Electromagnetic Theory

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Module - 3
Time Varying Field (Introduction)
Lecture - 28
Magnetostatics(Contd)

In the last lecture, we had seen how the method of scalar potential can be used to find out the magnetization and the magnetic field due to a uniformly magnetized sphere. What we do today is continue in the same line and try to solve the same problem, namely when we have a uniformly magnetized sphere, what is the corresponding field d and the field h due to it. But this time we will be using the method of vector potential of course, both of them being equally good in this case will lead to the same result.

And first we will do that, after that in the second half of this lecture we will be introducing, we will be till now we have been talking about static phenomena, we started with electrostatics and then we went to magneto statics. We will be for the first half we will still be with magneto statics, but in the later half we will bring in some Time Varying phenomena and introduce some other aspects of the Electromagnetic Theory.

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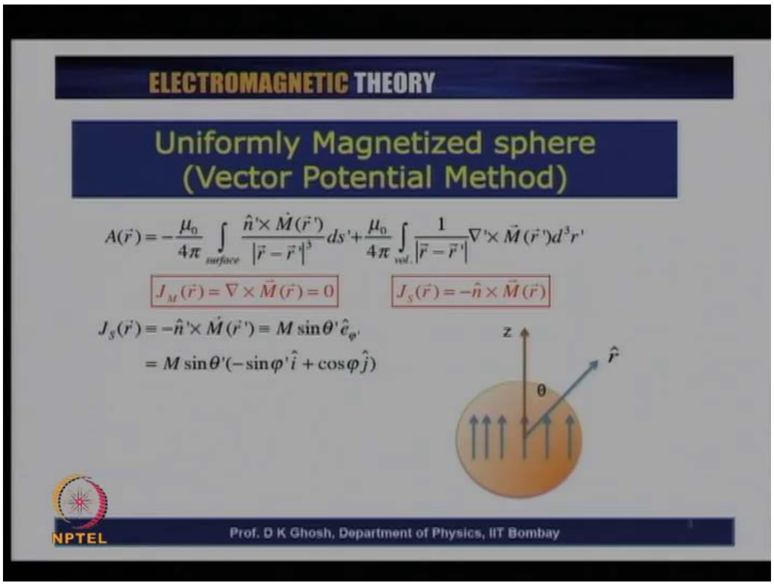

So, let me begin by reminding you, that we had seen that if I have magnetization m the corresponding magnetic vector potential due to it.

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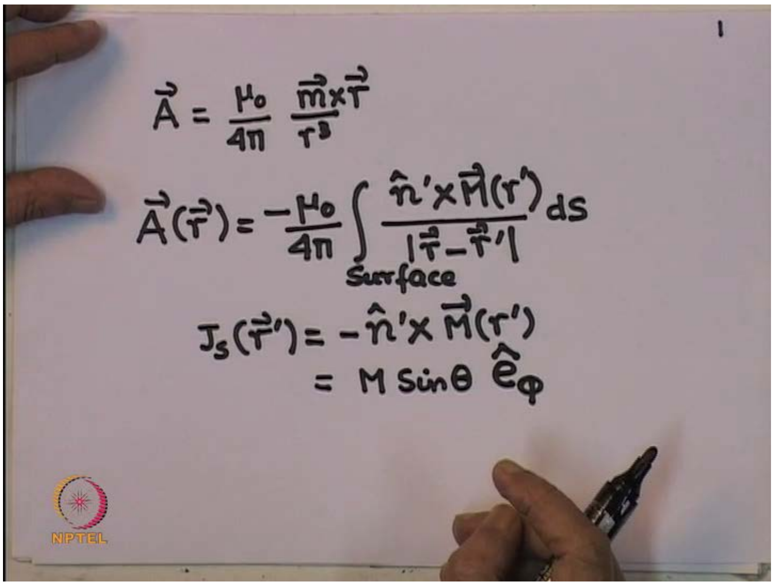

A was given by mu 0 by 4 pi, well basically m divided by r cube. Now, what we did is to say that if I have a magnetization distribution, then there is actually M cross r divided by r cube. Now, if I have a distribution of magnetization, then we are able to write it in two parts, one was a surface integral part and another was a volume integral part like here. And we had seen that the vector potential a can be written as a surface integral of n cross M by r minus r r prime cube.

So, that tells me that there is an equivalent surface magnetizing surface current if you like, which is equal to minus n cross M, remember inside the integral all my quantities are primes. Now, and there is a volume density, which is given by del cross M, but in this case I have got A uniform magnetization. So, as a result my del cross M term is 0. So, I need to simply concentrate on the term which is surface integral term, which is written as A of r equal to minus mu 0 by 4 pi integral over the surface of n prime cross M of r prime over r minus r prime and d s.

Now this is quantity that we will like to calculate and this n cross M or with a minus sign, that is my equivalent surface current due to magnetization. So, J s of r in this case r prime is given by minus n prime cross M of r prime and what we will do is if you refer back to this picture, we will take the direction of the magnetization as the z axis.

So, that the angle between the direction of magnetization and the radial vector will be taken as theta. So, therefore, this quantity is nothing but M sin theta and the direction of that is clearly the azimuthal direction e phi this is a direction on the surface of the sphere, which is perpendicular to both the radial direction and the z direction.

