Electromagnetic Theory
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Module - 3
Magnetostatics
Lecture - 27
Magnetized Material

In the last lecture we have talked about a vector potential and seen some cases where we could calculate it and using a vector potential we also obtained the magnetic field. Today's lecture is going to be on magnetic materials. If you recall when we did electrostatics we also had a discussion on materials which have polarization, there were materials which had electric dipole moment either induced or permanent and exactly the same way we have magnetic materials where there are magnetic moments which could be introduced by application of a magnetic field or there could be intrinsic magnetic moments which in a substance align in such a way so, as to give a net magnetic moment to the sample.

So, today we will see how the presence of magnetized material that is material which have intrinsic magnetic moment alter our understanding of the magnetic phenomena. So, we will, but before you do that let us introduce another simple concept. If you recall we had seen that since electric field was a conservative field we have a scalar potential associated with it. However, in case of a magnetic field we could define a vector potential whose curl was the magnetic field. Now, in some situations it is also possible to define a scalar potential corresponding to a magnetic situation. So, for example, if I do not confine myself to the location where there are currents for example, there is current flowing, but on the other hand I am looking at the magnetic field at in let us say the space around it which does not have a current then in that region the current density is 0.

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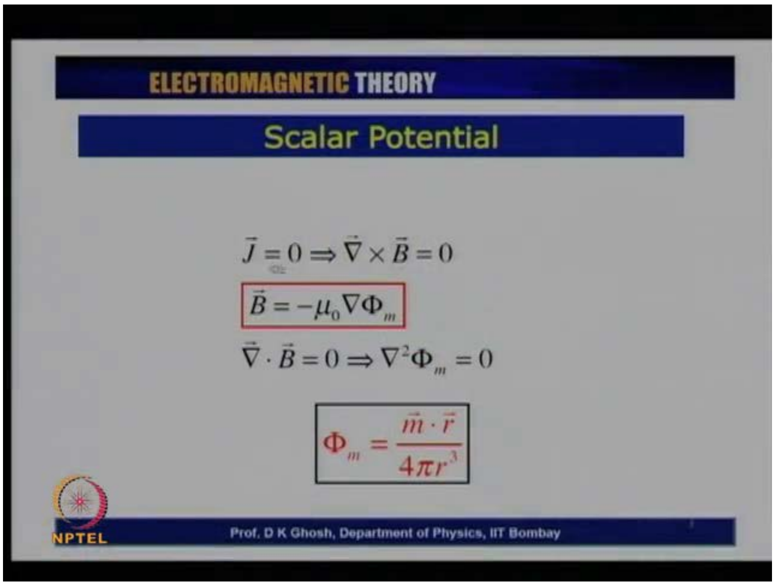

Now, if the current density is 0 del cross B which was shown earlier to be equal to mu 0 J the Ampere's law that turns out to be 0. Now, in such a situation like we defined for an electric field we could also define a scalar potential. So, what we do is for the moment we define a scalar potential as minus mu times minus mu 0 times gradient of the scalar potential giving me the magnetic field. Now, we will see later that the real definition should be to express the field h which will define in this chapter in terms of the scalar potential by simply saying h is equal to minus grad phi m, mu 0 has been introduced for dimensional purpose.

Now, so, notice that del dot of B equal to 0, which is Gauss's law converts this equation to del square phi equal to 0, phi m is the magnetic scalar potential. Now, this is an equation which is identical to the equation for the potential Laplace's equation for the potential which we had derived for the electrostatic field and we had seen there that when I have electric polarization P the expression for the scalar potential was given by P dot r by 4 pi r cube. I am not going to repeat that argument because this is the same Laplace's equation.

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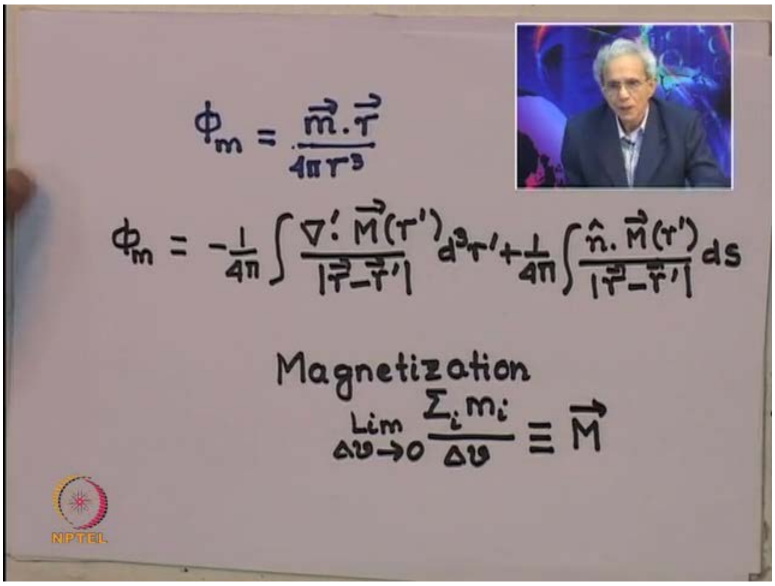

And so we could obtain in a parallel way that phi m the magnetic scalar potential is given by the magnetic moment m dotted with r over r cube and there is a 4 pi which comes in because of scalar potential’s definition. So, this my scalar potential corresponding to the magnetic field. Now, once again I will fall back on from this type of an equation how we divided the situation into two parts. One, where we talked about bound charge densities in case of electrically polarized medium and we had seen that minus del dot of p was the bound charge density and the charge on the surface was given by the normal component of polarization vector.

Now, exactly the same way parallelly following that what we will get is that the scalar potential phi m for the magnetic field can be written as sum of two parts, one is given by minus 1 over 4 pi divergence as I explained last time the quantities which are being integrated that is the material that is being written in terms of the primed index. I will just now come and define what is this M is. This is M r prime by 4 pi r cube so by r minus r prime d cube r prime plus 1 over 4 pi normal component dot M of r prime divided by r minus r prime and this is over a surface.

