector potential is produced due to the first circuit, the loop integral is taken over the second circuit.

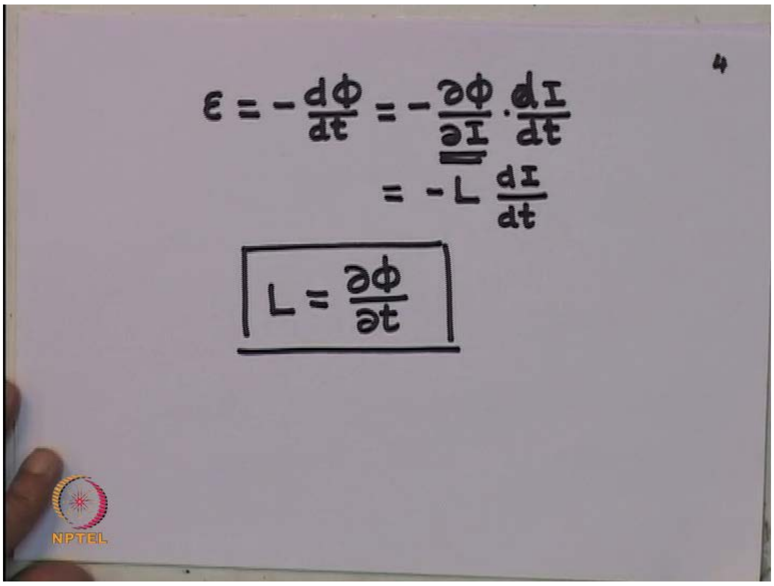
But, I know that A 1 that is the vector potential, produced by the current I 1 in the first circuit is given by mu 0 I 1 over 4 pi, line integral of d l 1 over r, this is something which we had discussed, while we defined the vector potential. Now, if I plug this into this expression, so this tells me that phi 2 is given by mu 0 I 1 by 4 pi integral d l 1 dot d l 2 over r; so this tells me that M 2 1 which is what we wrote down, which is proportional, the proportionality constant of phi 2 with I 1 is given by mu 0 by 4 pi double loop integral of d l 1 dot d l 2 divided by r.

Now, you can see that this expression is symmetric in one, and two because, this r is actually r 1 2, so therefore, this is identical to what you would have, if you reverse the procedure; namely M 1 2 is equal to M 2 1, thus this is what is known as Neumann's formula. Neumann's formula establishes that, the number one the structure tells us, it depends only on geometry and the relative position, the fact that M 2 1 is equal to M 1 2 tells us that the mutual inductance is symmetric in the indices of the loops.

Now, now I come to a rather curious facts, if you change the current in loop number 1, the flux through loop number 2, it cuts the flux through loop number 2 if it change, and establishes a emf. But, let us now ask the following question supposing I do not have two loops, I just had one loop, now this loop which I have when I change the current through this loop itself, the magnetic field produced by this current changes, and as a result flux through this circuit also changes. So, in some sense it is a self effect we are talking about, and since the when there is a change in magnetic flux through a loop, the loop does not care about what caused this changing flux, as long as there is changing in flux it will generate an emf.

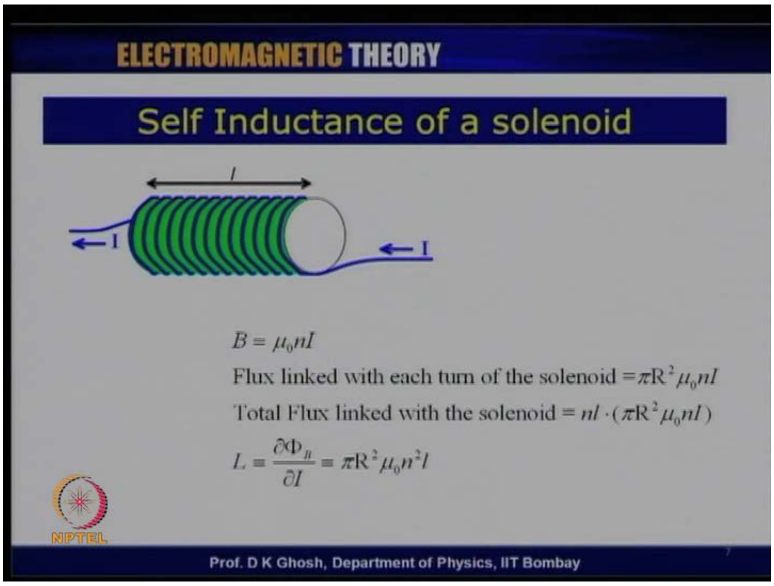
So, what will happen that you change the current through a loop, it leads to changing magnetic field produced by that loop, and which results in changing magnetic flux through the loop itself. So, this would mean that you have what is called a self effect, that is the magnetic field when it changes and leads to changing magnetic flux, there will be an emf generated in the same circuit.

(Refer Slide Time: 13:45)