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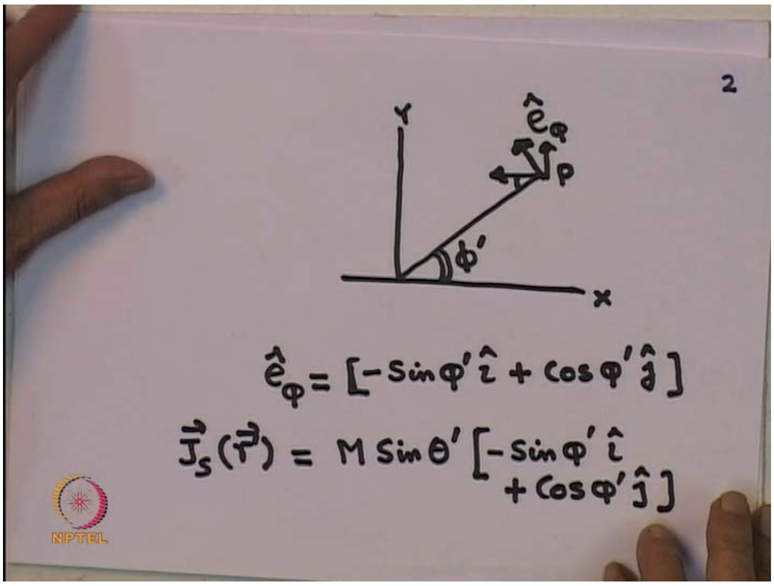

Now, if you look at how the phi is defined I will just recall for you, I am not drawing a three dimensional picture, but I am drawing a two-dimensional picture. Remember that the radial vector is along the radial direction is the r vector and what I do is to drop a perpendicular on to the x y plane. And it is the foot of the perpendicular which is, so this P is the foot of the perpendicular from the tip of the radius vector and this angle, which you make with the x axis with the x axis that is my azimuthal angle phi.

Now we had seen earlier that whenever we define a unit vector it is defined in the direction of the increasing quantity. So, in this case; obviously, my unit vector will be this, so this will be my unit vector which I wrote as e phi. So, as a result this e phi is can be resolved into a component along the x direction which is minus, so this angle is being phi this angle is phi this is 90 minus phi and a component along the y direction.

So, therefore, let us write down e of phi to be equal to M sin e of phi is minus sin phi well I am using phi prime, so minus sin phi prime unit vector I minus because that is that direction plus cos phi prime unit vector J. So, this is my e phi and we had seen that J s of r prime, which is what I am going to write down is given by M sin theta prime times this quantity e phi there which is minus sin phi prime i plus cos phi prime J and I need to now calculate the integral, which is involved in the expression for the A of r.

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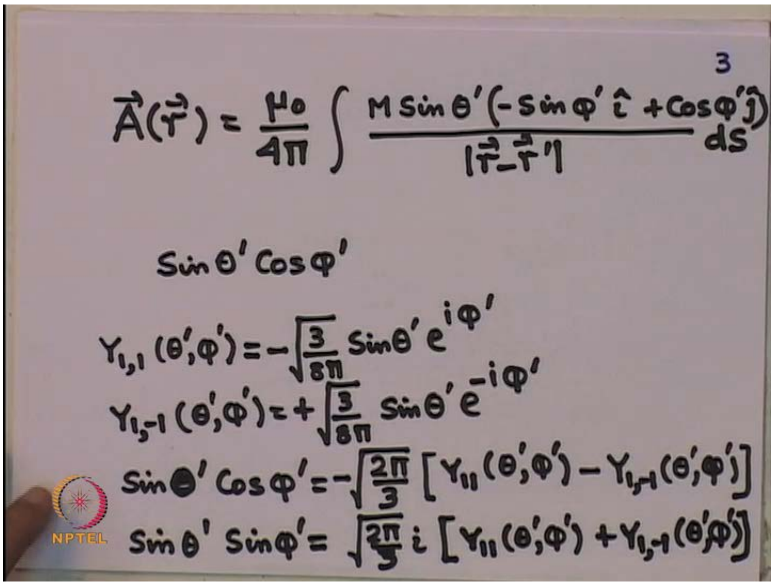

So, let us go back to A of r, so A of r is given by mu 0 by 4 phi I am taking care of the minus sign, which is outside and this will is equal to M sin theta prime into minus sin phi prime i plus cos phi prime j divided by r minus r prime and d s of course. So, the question is how do I evaluate this integral, now what I will do is I will calculate the part, which has which is along the y direction.

And the reason is as I go along I will make a comment, which will show that this term can be taken- shown to be is equal to 0 and we will see why but at the moment I will be just evaluating, this term with a cos pi prime j, now notice that in doing, so what I do is to use the property of the spherical harmonic the orthogonal property of spherical harmonic. So, we know that 1 over r minus r prime can be expanded in spherical harmonics. So, what I will do is this that since I told you that I am going to do only the y component. So, we will we have in y component in the numerator, where the quantity which is sin theta prime cos phi prime our idea is to express this quantity as in terms of spherical harmonics.

Now, you can check that the from the spherical harmonics table that I can write Y 1, 1 theta phi in this case theta prime phi prime does not matter is equal to minus square root of 3 by 8 pi sin theta prime e to the power i phi prime. And Y 1 minus 1 theta prime phi prime is plus square root of 3 by 8 pi sin theta prime e to the power minus i phi prime. So, what are you do is this, I can use the definition of sine and cosine that sine is of course, already there and cosine, I know is e to the power i phi plus e to the power i phi minus i phi by 2.

So, therefore, sin theta prime cos phi prime can be written as square root of 2 pi by 3 there is a root 3 by 8 pi, but I need also a factor of 2 in my definition of cos phi and sin phi. This times it turns out to be Y 1 1 theta prime phi prime minus well actually there is a overall minus sign also minus Y 1 minus 1 theta prime phi prime. Well, sin theta prime cos theta phi prime is this, sin theta prime sin phi prime, which is there which is I am writing it down, so that you can see why that term turns out to be 0.

So, sin theta prime sin phi prime you can easily show that it is given by root to pi by 3there is an i there, because sin is defined as e to the power i phi minus e to the power of minus i pi by 2 i. So, this into this is Y 1,1 theta prime phi prime plus Y 1 minus 1 theta prime phi prime. So, basic idea is like this that I am going to express 1 over r minus r prime in terms of spherical harmonics recognize the fact that is in the numerator also I have spherical harmonics.