Recall, that we had defined a polarization as the electric dipole moment per unit volume. Now, what I do here is to define a quantity called magnetization. So, magnetization is basically the net magnetic moment per unit volume. Therefore, if I take a small volume delta v so delta v is going to 0 and some over all the magnetic moments that happened to be in that volume and take this limit. Then this is the net magnetic moment per unit volume and this is my definition of the magnetic moment vector. So, what I have done here is this.

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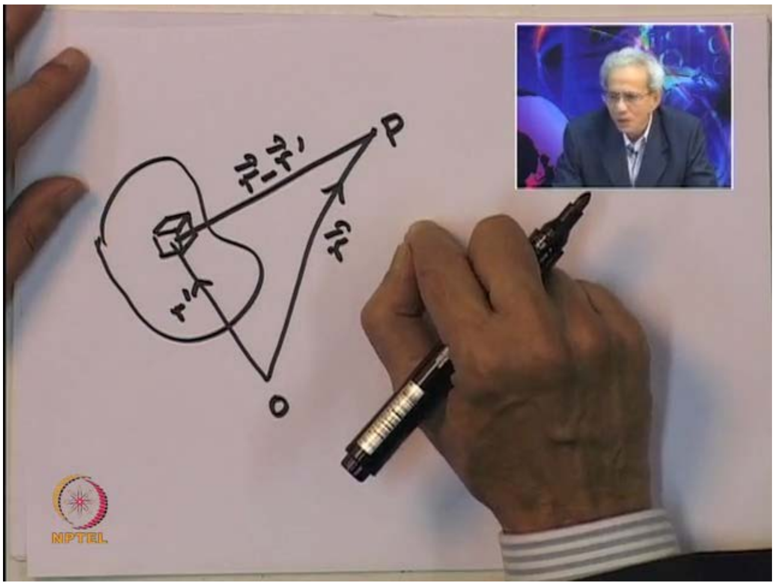

We have said that I have a distribution, when I have a magnetic moments all over so I take a small volume in that and with respect to an arbitrary origin O this is my r prime. Prime quantities are the quantities relevant to the material which is going to be integrated out and I make an observation at the point r. So, this is r minus r prime. So, this is the point of observation. So, what I do with this. I calculate how much is the magnetic moment in that volume, find out what is the contribution to this scalar potential of that magnetic moment at P and then following the what I did for the polarization case we can divide it into two parts and this is what we talked about earlier that I have one part which is a volume part and where I notice that is written as minus del prime dot M r prime and the other one is this surface part.

So, exactly the way we did for the polarization case we notice that I can define, I use the word magnetic charge density. We have seen that isolated magnetic charges do not exists, but I am using this definition more to use the same terminology as we used for the electric polarization case.

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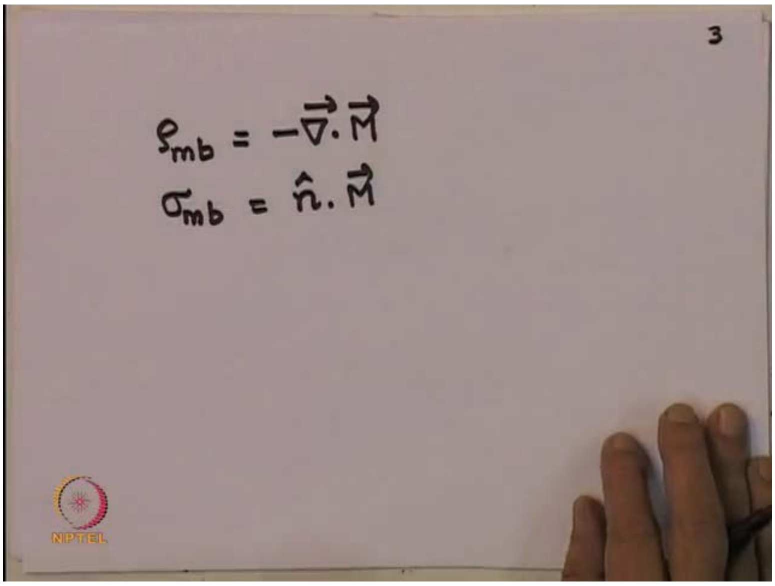

So, like that case I have a volume charge density of magnetic case. If you like, I will call it charge density, magnetic bound charge density which is given by minus divergence of M and a surface charge density which is given by M dotted with n. Since, these are general definitions I have removed the primes, but inside the integral all the quantities are prime including this n prime.

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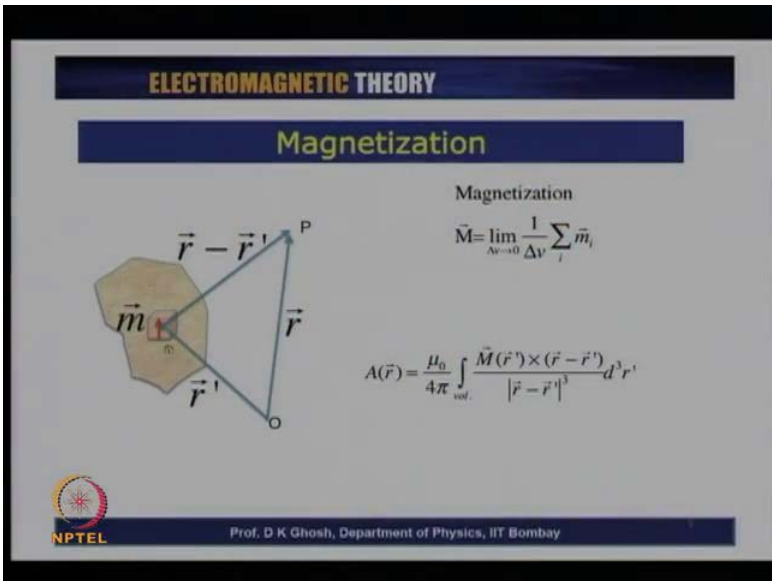

So, this is this is what we talked to you about, that I have this material here and there is a small element of volume d cube r prime and that contains d cube r prime into capital M amount of magnetic moment and I calculate p there. So, that is that is scalar potential part. We had also seen that for a magnetic moment M, how to write down a vector potential and I can now sum it over that again doing exactly the same way I write down the magnetic vector potential corresponding to this volume element which has an amount of magnetic moment equal to d cube r prime times magnetization and therefore, the vector potential corresponding to this situation is given by mu 0 by 4 pi integral M r prime. So, let me rewrite it again.