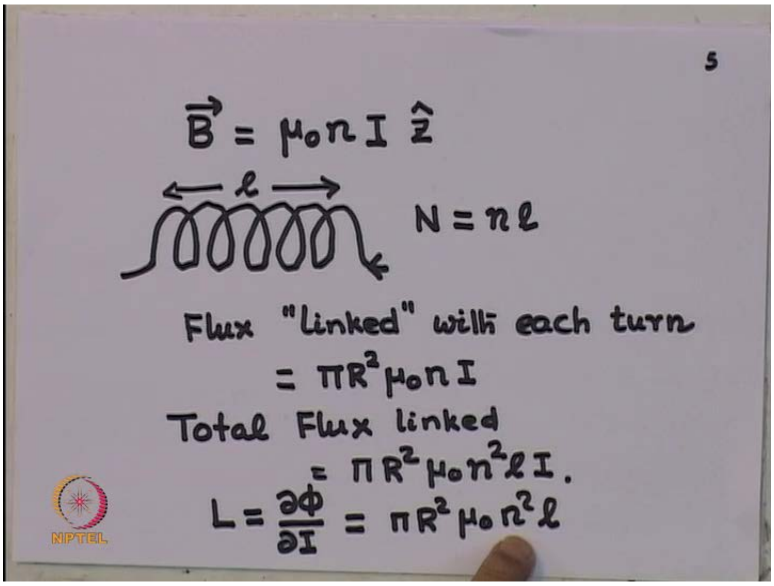
So, I will write that, this emf this emf is minus by definition minus d phi by d t, and let me write this as minus delta phi by delta I and of course, delta I by d, d I by d, now this proportionality constant that you have got here, rate of change of flux through a circuit with respect to the changing current in that circuit itself is known as a self inductance, and it is denoted by letter L. So, therefore, the definition of self inductance is d phi by d t, where phi stands for changing flux in a given circuit, so we have talked about what is self inductance and what is mutual inductance, but let me first calculate just as illustrative example, let me talk about calculation of some simple cases of self and mutual inductance.

(Refer Slide Time: 14:45)



So, here in this picture, you have find that there is a current through a solenoid, and this current is going to be changed. Now, I know f there is a current in a solenoid current I, then this leads to a magnetic field B given by mu 0 n I.

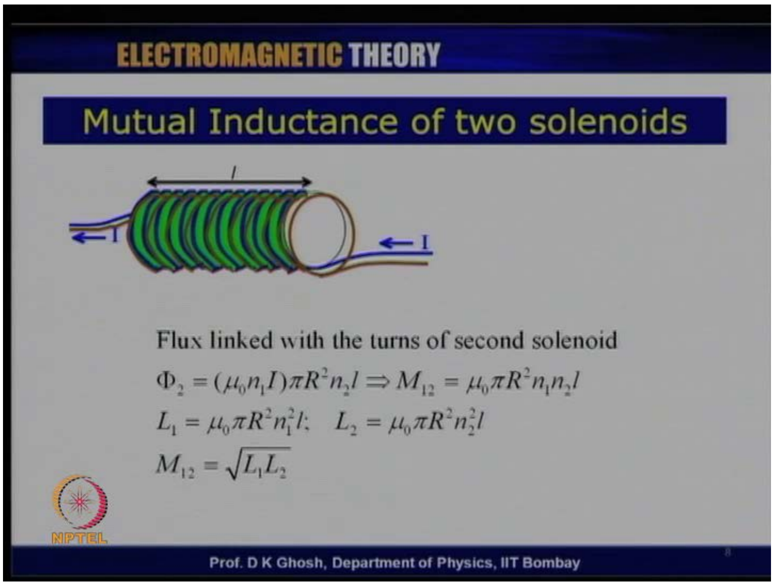
(Refer Slide Time: 15:17)



So, magnetic field B in a solenoid is mu 0 n I, and actually it is direction I should specify that direction is along the axis, let me take it along the z axis. Now so basically, what is happening is this, so I have this solenoid, now when the current changes through the circuit of course, the magnetic field changes, but then there is a concept of how much of flux is linked with each loop. So, we say it is flux linked with each loop, with each turn if you like So, this is equal to simply the pi times R square, which is the area of that loop, multiplied by the magnetic field which is mu 0 n I. So, if you take a length L of the loop, let me write it as small l, so that I do not confuse it with the notation for the self inductance itself, now I know that these many, these length since the number of terms per unit length is small n, then number of turns is equal to n times l. So, total flux linked, this is equal to pi R square mu 0 n I have a another n from there, so I have a n square l I, and the now of course, the self inductance is simply d phi by d I and this is linear n I, so this works out to pi R square mu 0 n square times l.

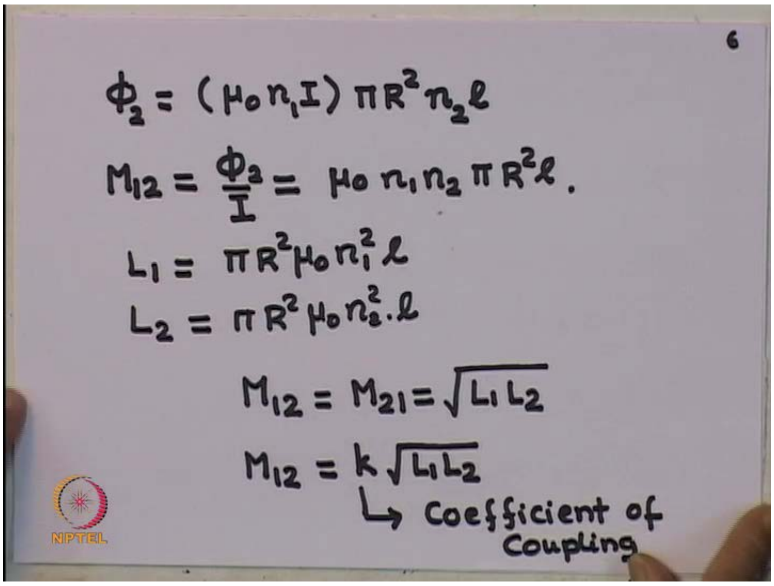
So, this is the expression and you can see, it depends upon only the dimensionalities the geometry of the radius, there are number of terms, the length of that loop. So, this is an example of calculation of self inductance of a solenoid, now let us do the following, let us calculate the mutual inductance of the solenoid.