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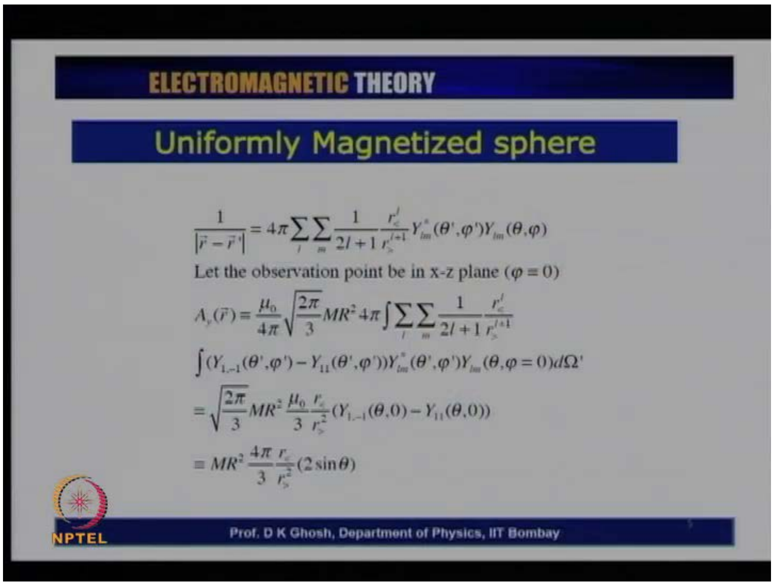

So, as a result what I am going is to write down the entire thing in terms of spherical harmonics.

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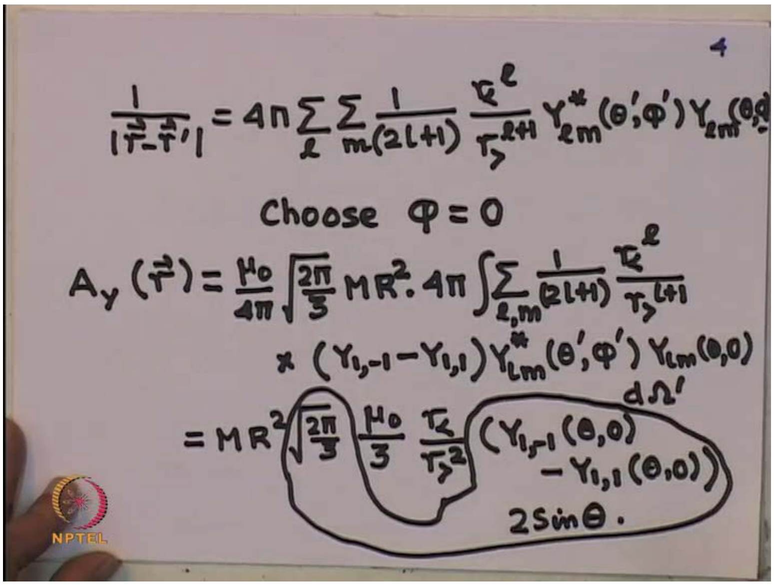

I recall that 1 over r minus r prime can be written as 4 pi sum over l sum over m 1 over 2 l plus 1 this is something, which you did last time also r lesser raised to the power l by r greater raised to the power l plus 1. Y l m star theta prime phi prime which are the angles corresponding to the vector r prime into Y l n theta phi which are the angles corresponding to the vector I. Now, what we do is this, I define my point of observation as in the x z plane. See, remember that I can always do that because the only direction, which is given to me is the z direction along which the magnetization direction is there. So, I have the liberty of choosing the x y plane.

So, therefore, I choose my observation point to be in the x z plane, which means I choose phi to be equal to 0 this is this is an important point which simplifies the problems of transit. With that A remember, I told you that the x component will go away, which I will make you comment on later, but let me let me say what is A Y, which is which comes from sin theta cos phi prime pi. So, I had this mu 0 over 4 pi I have viewed up 2 pi by 3 M R square came out of R square d omega which is the angle integration, 4 pi from here.

And then I have an integral to be performed integral over l m 1 over 2 l plus 1 this is a rather long expression, but terms are to be reasonably easy r lesser raised to the power l r greater raised to the power l plus 1. Then, I have actual integration will be over well if you recall I had shown this to be equal to Y 1 minus 1 minus Y 1,1 I had taken care of this minus sign by interchanging the Y s. Then, I have the 2 Y l m’s which came here, Y l m star theta prime phi prime Y l m theta and 0 and of course, d omega prime. Remember that, these also have theta prime phi prime as their arguments why because it came from sin theta prime cos phi prime.

So, if you use the orthogonality condition of this, then the integral see for example, when you integrate this one with that one, then it tells you that l must be equal to 1 and n must be equal to minus 1. And similarly in this case l is equal to 1 and m is equal to 1, so I get two terms and. So, what I am left with will be is something like this (( )) of course, M R square the I had this root 2 pi by 3 I will retain, because this term here.

So, I got still root 2 pi by 3, 4 pi has cancelled out with that, so I have a mu 0 since l is equal to 1 I get 2 l plus 1 which is equal to 3. Then of course, I have r lesser raise to the power l and l is equal to 1. So, therefore, it is simply r lesser by r greater to the power l plus 1, which means r square times this time it is Y 1 minus 1 of theta not theta prime 0 minus Y 1,1 and theta 0. Now remember, y both Y 11 or Y 1 minus one there e to the power i phi and e to the power minus i phi respectively.

But, since phi is equal to 0 each one of them each one of them turns out to be a sin theta, but Y 11 had a minus sign in front of it. So, therefore, this quantity turns out to be 2 sin theta, so this along with this factor of 2 pi by 3, this turns out to be 2 sin theta. Well at this stage I will also point out y the x component would minus in the x component I have a very similar integral. But, I have I will have Y 1 minus 1 plus Y Y Y 1,1 and so that would be sin theta minus sin theta which will be equal to 0. So, other than that this is essentially identical.