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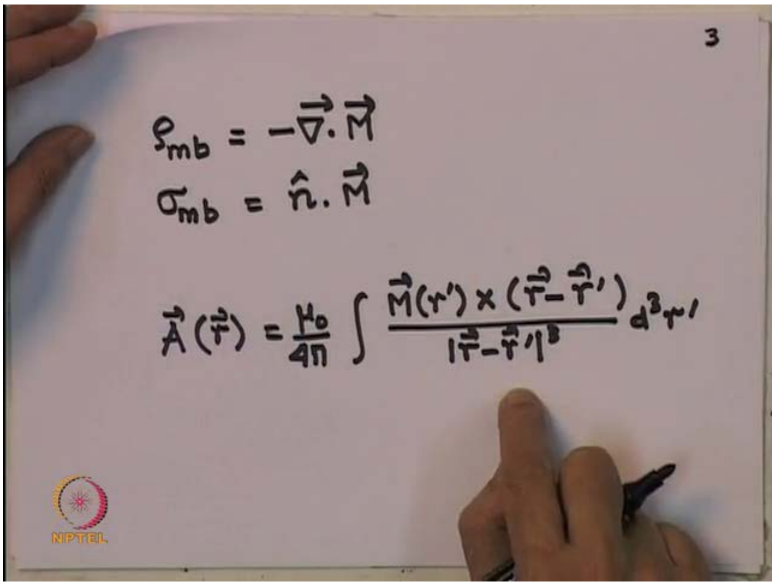

So, A of vector r is mu 0 by 4 pi integral M r prime cross r minus r prime divided by r minus r prime cube d cube r prime. So, we have got two parallel ways of handling the magnetization thing. One is if there are no sources then to talk about scalar potential and a of course, I also have the possibility of working with vector potential. But once again we had seen that if I have a situation like this I can once again convert this into two parts.

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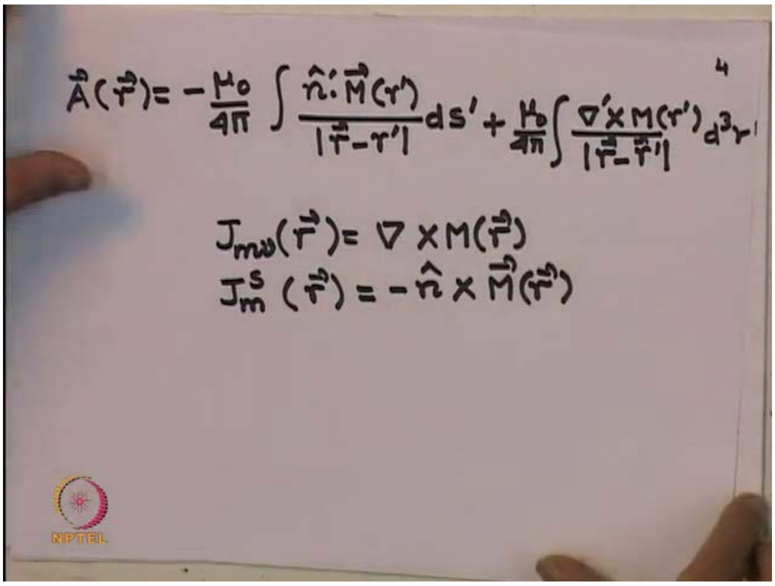

One part which is again a volume part and that will be given by so A at the position r is equal to minus mu 0 by 4 pi integral of well I will write down the scalar part the surface integral part first, n prime dot M r prime by r minus r prime d S prime S prime meaning surface plus mu 0 by 4 pi 1 over r minus r prime and of course, I have got del prime cross M r prime. Derivation is very simple, in case of the difference between phi and A is A has a cross product their therefore, there are some differences. Otherwise, the derivations are the same. Now, if you now look at these expressions and compare it with the expression that we had for the vector potential corresponding to a current distribution, corresponding to a current density then you notice that this appears del prime cross M r appears to take the place of the current density volume current density J.

I will still put a subscript m to indicate we are talking about magnetization current density. So, J m of r is del cross M of r. So, this is simply J m of r prime and the J m of r if you like we will call it surface and this is volume. So, this is equal to minus n cross M of r. So, in both the cases we have identified a surface term and a volume term and so let me before I proceed, I want to simplify these expressions a little bit so as to give some interpretation.

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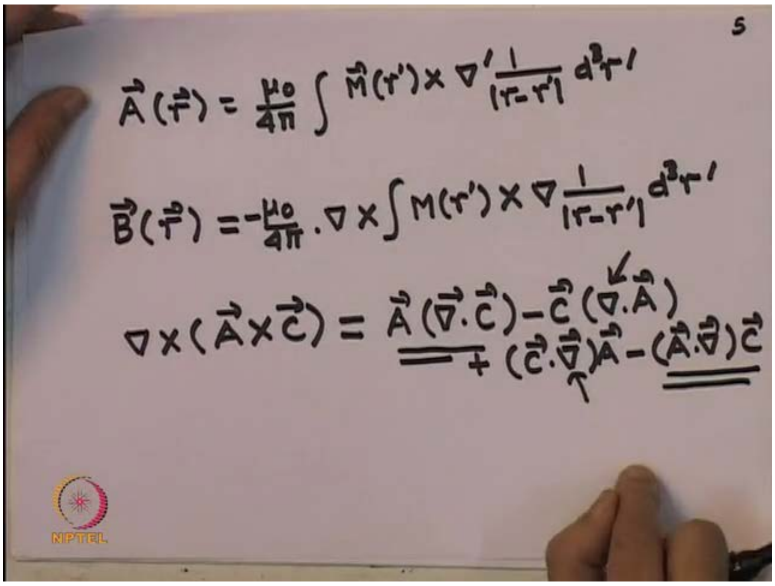

So, notice that the vector A of r is equal to mu 0 by 4 pi and we had seen this is given by M of r prime cross del prime of 1 over r minus r prime d cube r prime this is actually my original expression. All that I have done is that I have written that original expression that I started with namely this expression M r cross this and since this dependence is r prime, I am writing this as M r prime cross del of 1 over r minus r prime with a minus sign and then I am changing over from del to del prime by removing that minus sign and hence I get this expression. So, which is the same as this expression as written here.