(Refer Slide Time: 17:58)



So, here what I done is this, simple I made it somewhat simple, I have put one loop over the other, you can see this in this picture, there is blue colored loop, blue or green which I was talking about earlier; and now I have superposed on it rather tightly. The another solenoid, which let us say we will assume that the first solenoid has let us say n 1 number of turns, the second solenoid will have let say n 2 number of turns, and they are tightly bound. So, that we assume all that magnetic field that is produced by one of the solenoids passes, or a will be linked with the flux will be totally linked with the other one, So, let us look at how much flux will be linked.

(Refer Slide Time: 18:54)

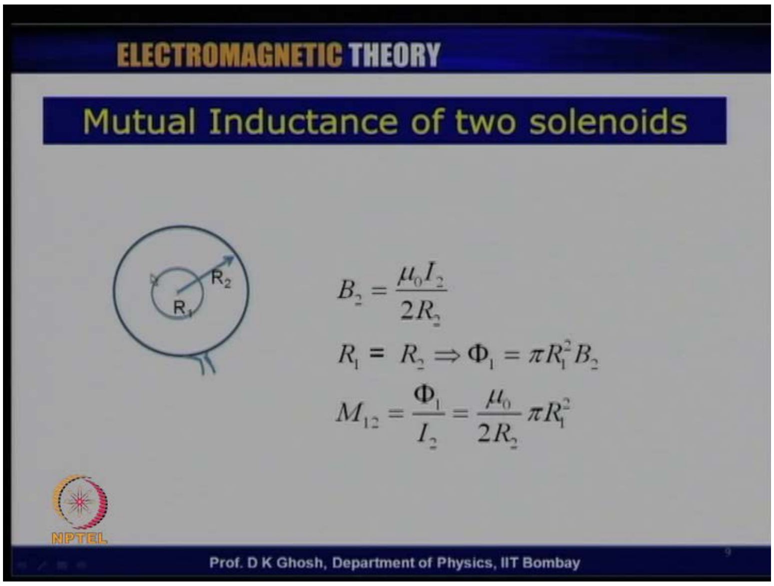


So, I am talking about now, phi 2 I am changing, I am passing a current I through the first loop, and which we had seen gives me a magnetic field mu 0 n I, I is the current through the first loop, and the total area is the area of each turn is pi R square, then total number of turns that I have in the second loop is n 2 into l. So, since n 1 2 or M 2 1 is nothing but, phi 2 divided by I that is simply given by mu 0 this is n 1, mu 0 n 1 n 2 pi R square l; now this is a very interesting relationship that can come out of here, remember that we showed that the self inductance of a solenoid is given by pi R square mu 0 n square l.

So, therefore, self inductance of first solenoid is given by pi R square mu 0 n 1 square into l, self inductance of the second solenoid will be pi R square mu 0 n 2 square into l, remember I made an assumption that the two solenoids are tightly bound to each other. So, therefore, whenever flux changes, because of change in current the entire flux is linked with the turns. So, if you compare this with this, you find M 1 2 can be written as, M 1 2 which we had seen as same as M 2 1, can be written as square root of L 1 into L 2.

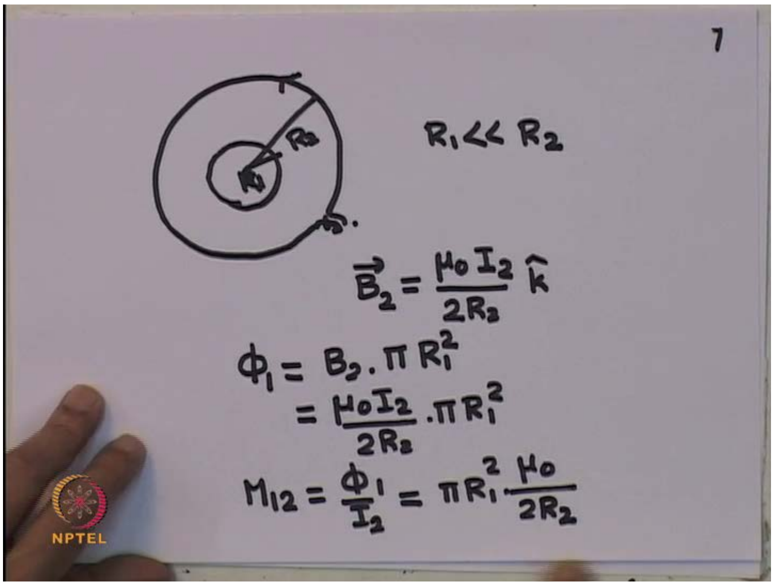
This is not really a general expression, because we have assumed that the two solenoids are tightly bound to each other, but what have what we find is that the relationship between M 1 2. And the self inductances of the two loops can be generally written as some constant kappa times root of L 1 into L 2, and this coefficient kappa will depend upon relative orientation of the loop, and this I can call it as a coefficient of coupling.

(Refer Slide Time: 21:50)



Let us do another exercise, I am talking about, I am taking two circular loops, and again I made the problem somewhat simple, there is two co-planner are co-planner and concentric circular loop, that is their centers are the same and they are in the same plane. And I have to assumed that the current is changing through the second circle the bigger circle.