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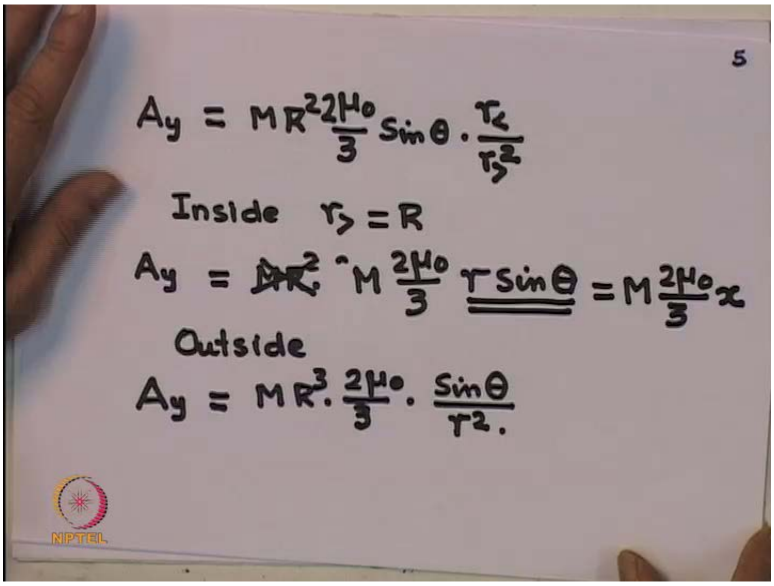

So, let me then with that write down what I get for my A y. So, A y so vector potential only has y component, which is equal to M R square mu 0 by 3; well actually 2 mu 0 by 3 sin theta and r lesser divided by r greater whole square. So, this is this is the expression for the y component of the vector potential. And so what I now do is that if I am inside the sphere, I know r greater than is capital R.

So, the potential inside, so I get a R square which will cancel with this R square there. So, I will be left with M R square sorry I will be left with M 2 mu 0 by 3 and r sin theta. So, this is what I would have inside the sphere and outside the sphere I will get A y again is given by M outside this is r, r lesser is r. So, I get a M R cube then 2 mu 0 by 3 and r greater a small r, so therefore, I get sin theta over r square.

Now, if you remember that since phi is equal to 0 my r sin theta is nothing but, my x. So, it is M 2 mu 0 inside only by 3 x, I am not doing this, but it because it has slightly more complicated, but you could do it as an exercise. Now, so therefore, inside this sphere my vector potential has only y component and it is proportional, the y component of the vector potential is proportional to x. If you recall we had shown earlier that, if y component of the vector potential is linearly x it means a constant magnetic field.

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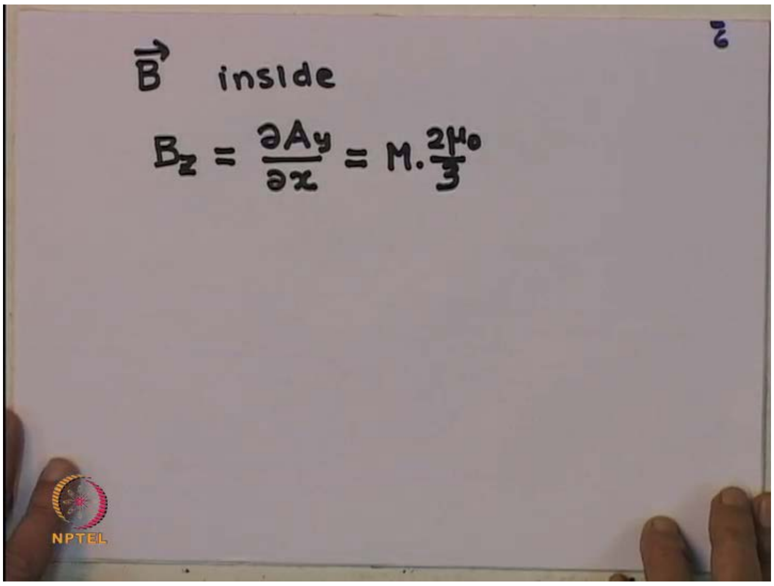

So, I can immediately evaluate the magnetic field inside and obviously, it only have z component because B z will be than d A y by d x minus d A y d A x by d y which is equal to 0. And since the vector potential is proportional to x this simply gives me M times 2 mu 0 by 3. Now this is this is the identical to the expression that we had obtained using a scalar potential, of course, it has to because it is the same thing.

Before we conclude the subject of magneto statics, let us do something else, let me put this magnetized sphere in an external magnetic field of strength B 0. Now, since the equations are linear I can simply add, so we had in the absence of the external field.

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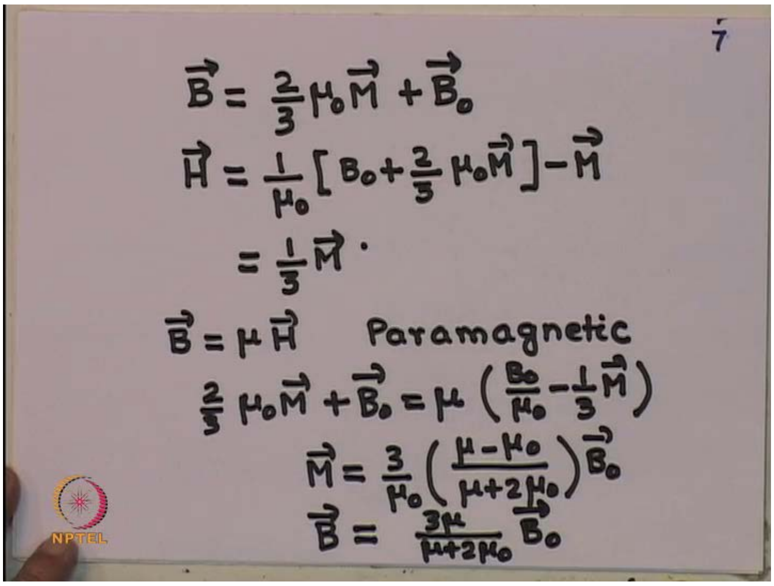

So, I am doing magnetized sphere in an external field. So, in the absence of the external field I had this as 2 by 3 mu 0 M, M is the magnetization and if I have an external field I of course, add d 0 to it. Now, I will define I will get the corresponding value of H in terms of 1 over mu 0. So, I have got B 0 plus 2 by 3, which is simply the value of B minus M this is simply 1 over mu 0 B minus M this expression and if you simply it turns out to be simply equal to 1 3 rd of M. So, this is this what I would get if I put in an external field now couple of comment, that suppose this material of the material of the sphere happens to be paramagnetic. Now, we had said that the just as in the absence of a magnetic material, my relationship between H and B is B by mu is H. Now, if I have a paramagnetic material then my B is related to H through mu H. So, this is for paramagnetic material, where mu is the permittivity of the material. So, let us see what does actually it mean.