Now, the magnetic field B at the position r which is simply del cross of this quantity. So, I will write this as mu 0 by 4 pi del cross M r prime. Let me write it as a del of 1 over r minus r prime. Notice, that I am shifting between r and r prime differentiation frequently that should not cause any confusion. So, the point is that this del cross is with respect to unprimed index. So, that is the location where I am making an observation. The del, the r prime is with respect to the integration index.

So, what I want to do now is this. So, this obviously this del cross can go inside. Now, I am going to use a mathematical expression of calculus which says if I have del cross A cross C this A is not to be confused with the vector potential, but is just an arbitrary vector. So, if I have this then I can write this as A del dot C minus C del dot A plus C dot del A minus A dot del C.

So, notice that my del operator r only with respect to r whereas the dependence of M is one r prime. So, if I bring this del cross there and identify A to be M and C to be del of 1 1 over r minus r prime. Then I notice the following that this term will be there because it is A dot A del dot C. So, I will get this term, but I do not have this term which is del dot A because this del is with respect to unprimed index. So, del dot M r prime will be 0 and likewise I will not have this term as well because there is del of A itself so A does not get differentiated with respect to unprimed n x, but the other two terms will be there. So, in that case I can write this expression as the following. So, start with this use del cross A cross C and you notice what do I get.

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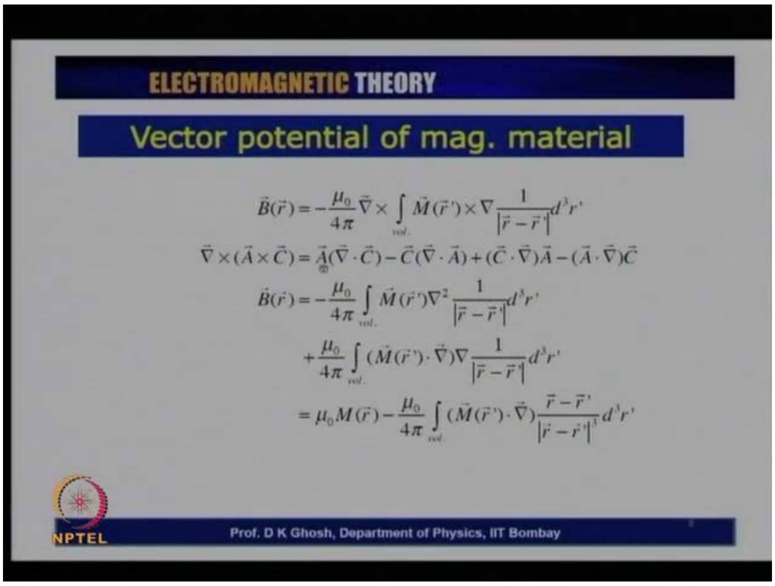

So, I have got del cross del cross this. So, I have got A del dot C term A del dot C term of this.

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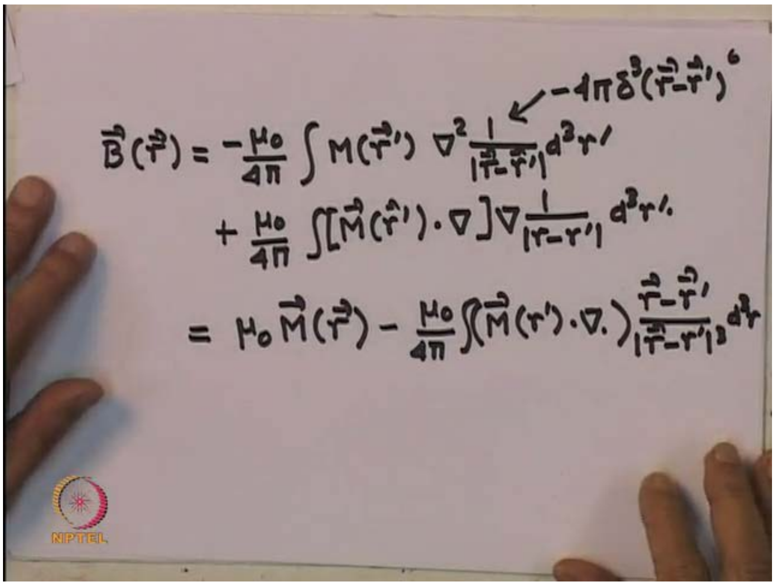

I will rewrite it that is B of r is equal to minus mu 0 by 4 pi A is M r prime del dot C is del dot del of 1 minus 1 over r minus r prime which is del square of 1 over r minus r prime d cube r prime. The other term that survived was mu 0 by 4 pi with a minus sign A that is M r prime dot del acting on 1 over, acting on del of del of 1 over r minus r prime d cube r prime. So, these are the things that we got. Now, notice the first term I have a del square 1 over r minus r prime and this term is minus 4 pi delta function, three dimensional delta function of r minus r prime because del square of 1 over r is minus 4 pi delta function.

So, because of this delta function I can write this, do this integration and I simply get minus 4 pi which cancels with this minus 4 pi there so I will be left with mu 0 M of r and I am left with this term there. That is my minus mu 0 by 4 pi integral M r prime dot del, this of del of 1 over r minus r prime. So, that is why this minus sign came in and therefore, this is r minus r prime divided by r minus r prime cube. So, this is what I got. Now, I need to simplify this expression little little bit and for that I use another complicated vector calculation relationship and let me write it down. These can be obtained from any standard text books.