(Refer Slide Time: 22:22)

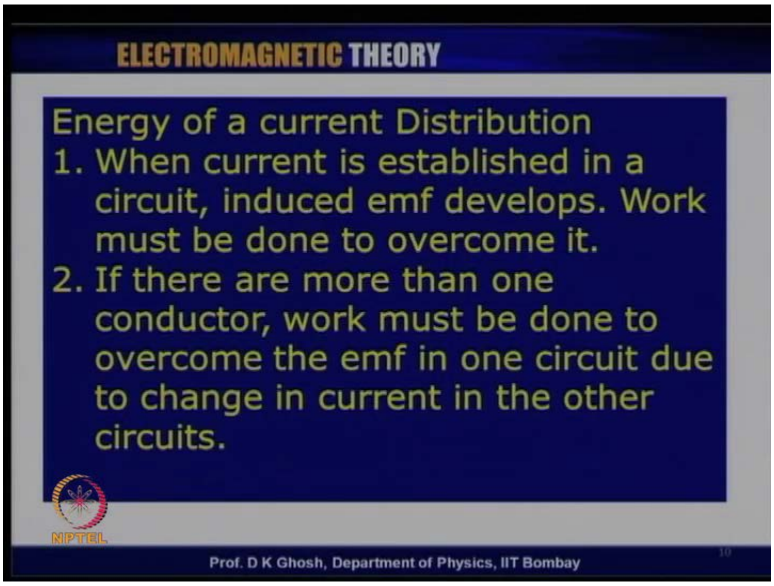


Now, if I assume that the current is changing through the bigger circle, and there is a smaller circle inside I have, I am magnified it but, let us assume that R 1, R 1 is much smaller than R 2. Now, when that happens, I know how to find out the magnetic field due to a circular loop at it center, and that expression the magnetic field due to the bigger circle is given by mu 0 I 2, because I 2 is the current in that divided by 2 R 2. Once again I could give it a direction, which is let us say z axis, because I could change the make the current flow that way, so that if I, I use the right handed rule and so therefore, thus that could be this direction. Now, it is reasonable to assume, that if R 1 is much less than R 2, then the magnetic field strength at the center of the bigger circle is roughly the magnetic field strength over the smaller circle. So, the so once I make this statement that then the flux through the first circle, will be simply B 2 into pi R 1 square, and B 2 we have seen is mu 0 I 2 by 2 R 2 into pi R 1 square, so therefore, M 1 2 is divided by phi 1 divided by I 2 is given by pi R 1 square mu 0 divided by 2 R 2; once again it shows that mutual inductance is a property of the geometry, and mostly the dimensionality of the fix.

So, with these examples, let me now go over to a slightly different idea, let us talk about how to find the energy of a current distribution, recall that when we did electrostatics, we discussed how to find the energy of a charge distribution. And what we did at that time was, assume first initially all the charges are away at infinity, and that I took as the situation of 0 energy, then I brought one charge put it in its place, no work is done, because there is no field.

But, once the first charge comes, there is another electric field than of course, I as I bring in the charges, I would require it would be necessary for me to do work against the electric field that have been established, and we calculated how much is the energy. Now, I have a problem now, I cannot remove the current distribution to infinity so I cannot do that but, so this technique of starting with a 0 potential energy, and then establishing the charges will not work here.

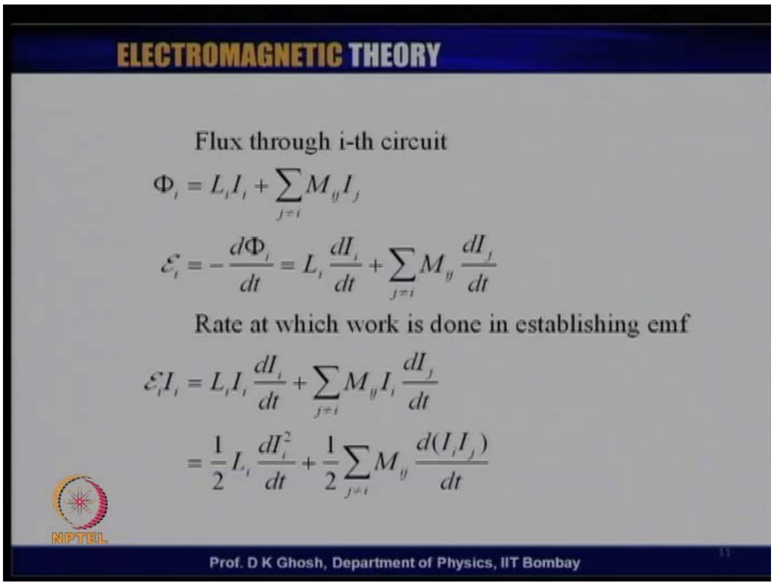
(Refer Slide Time: 26:04)



So, let us let us see what is the way in which we will do it, now so these are the basic principle, when we when we establish a current in a circuit, we have just now seen an induced emf is different, now I we must do work to overcome this induced emf, this induced emf we have seen by (()) opposes the cause. So, therefore, if you are moving, for example if you are moving a magnet towards a circuit, the circuit will generate an induced emf, circuit will have an induced emf, and if that happens to be a conducting circuit then of course, there will be a induced current.

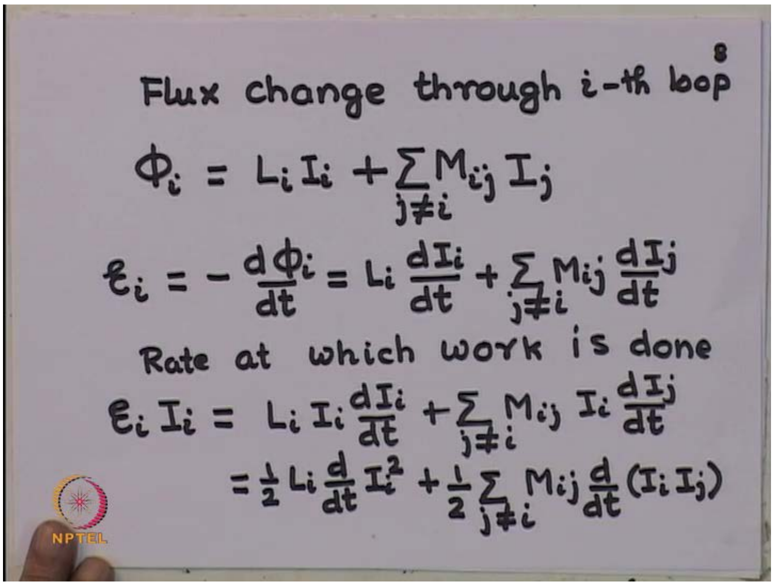
So, therefore, if you want to move for example, this magnet with a uniform velocity, you will have to do work on it; now unlike the case of resistance, this work is completely recoverable, because the you know you get if you are moving something, you need to do some work, but then you can always get it back in the reverse situation, that is situation number one. And even if there is one circuit, there will be an emf in that circuit and we will have to overcome, do work to overcome it, if there are more than one sub conductors, then we had seen there are mutual effects that is supposing, I concentrate on a particular loop, let us call it loop number 1. Now, loop number 1 will have an emf generated, because of changing current in loop number 1 itself, secondly when there are changing currents in other conductors loop number 2, 3, 4, etcetera, then also there will be an emf generated in loop number 1, and we must do work for that question. And so in order to calculate the total energy, we have to take into account both of these situations, so let us see how it works.