So, this means in that case what I will get is this B which is 2 by 3 mu 0 M plus B 0 will be equal to mu times B 0 by mu 0 minus 1 3 rd M this is what it would be. Now, you can solve this equation and you get that magnetization M is written as 3 by mu 0 mu minus mu 0 by mu plus 2 mu 0 times B 0. And corresponding value of the magnetic field b will be 3 mu by mu plus 2 mu 0 times B 0. I argue to look up the expression that we had derived, when we had put a polarized electrically polarized material in to an electric field.

And you will find that there is almost a direct similarity between these two expressions. The last comment that you want to make is that, what happens if I have what are known as ferromagnetic substances. So, we had see that in case of ferromagnet, we have a relationship where B and H are parallel the if you reduce the external field to 0 the B field also goes away. Now, there are materials which are known as ferromagnets.

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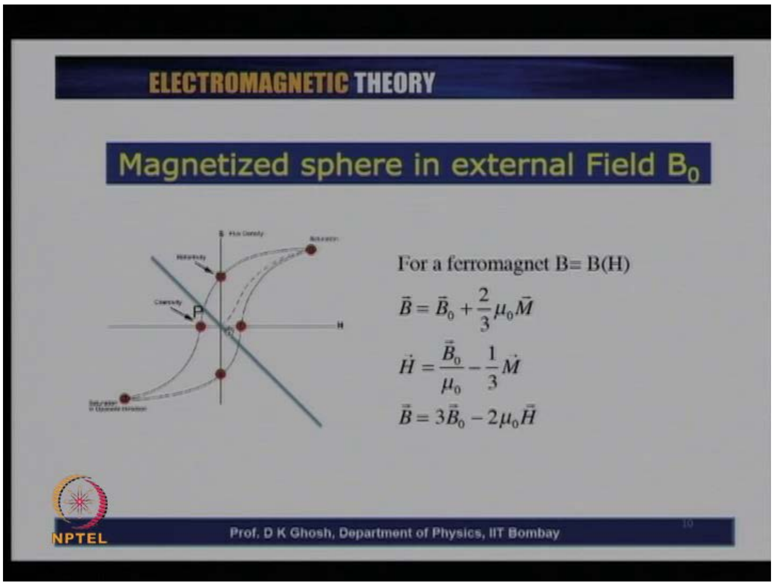

And this is a picture what happens in case of a ferromagnet. So, I am as I apply external field the initially B raises, but for a certain strength of the external field, the value of B saturates. Now, the reason is that the ferromagnetism appears due to the magnetic movements being aligned. Now, once in a particular material all the magnetic movements are been aligned, I have nothing more to align it. And that then is the limit after this, if you increase the electromagnetic field nothing will happen, because saturation has been reached.

Suppose, when you saturate a ferromagnet by applying a magnetic field and then you start reducing the strength of the magnetic field. Now, something interesting happens unlike in a case of a paramagnet, the as you reduce the external field, the magnetization or the B direction does not retrace it is own path. In fact, it picks up in different path and when it picks up a different path, it means still have some memory even after you have reduce the external field to be equal to 0, the magnetization is not lost..

So, this is what is known as retentively or the process of hysteresis, the material in some sense remembers having been subjected to a magnetic field in the past. So, you prepare a magnet by applying a magnetic field, then gradually bring the strength of the magnetic field to 0, you will find the magnetization will still exists as is earlier. In fact, you will need a magnetic field to be applied in the negative direction, if you want the magnetism to vanish and this is actually this value is called the cohesive field.

And of course, there is then a symmetry of this loop is known as a hysteresis loop. Now so therefore, we have now obtained two expressions. So, there is an expression, which we derive namely d the total magnetic induction field is B 0 external field plus 2 by 3 mu 0 M, but this expression, this curve gives me a dependent of B on H which is some function or relationship. So, what I do is this that the when I have this let me write this equation for H you go back you can find that H is written as B 0 by mu 0 minus 1 3 rd M.

And if you solve this 2 equations together I by eliminating M you can get this expression, which is B is equal to 3 B 0 minus 2 mu 0 H. This tells me that the relationship between B and H is a straight line with the slope being equal to minus 2 mu if you are plotting it against mu 0 H. Now

so therefore, the I have a relationship which is given by the hysteresis and I have derived a relationship between B and H. So, what I can do is this, that I have once I have subjected the ferromagnet to saturation and reduced then I find out where is the intersection of this with B versus H relationship. And this is that point P there which will sort of tell me how much is the value of the magnetization at that point. So, this is the way one determines the magnetization for a ferromagnetic substance, that brings us to an end of the magneto statics.

And what I am going to do now is to switch over to phenomena time dependent phenomenon. So, but before I do that let me start with a few definition. So, one of them which we talked about in case of electric field very similarly one can define magnetic flux.

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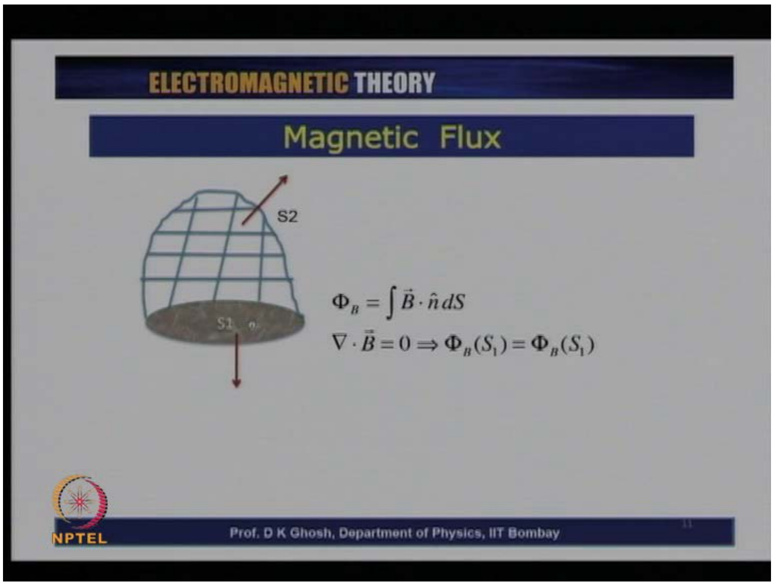

The magnetic flux will be defined as this B dot d S, now look at this situation here I have a surface S 1, which is this oval shaped plate and on the same rid of S 1 I can have a perseverance of net type of a surface as well I call this S 1 I call this S 2. Now, since the direction of the surface is always defined as an outward normal, the normal to the surface S 1 is like this and the normal to surface S 2 is like that. Now, if you now use the fact that del dot of B equal to 0 which means the surface integral of B B dot d S is equal to 0.