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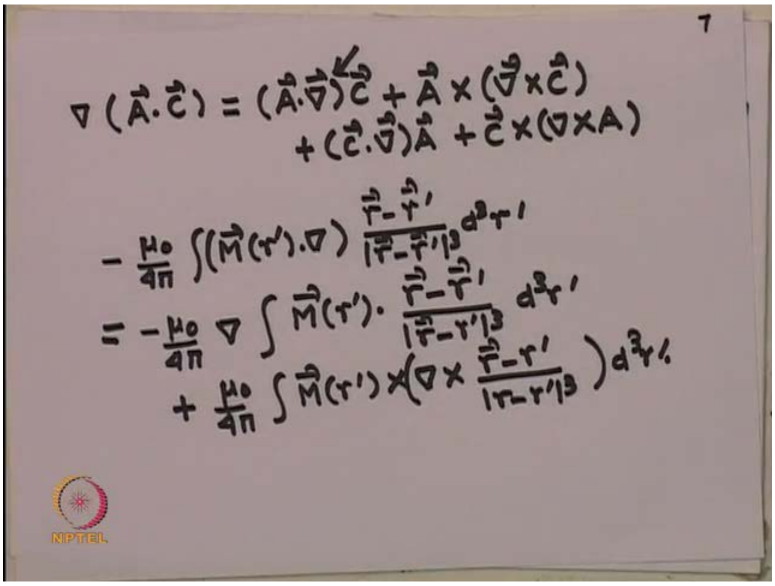

So, this is if you want to write the del of A dot C. So, you can write it as remember del of A dot C, this is a scalar and the gradient of that will be a vector. So, I can write it as a A dot del C plus A cross del cross C plus C dot del A plus C cross del cross A. I repeat exactly what I did here. Notice, this that this is of the form A dot del of a scalar function. So, I have sorry A dot del of a vector function. Now, remember what is meant by del of a vector this is actually three quantities. So, take the scalar components and apply del operator on it and the whole thing will then be a vector.

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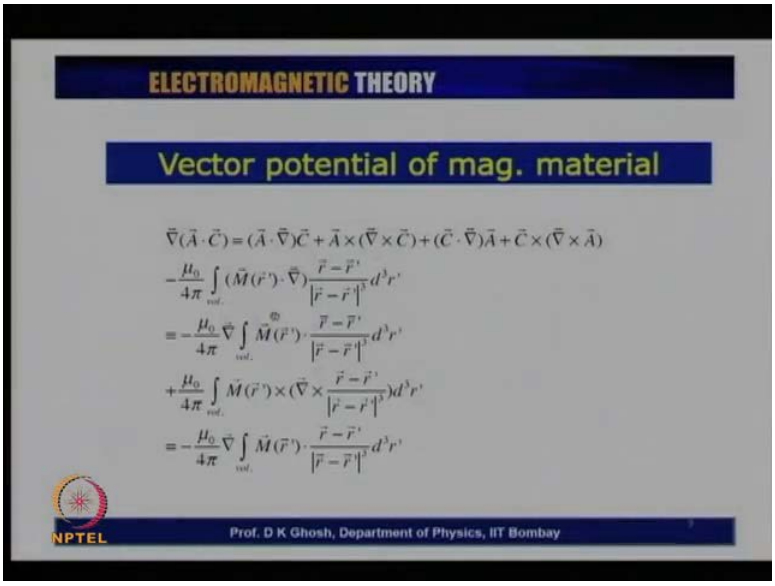

So, if you do that and use this expression here notice that I have M dot del. So, I I will write down minus M mu 0 by 4 pi integral M of r prime dotted with del of r minus r prime divided by r minus r prime cube d cube r prime, this is identical to similar to this expression identifying A as M del of course, is there and r minus r prime by r minus r prime cube as C. So, I can write this in terms of this quantity and the three remaining quantities, but then I have to realize that since I have identified A as M and which is a function of r prime the del cross A term or del of A term, this two del terms do not contribute.

So, I am left with only this term minus that term and so we will write that down. So, that is equal to minus mu 0 by 4 pi so del of I am writing now del of A dot C type of thing. So, this will be del of since A is M r prime dotted with r minus r prime by r minus r prime cube d cube r prime and there is another term which is minus mu 0 by 4 pi integral M r prime cross del cross r minus r prime divided by r minus r prime cube. So, this is this is what I have written as my first term.

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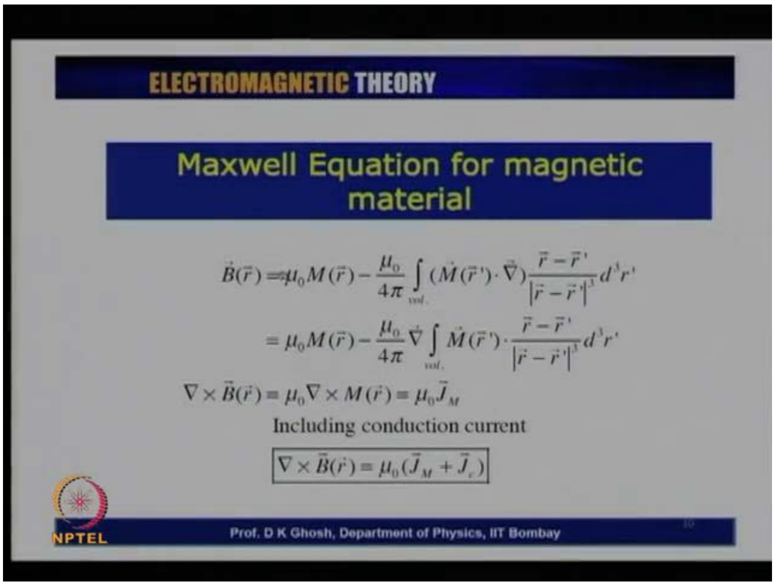

Now, so that tells me that recall my expression what I had here this was my B and I wrote B in terms of this expression and this is the expression which I have now simplified and I have written it like this and therefore, my B of r turns out to be given by an expression like this and notice this is mu 0 M minus mu 0 by 4 pi grad of something.

Therefore, if take curl of B I get this term goes away and I am simply left with mu 0 del cross M and del cross M, I have identified as the volume current density. Now, so this was due to magnetization. Now, if I include now the current density due to the charge transfer namely the conduction current.