(Refer Slide Time: 28:11)



So, let me I am going to calculate, how much is the flux through I th circuit, supposing I have got very large number of loops, what is the flux changed through the I th circuit

(Refer Slide Time: 28:22)



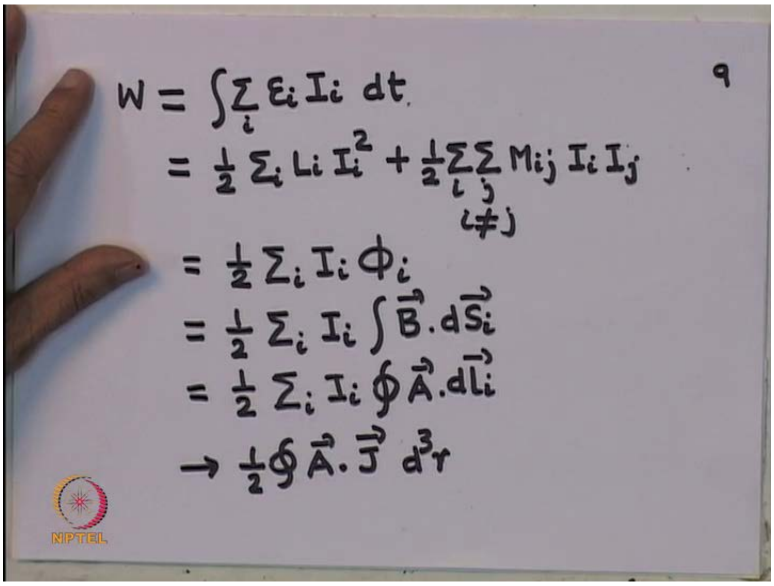
I th loop let us say, so we did that just now, so let me call it phi i. Firstly, we notice that, there is a there is a flux generated, if there is a current I is in the circle loop, I th loop itself, and we had seen that the proportionality constant was the self inductance, since it is the I th loop I will call it L i times I i. The second part is that, we have seen that, when there are currents in the other circuits j not equal to I, then I have a flux generated in the I th circuit which is given by M i j i j, where M i j is the mutual inductance between i and j; and I am assuming that M i j is equal to M j i.

So, the emf through the first circuit, I th circuits which is minus d phi i by d t that is equal to L i d I i by d t plus sum over j not equal to i M i j d i j by d t. So, this is the emf, and this is the emf established in I th circuit, so the rate at which a work must be done to establish this emf or to overcome this emf. So, rate at which work is done that is obviously, given by E i times I i emf times iso this is simply multiplying these with I i all over let us do that, L i I i d I i by d t plus sum over j not equal to i M i j I i d i j by d t.

Now, I will write it in a little more symmetric fashion, firstly notice I i d I by d t can be written as a half of L i of course, is a constant d by d t of I square, I i square. So, that d by d t of I i square is 2 I i d I i by d t 2 and half cancels out, the second term requires a little bit of explanation, but it turns out I can write it like let me, first write it down and then we will see why half sum over j not equal to I M i j d by d t of I I i j.

Now, this is really nothing great, because what we have done is to say this is d I i by d t multiplied by i j plus i i times d i j by d t; now because, M i j is equal to M j I,I could interchange like this, that is write it as your d I by d t I times d j by d t I j times d I by d t. So, you notice, so that these are written very symmetrically, these are written very symmetrically, so with this, we can now calculate how much is the total work done, so this is the rate at which the work is being done, so that you need to integrate it over d t.

(Refer Slide Time: 33:07)

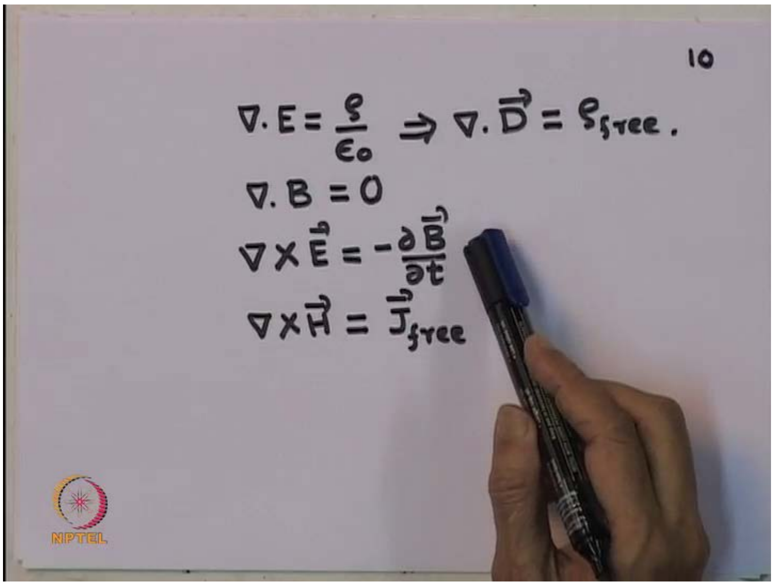


So, that this will give me the work done, which will convert to the potential energy, which is integral sum over i E i I i strictly speaking, this last line that I did it is valid only if I sum over , so that you can sum over this, and there is a sum over I here, and then I use M i j equal to M j i. So, then this once I integrate over time, I find I get half sum over i L i I i square and plus half sum over I sum over j, I not equal to j of M i j I i I j.