Then, you can easily show that as long as the boundary is fixed it does not matter at what surface you take to define flux. So, phi n of S 1 equal to phi n of S 2. The second point that I want to talk to you is what is known as electromotive force. Remember or let me alert to you that what I am going to talk about is not a force at all it is called electromotive force, because of historical reason, but let us see what happens. Supposing you have an electric circuit and the electric circuit connected to a battery and in the external circuit you have a current.

Now, you can assume something like a ohms law, so that you know that the current is nothing but, the electric field divided by the resistance. So, what I know is the integral of E dot d l is not equal to 0 because there is a current in fact, flowing. However, we have seen in electrostatics the integral of E dot d l must be equal to 0. So, how does one understand this apparent contradiction, so one over things that I would like you to realize is that integral E dot d l is equal to 0.

Alternatively del cross of E is equal to 0 was a consequence of the fact that I was talking about electrostatic fields which are conservative. Now, what happens in a circuit is that for instance let us say that the current is flowing, because of a battery connected to the circuit. So, I have an external loop where I can calculate E dot d l going from let us say the positive terminal to the negative terminal and I get some value. Now, when I complete the circuit through the battery, the battery provides the motive force by usually in case of a battery converting chemical energy to electrical energy and that is a non conservative field.

So, therefore, a the this will provide let us say an integral E prime dot d l which inside the battery will exactly cancel the E dot d l that I have calculated on the outer circuit. So, therefore, my electromotive force is actually integral if you like inside the battery from terminal to terminal of integral E dot l; however, this non conservative field is 0 outside the battery.

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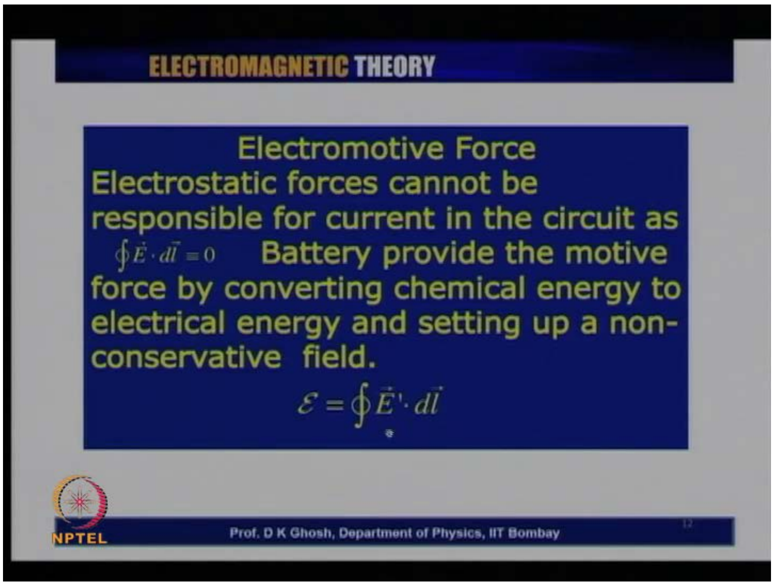

So, I can actually complete the loop and redefine my electromotive force through this relationship, that is electromotive force which I use script E for it is loop integral of E prime dot d l. Where E prime is the electric field that the battery has produced.

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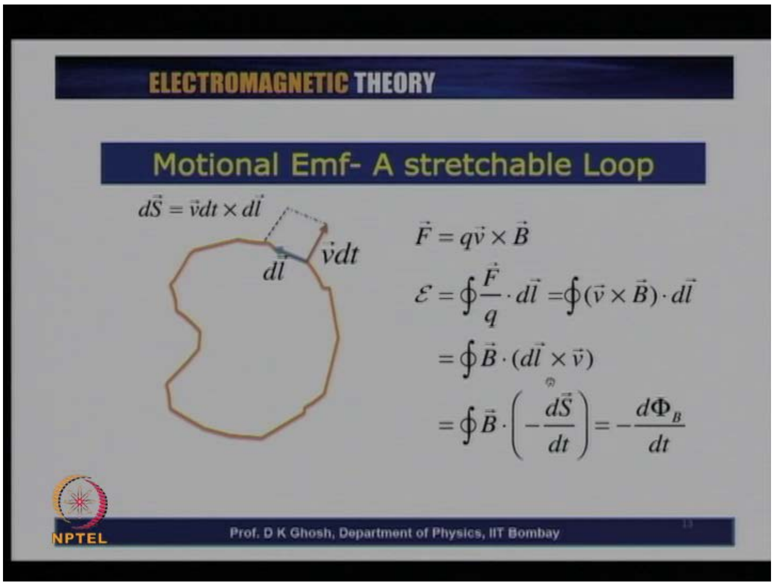

Now, as I said we are talking about time dependent phenomenon Faraday has had enunciated a law from observation. And what he had said is that he found by a sort of experiments that whenever there is either a relative motion between the source of magnetic field and a circuit. Or if by some reason even if there is no relative motion the magnetic field is changing with time. In both the cases it gives rise to an equivalent or gives rise to an emf and what Faraday had said is that the emf is equal to the rate of change of magnetic flux through circuit.

There is a another law which goes along with the faraday's law and that is why you will find when we write down the faraday's law there will be a minus sign that it electromotive force is equal to minus d phi by d t. This minus is not be treated as an algebraic quantity, but it is a remainder of the fact that there is another law called the Lenges law, which tells us that if this is connected to a circuit conducting circuit.