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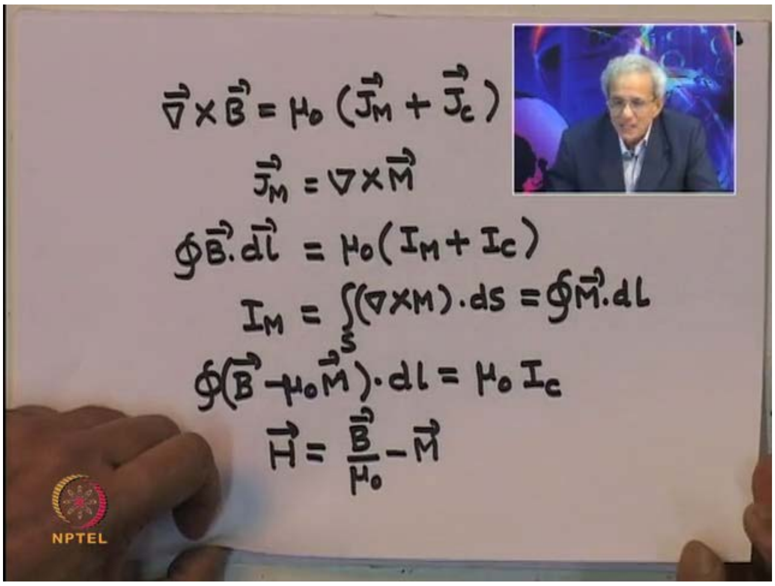

Then my equation becomes del cross B given by mu 0 into the magnetizing current or magnetization current plus the conduction current which we have been talking about all the time. Where we have defined the conduction current as del cross M as you have done earlier. So, J M is del cross M. Let us look at, how does it change my equations? So firstly integral B dot d l which was obtained from surface integral of del cross B using Stoke's theorem. So, I could write this now as mu 0 times if you like a term which is I M magnetizing current and plus the normal current which is like I conduction. Where I M is defined as surface integral of del cross M dot with d S which according to Stoke's theorem is line integral of M dot d l.

Now, what is this enables us to rewrite the Ampere law in this form, B minus M mu 0 M dot d l is equal to mu 0 I conduction. So, the Ampere's law which is integral B dot d l equal to mu 0 I or I conduction is changed because of magnetized medium or magnetic medium by this quantity. Now, so what we do is this that in the presence of magnetic material we define a new vector which is called vector H is through this relationship B by mu 0 minus M. Now, there is a lot of confusion on what these things are called. The convention that is followed in many textbook is to call the vector B and the magnetic field of induction or magnetic induction and the vector H is usually called the magnetic field.

Now, however whenever there is no confusion calling B as the magnetic field has been very standard. Therefore, we will using both, but very clearly specifying whether we are talking about H field or B field. Therefore, in terms of this mu vector that I have defined.

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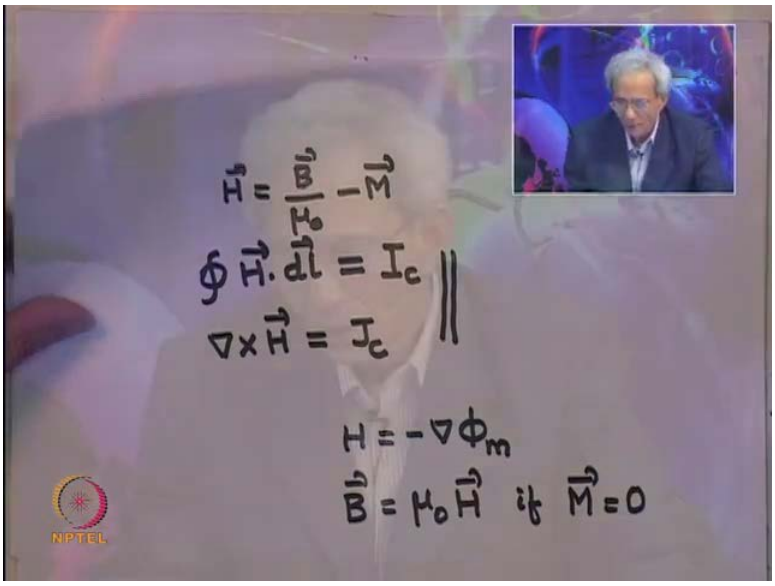

So, we have said that vector H is equal to B by mu 0 minus M and in terms of this vector I have integral H dot d l is equal to I. You notice what this is doing. Now, when I have a magnetic material it is difficult to separate out what is the contribution due to the magnetizing current and conduction current, but supposing you could do it. So, in some sense the vector H, the magnetic field H is thought to arise by conduction current alone whereas the magnetic field B is the actual field that will be measured at any point because it is the super position of the field due to the magnetization and the charge transport and exactly the same way del cross of H will turn out to be J C instead of del cross B equal to mu 0 J.

So, this is the way the Maxwell's equation will be stated in the integral and differential from in terms of the quantity H and in fact if you are defining a scalar potential then you would like to define H is equal to minus gradient of phi m this will be our operating definition because earlier we had put in a mu 0 for B because mu if there are no magnetic material then H and B are proportional to each other, if magnetization is 0 then you can see B is equal to mu 0 H. Magnetic materials are of different types. Now, if you apply sole magnetic field then it turns out that the magnetization is proportional to H. There is the linear response and as the result a quantity which is magnetization, I write it as a quantity chi M times H.

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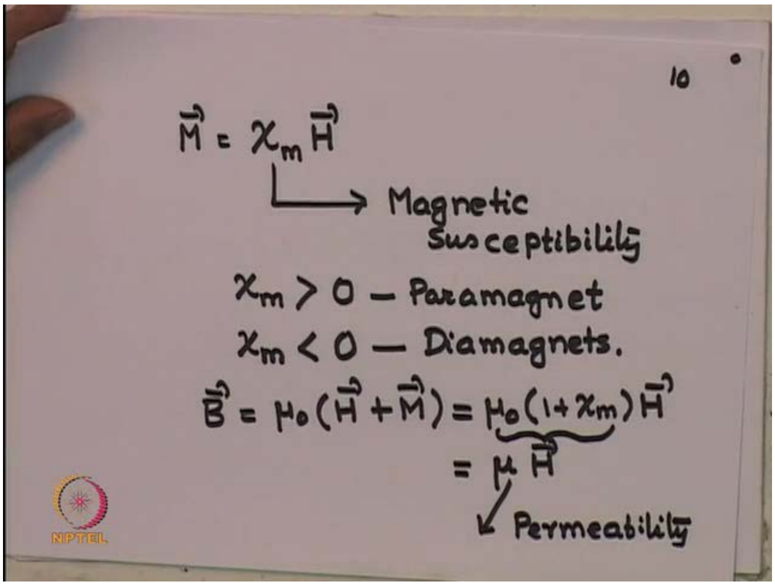

So, magnetization M is given by a constant chi times H and this is known as the magnetic susceptibility. Magnetic materials are generally divided into three broad classes, the there are what are known as paramagnets I will talk about it little later, but paramagnets are those material for which the magnetic susceptibility is greater than 0. And we will realize that because of a law called the Faraday's law there are. So, this paramagnet and these are known as diamagnets.