Now, if you bring back the expression for the flux, this is the expression for the flux, you notice this expression is nothing but, sum over, this is nothing but, sum over I of I i phi i multiply I i with I i with I i square, and multiply here you get I i I j and sum. So, this is equal to half sum over i I i times phi I, now we will write it in a slightly different fashion now, this is half sum over i I i this is flux, so flux by definition B dot d S, so I write it as B dot d S , so this is the flux through the I th circuit. And I will write this again, write B as del cross of A, so del cross A dot d S I, and then I use Stokes's law, so that this becomes loop integral of A dot d l I, this is for discrete. Now, how will it change if I have it continues distribution, now see if I have continues distribution then of course, all that I need is this that, this I i with a surface will give me the current, and so therefore, this will go to half of loop integral of A dot J current, this is current density and d cube r, the current density because, this is per area I mean we are current by area, so here since there is just a length, so therefore, I have got it d cube r there, because I have divided by this thing.

So, this is the expression for the work done by did by establishing, while establishing currents in various circuits, and this is corresponding continues version, with this much of introduction of Faraday's law, let me go over to a slightly different story. Now, let us recall where are we with respect to the electromagnetism at as of now.

(Refer Slide Time: 36:59)



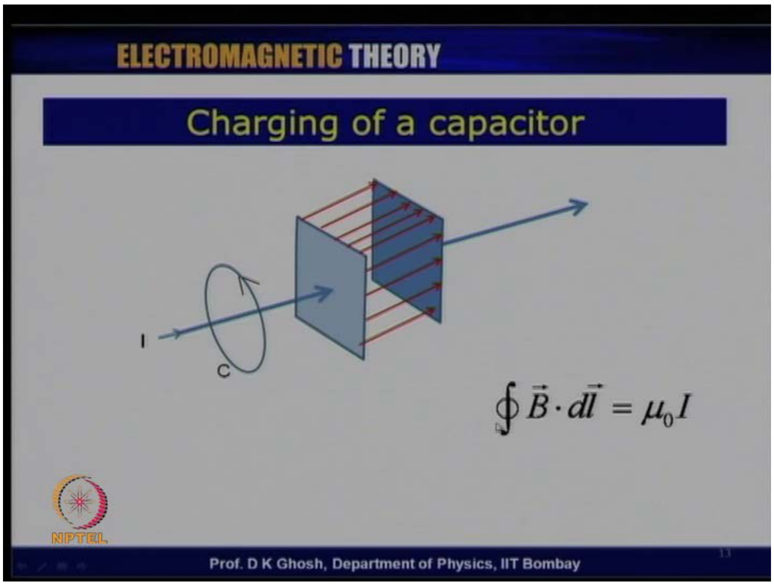
So, we had del dot of E equal to rho by epsilon 0, del dot of B equal to 0 this actually will not change generally, then we have just now seen that del cross of E is minus d v by d t, we have not yet talked about, what happens to this del cross of B equal to mu 0 j or the corresponding expression del cross H equal to just with J free. Similarly, here if you want to talk in terms of the displacement field than this is simply equal to rho free, so this is where we stand with respect to Maxwell's equation.

But, you notice that we understand the a symmetry between del dot of E and del dot of B, because this tells us there are charges, this tells us that there are no magnetic monopolies. Whereas this seems to be some difference between the way these appear also, once again I understand why the term corresponding to J free is not there, because there are no magnetic monopoles which are moving around, but what is not clear is why is in there a time dependence here. So, this was actually Maxwell’s contribution to the electro dynamics, and that is why these set of equation as modified later are known as Maxwell’s equation, so, let me go through that.

Firstly, let us look at a simple example of charging of a capacitor, now remember that when I pass a current through a circuit which contains a capacitor, the there is no current which passes through the gap of the capacitor. But, however, there is an electric field established in the capacitor, we will take the simple example of parallel plate capacitor; and we know that l I am charging a capacitor, the electric field through this changes.

Because, electric field will depend upon what is the instantaneous charge on the plates of the capacitor, and as you connect a battery to a circuit, even though it is a small time during that time the charge goes from 0 to whatever its final values is. So, during that time there is a change in electric field through a capacitor.

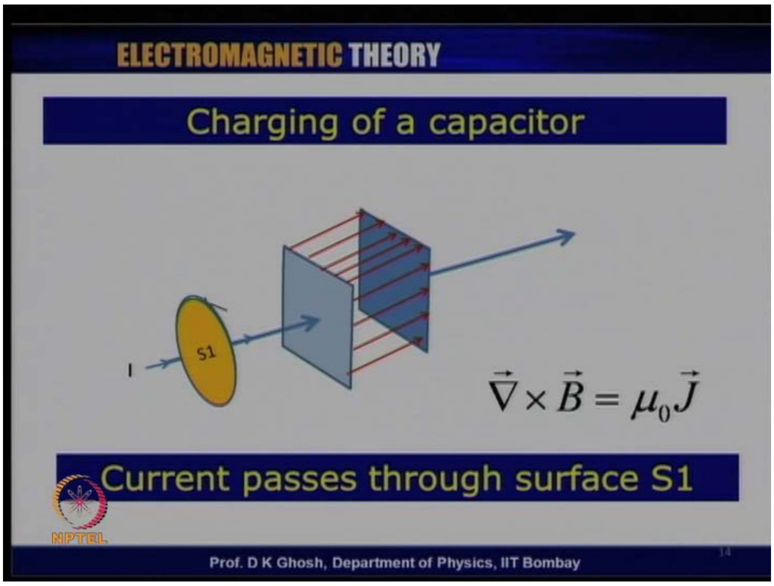
(Refer Slide Time: 40:08)



But, there is a problem and the problem comes up, let us look at the charging of a capacitor, there is a capacitor with a current I flowing in, and this current I is let us say momentarily changing, because of my having connected a battery to the circuit, and the electric field is of course, given like this. Now, consider a loop which I have called as C like this, now I know that I could calculate I could calculate the magnetic field by using the standard Ampere's Law, which says integral B dot d l equal to mu 0 I.