The direction of the current will be such that, that this current would generate magnetic field of it is own which will tend to oppose the cause which generated, namely if the flux was increasing as a result of over action the induced current will be in such a direction that the flux would tend to decrease as a result. So, let us first talk about couple of cases connected with this production of E y. Now if my circuit is movic, if there is motion in the circuit or relative motion I find it easy to explain the Faraday's law and the explanation comes by the Lawrence force that acts on the charged particle.

Remember that is a circuit is moving, the circuit has charges this charges themselves then are physically moving along with the circuit. Now, so therefore, when a magnetic field is applied on a circuit the charges that a that exists inside the circuit are now subjected to a Lawrence force. And this Lawrence force can make the charges move, so that would be a source of my electromotive force.

Now, let us look at that situation here, look at the circuit which I have indicated in origin pattern and let us suppose that this is my d l that is the circuit will be moving like this. Now, when this happens the circuit is for example, instantaneously moving by an amount d l. So, the charges in that will be subjected to a force, suppose a charge q in that will be subjected to a force equal to q times v cross B.

So, electromotive the mmf which is the integral of electric field dotted with d l which is F by q dotted with d l, which is simply v cross B dotted with d l. Now, remember that what we have is the charges are subjected to a transverse force. So, the velocity direction is shown like this and supposing as a result this portion of the circuit tends to stretch like this, then the idea is that this d l would suit an area this d l will suit an area and so this expression here is integral of B cross B dot d l that came from the Lawrence force.

I use the fact a dot b cross c is B dot c cross a etcetera and rewrite it as equal to integral B dot d l cross v. Now so d l cross v this is d l, this is v is the rate at which the area changes, because d l times the distance along this would be the area. So, therefore, this quantity can be shown to be that there is a minus sign you can see it from the v d l cross v is done. And so therefore, this is B dot d S by d t where s is the area and this is nothing but, the rate of change of flux with time.

So, therefore this tells me the way Faraday's law was treated that the emf is minus d phi by d t. Now, this is a case which we used the fact that I have a loop, which is being stretched now supposing when loop is not getting stretched.

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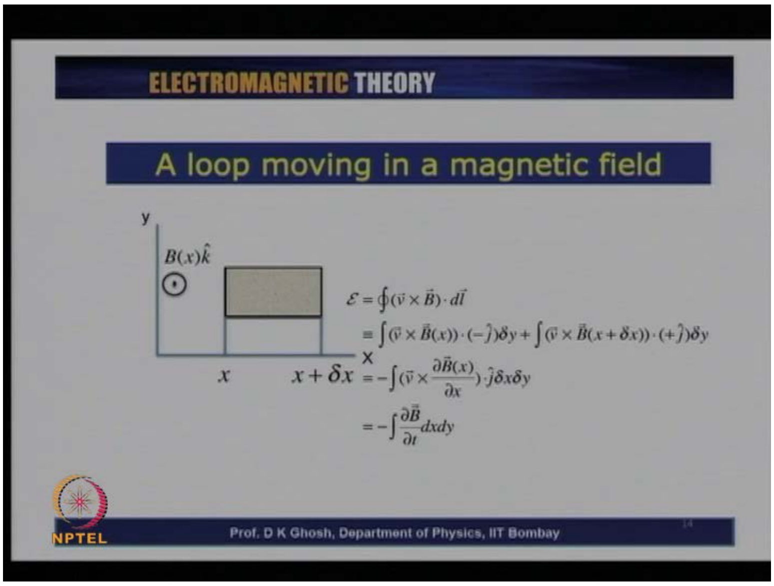

But, I have just a simple question in this picture what I am showing is that I have a magnetic field which is non uniform I have taken the magnetic field along the z direction, But, let us suppose that I have, so that magnetic field is outward along the z direction perpendicular to the screen and I have a rectangle which is moving with a velocity v along the x direction. And let us say at a particular instant the left edge of the rectangle is at the position x and the right edge is the position of x plus d x. Now, what I am going to do is this, that let me calculate then how much is the electromotive force.

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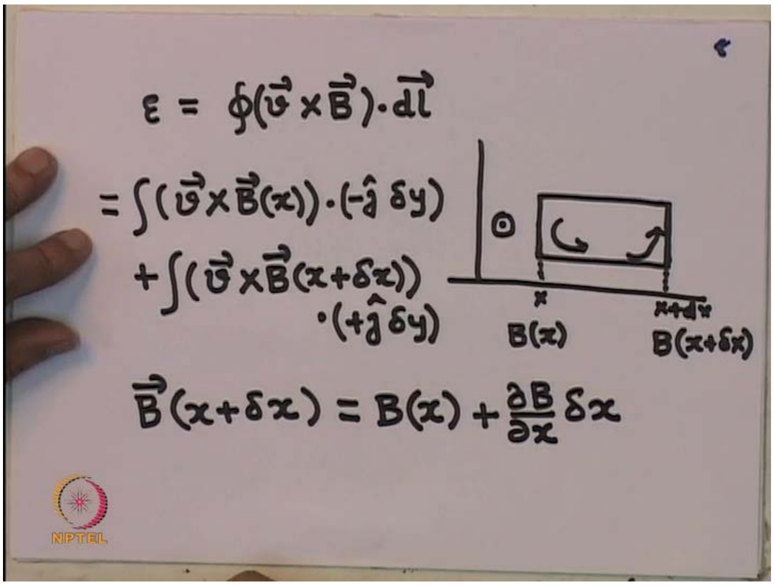

So, this is done remember that electromotive force is integral of v cross B dot d l I am still with what I called as the motional e m f. Only in this case, the velocity is the physical velocity of the loop, which is moving with a velocity v along the x direction. Now, in this case what I have said is that if you refer to let me draw this picture here again. So, this is a rectangle it is instantaneous position is x here and an x plus d x there and supposing I am taking a loop direction like this. So, the magnetic field strength on this edge is B at x the magnetic field strength here is B at x plus delta x.

Now, what am going to do is to calculate how much is v cross B velocity is in the x direction the magnetic field is in the z direction. So, v cross B will be in the y direction x cross z, so if it is x cross z it is minus y direction. So, I have contribution to the integral only from these 2 sides because these are parallel to the y direction. So, this emf can be written as integral v cross B at point x dotted with now this d l is because I am going like this. So, it is minus j if you like delta y plus integral v cross B of B at x plus delta x dot in this case it is along the plus j direction.