We will also see that there is class of material known as paramagnet, it is a little more complicated so I am leaving it for the moment. So, B which is written as mu 0 H plus M can be written as mu 0 into because M is chi m H so mu 0 into 1 plus chi m into H and this quantity mu 0 into 1 plus chi m is usually written as mu H. Remember, mu was called the permeability of the vacuum and this mu is the permeability of the medium. 1 plus chi is occasionally called as the relative permeability which is a number which sort of shows that what is the factor by which the permeability changes when there is a magnetic material present. We could continue this and obtain the boundary conditions for H.

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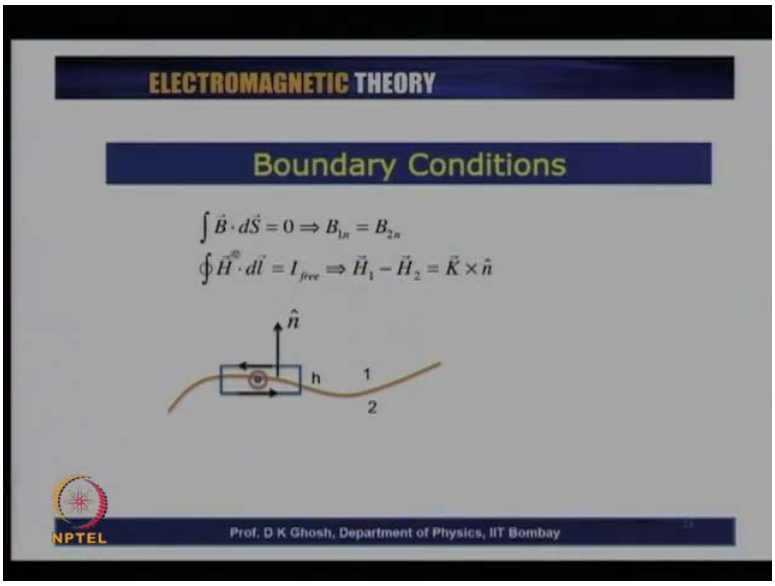

And this is integral B dot d S equal to 0. We have seen this is always valid and that give us B the normal component of B to be continuous. We had shown that integral B dot d l equal to mu 0 i leads us to the tangential component of B to be discontinuous by an amount K there was a mu 0 there. Since, this equation is very similar it tells me that H 1 minus H 2 tangential that is equal to K cross n, that is K is the surface current density.

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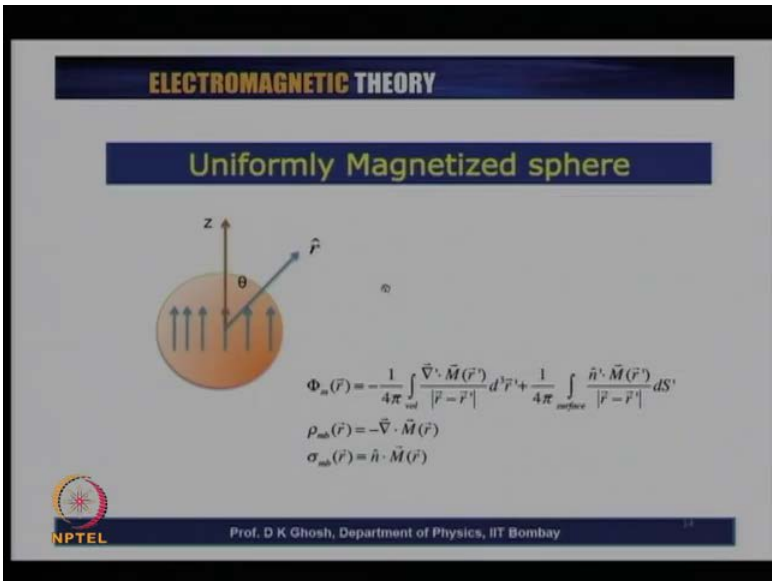

I will discuss the application of scalar and the vector potential to a specific problem and I will start with application to, application of the scalar potential method to a uniformly magnetized sphere. So, here in this picture I am showing a uniformly magnetized sphere and this is I am taking the magnetic moments all aligned along the z direction and let us suppose I am making an observation at a point r which is sort of located at a point an angle theta phi with respect to the z axis.

Recall, that we had said that this scalar potential can be written as a surface term and a volume term. The volume charge density, volume magnetic charge density was given by minus del dot M and the surface charge density was given by n dot M. So therefore, since my magnetization is uniform del dot M will be 0 in this case of course, because these are within integration they are all prime. So, I will be simply left with a surface term and let us look at what do I get?