Now, during the charging of the capacitor, the charge accumulating in the plates of the capacitor is changing, as a result there is a current in the external circuit. Now, if there is a current in the external circuit, and I look at this loop I can calculate the loop integral, and say it is equal to mu 0 times I,I is the instantaneous current, that instant at which I am calculating the magnetic field. Now, let us look at what is the problem, now if I have this loop, this is the artificial loop, mathematical loop, now let us suppose that I put S surface on the plane of the loop.

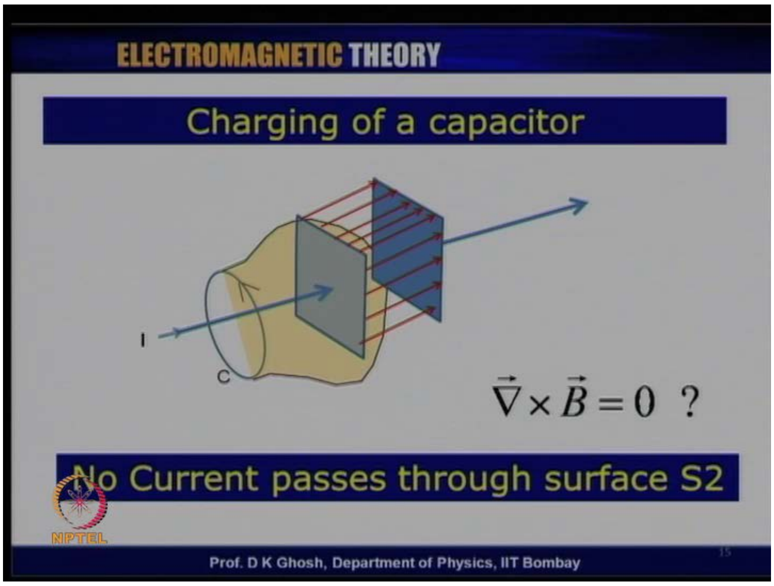
(Refer Slide Time: 41:55)



So, let me do that, this is what we said just now integral B dot d l is equal to mu 0 I, now let me put a surface now on that loop, which I have called as S 1. So, through that surface a current is flowing, and the magnetic flux line are intersecting that surface, so so far as this surface is concerned, I understand that del cross of B equal to mu 0 J, because B dot d l which I had written down integral B dot d l equal to mu 0 I that B dot d l could be converted to curl B dot d S; and over the surface which I have called as S 1, the I can calculate the magnetic field at every point.

And so therefore, del cross B dot d S is my mu 0 I, and you since this will be true for any surface of this type, I expect del cross B equal to mu 0 this is something which you done in the earlier, but let us pause for a moment. We have said several times, if I have a loop which I showed earlier, these result del cross B equal to mu 0 J is independent of what surface I take, as long as this is the boundary of the surface that has been defined. So, suppose instead of the surface S 1 through which the current passes, and as a result the magnetic field lines also passes, let us look at slightly different situation.

(Refer Slide Time: 43:40)

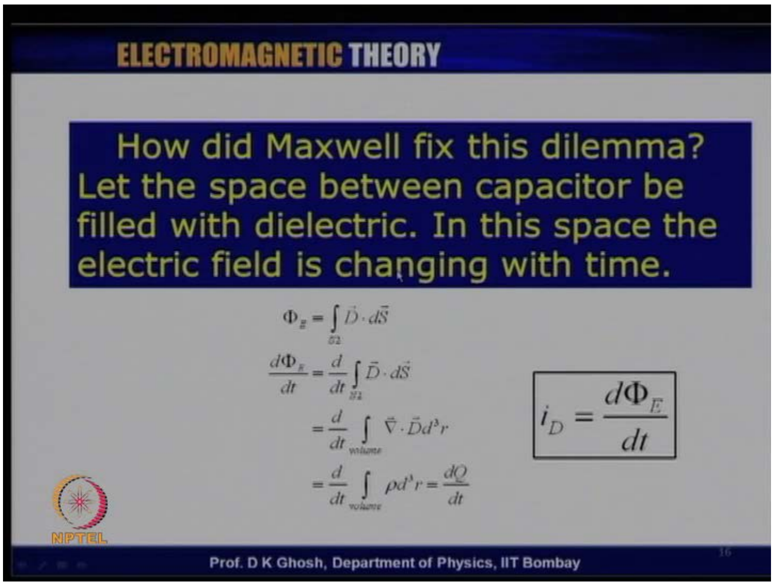


Again I had the same situation, I have however, a slightly different surface you notice this is a pot like surface, I have this same loop C, but this time what is happened is instead of choosing the surface on the plane of the loop, I had taken at like this. So, therefore, all this surface the through this surface there is no current, now if there is no current through that surface, I expect del cross the I dot d l to be equal to 0, as a result del cross B should be equal to 0, but there is a dichotomy we had said earlier the result should not depend upon which surface you choose.