So, plus j delta y now if delta x is small I can write B at x plus delta x is equal to I can use a Taylor series B at x plus d B by d x as you know homogeneous field times delta x. Here, I have got two terms one is minus another is plus and what I will now do is, I will also give explicit from to the direction of v cross B is along the x direction B is along the z direction. So, v cross B's direction is minus y direction. So, this is minus j dot minus j which is plus 1, but this term will be negative, because this is minus j dot plus j.

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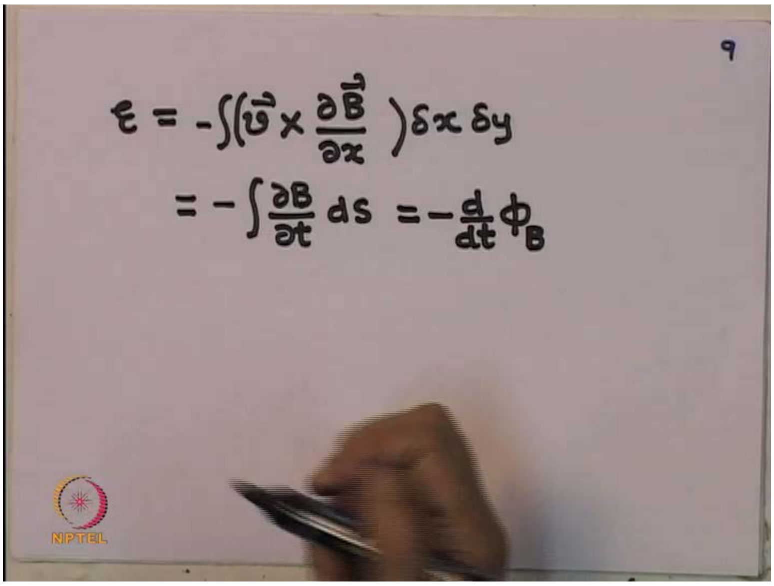

So, if you now substitute it here what I am going to get is emf is equal to integral minus v cross delta B x by delta x multiplied by delta x delta y, you know v is d x by d t. So, therefore, if I write this as d x by d t, then gives me d b by d t, so this will become integral of d b by d t and delta x delta y is my area, so this gives me a d S. So, which is again minus d by d t of the flux phi, but you see in when we do experiments, the charged particles that are there they do not realize they experience a force.

They do not realize whether this is happening this effect is happening, because that with respect to another observer the charges happen to be moving that there are part of a moving circuit or just the magnetic field is changing with that. So, whatever be the case there would be an emf if there is a change in the magnetic flux. And this change in the magnetic flux can come from either a physical change like stretching of a circuit, actual movement of the circuit or a changing magnetic field.

In the last case well there is a time dependent magnetic field I still have faraday's law valid, but I cannot give you an explanation in terms of Lawrence forces. So, for reasons that we do not understand at this moment, it seems that that if there is a change in magnetic flux for whatever reason, that is equivalent to an induced electric field. Because, I need an induced electric field to define my electromotive force, my electromotive force was integral loop integral of E dot d l. So, the cause could be physical movement in which case we give it a special name, we call it motional emf or it could be simply a time dependent magnetic field.

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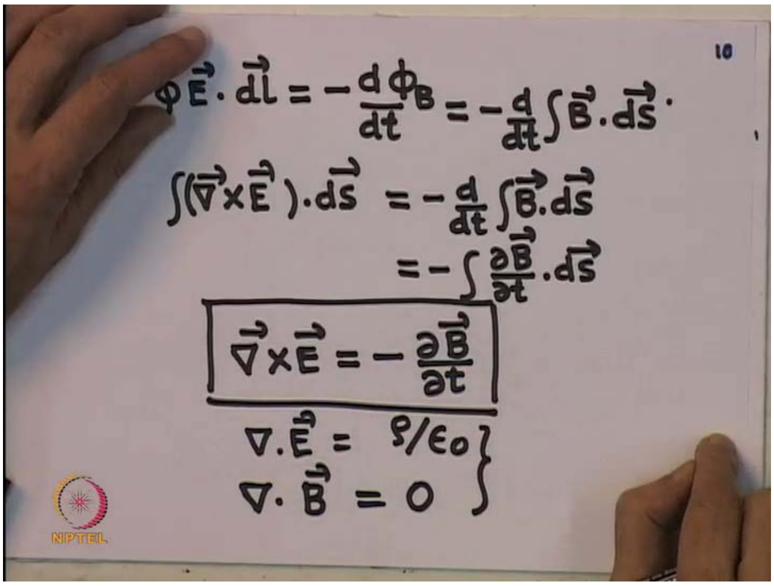

With this let us summarize the laws as they the Faraday's law as it stands. So, we had seen integral of E dot d l, which is my emf is minus d phi by d t and the flux is by definition integral B dot d S, this line integral I am going to convert using Stoke's theorem to del cross E dot d S. So, this is equal to minus d by d t of B dot d S and this integral is over arbitrary surface. Since, the integral is over the arbitrary surface well I can also write this as r c l d B by d t dot d S, what I get is del cross of E is equal to minus d v by d t.

So, Faraday's law which tells us which is the first time varying phenomena that we have introduced, tells us that if there is a changing magnetic flux, this produces an emf in the circuit and this is so in addition to for example, del dot of E equal to rho by epsilon 0, del dot of B equal to 0 these were the two Gauss's law I have this equation. Remember in the absence of time varying field del cross of E was is equal to 0 I am still left with one equation, which I have not yet talked about and that is del cross of B equal to mu 0 j the ampere's law.

What will do in the next lecture is to find out if there is time varying electric field how does Ampere's law change and once we have done that we would get a set of 4 complete Maxwell's equations. And that is that will be done next time that is talking about just as time varying l magnetic field will give us an equivalent induced electric field we will see that a time varying electric field can be shown to be equivalent. O r give raise to a an induced magnetic field as to why that is a rather weak thing or a difficult thing to observe and things like that this will be content of our next lecture.