Because, both of them represent the current passing through the loop C 1, so how can I get one result for this type of a service, and another result for the surface S 1 which was on the plane of the loop. Now, what is the problem, the problem is that there is no magnetic field here, so that B dot d S should be is equal to 0, but is there something inside that, now you notice there is an electric field inside that, and that electric field is changing. So, as a result, this is this is a dichotomy which Maxwell had realized, that I I get two different answers for the same problem depending upon surface I choose.

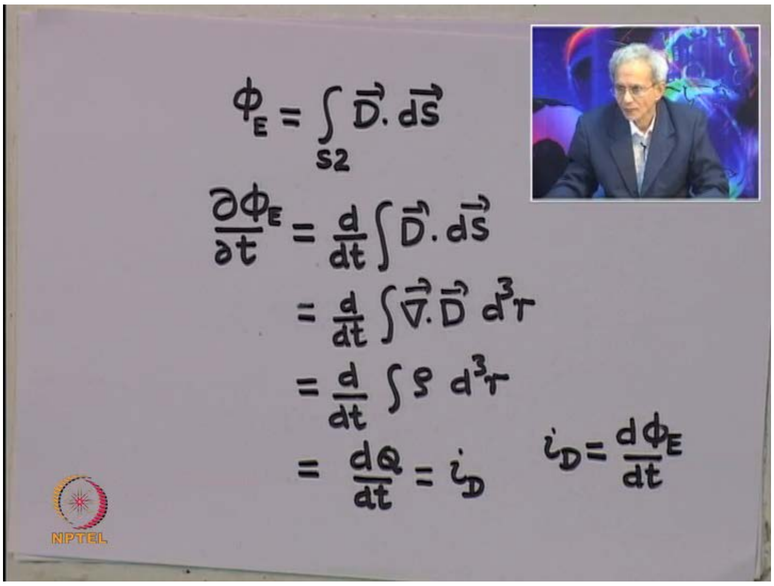
So, what he did is to conjecture, they just as by Faraday's Law, whenever there is a change in magnetic flux, where ever there is a changing it change in magnetic flux through a circuit, there is an emf generated he wanted alternatively a changing magnetic field, a time dependent magnetic field gives rise to an induced electric field. So, Maxwell now postulate, it going by parallelism that if I time dependent magnetic field gives rise to an induced electric field, it is reasonable to assume that a time dependent electric field of the type that I have here, the as the charging is taking place, the electric field through the capacitor is changing; so a time dependent electric field is equivalent to a magnetic field, the effects must be symmetric,

(Refer Slide Time: 46:47)



And this indeed was sort of proved or verified by experiment, So, how does one fix this dilemma, let us look at that, so just to understand let us assume that the space between the dielectric is filled with a space between the capacitor plates is filled with dielectric. Now, if there are dielectric, there are charges and when you change the current through it, the electric field changes that of course, pushes these charges. So, what we say is this, that this is the space in which the electric field is changing with time.

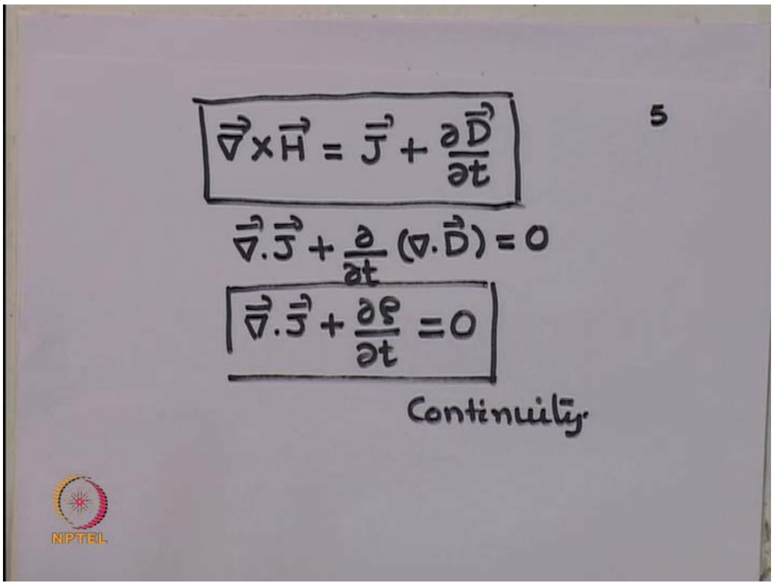
(Refer Slide Time: 47:38)



And let us calculate how much is the what is the effect, so the electric flux in this region is given by surface integral, I have this time I am taking surface S 2 D dot d S, I have taken the flux of the D field, because I have said it is filled with directive. Now, so therefore, d phi by d t, well I should write full, so this is equal to d by d t of integral of D dot d S, and this I will write this in a using the divergence theorem, as d by d t of del dot of D d cube r volume integral, but del dot of D is nothing but, rho, rho free d cube r which is nothing but, the total charge Q.

And so therefore, this is d Q by d t, notice d Q by d t that is the rate of change of charge on the capacitor plate is same as that of the current that is flowing in, so therefore, this is this as the dimension of current. So, what we have said is i D the name came from displacement current, this was Maxwell gave it a name displacement current, that is the rate of change of the electric flux with that time, so how did Maxwell handle it?

(Refer Slide Time: 49:43)



So, Maxwell said that let us still modify, the last law that is the Ampere's Law which was del cross H equal to J by adding a term d B by d t notice that del dot del cross H is the identically 0, because it is divergence of a curl. So, therefore, I get from here del dot of J plus D by d t of del dot of D this equal to 0, but del dot of D is rho, so that give we del dot of J plus D rho by d t is equal to 0, you identify recall this is nothing but, our continuity relationship, thus this is the equation which should replace del cross of H instead of J, should now be added with a term d B by d t. That completes our establishment of the four Maxwell's equation, and we will discuss them all together in a more coherent way from the next lecture.