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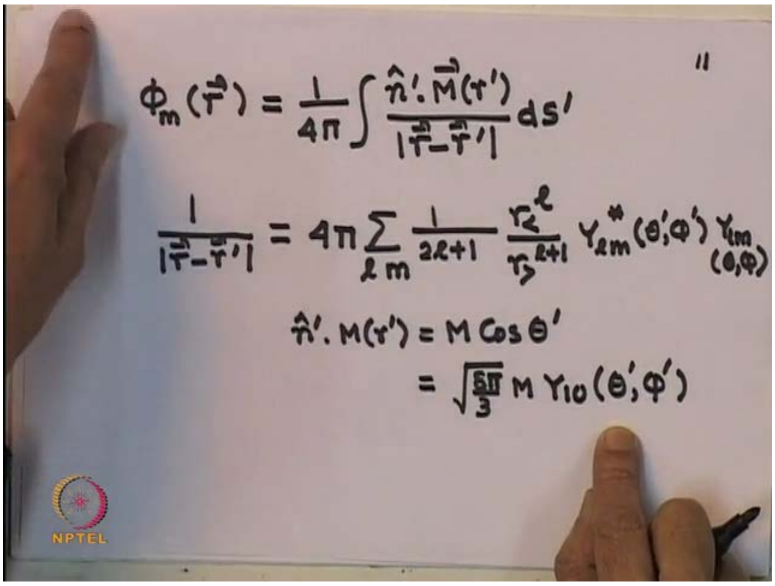

So, let me write down this as scalar potential, scalar magnetic potential due to a uniformly charged magnetic sphere is 1 over 4 pi integral over the surface n prime unit vector dot M r prime divided by r minus r prime and over the surface d S prime. Now, what I am going to do is this. We of course, know that this quantity is n dot M if you referred to this picture is nothing but M cos theta. So, this is M cos theta and I am going to expand 1 over r minus r prime in terms of spherical harmonics, this we have done several times.

So, this is 4 pi sum over l sum over m 1 over 2 l plus 1 r less raised to the power l r greater raised to the power of l plus 1 Y l m remember that r is fixed point of observation. So, the dependence is on theta prime phi prime and star and then Y l m theta phi. So, this quantity has to be introduced in that. The, before you do that we realize that n dot M r prime is M times cosine theta prime, but I also observed that cosine theta prime is nothing but Y 1 0 therefore, I will write this as equal to square root of these are normalization factors 8 pi by 3 M times Y 1 0 and since it is a prime there is a function of theta prime phi prime. I plug all these things there and I will have to do an integration. Since, it is on a surface the integration is essentially over a solid angle. Therefore, let me do that integration separately and I will plug in all the remaining quantities later.

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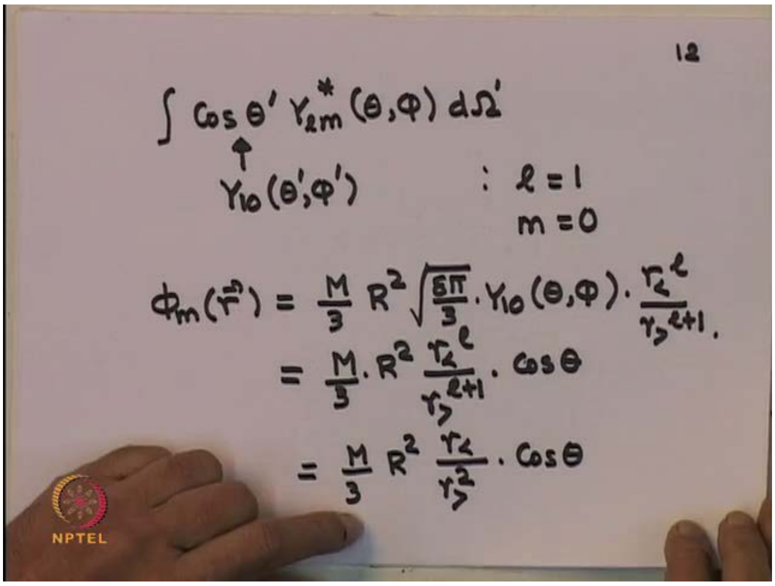

So, so what I have is this. I am integrating basically cosine theta prime Y l m star theta phi and d omega d omega prime. Since, cosine theta prime is basically a Y 1 0 theta prime phi prime. This tells me when I integrate this I am going to get l is equal to 1 and M is equal to 0 by the orthogonally of the spherical harmonics. So, in this expression here if I take l is equal to 1 M is equal to 0. So, I will get a 1 over 3 factor and I will be left of course, I will take care of all these constants and all that.

So, that will give me phi m of r is equal to 1 over collect everything M by 3, capital R is the radius of the sphere root 8 pi by 3 Y 1 0 theta phi which is nothing but and of course, I have r lesser raised to the power l r greater raised to the power l plus 1 that is M by 3 R square r lesser raised to the power l r greater raised to the power l plus l is 1 into cosine theta. Plugging in the value of l I get it as M by 3 R square r lesser by r greater to the power 2 times cos theta. Notice, that if I am inside the sphere then r greater is my radius r and if I am outside the sphere my r greater is the small r.

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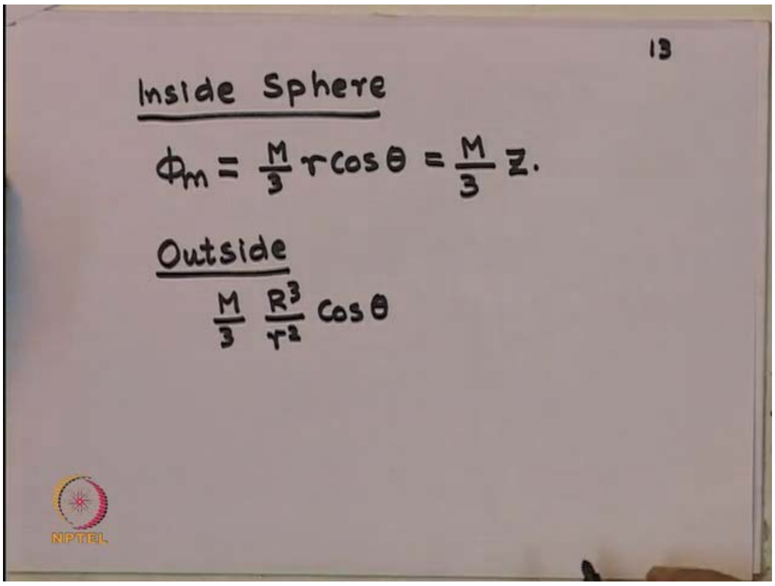

So, in this expression if I am inside the sphere I write down this scalar potential phi of m. So, this will cancel and I will be left with simply M by 3 r cos theta which is nothing but M by 3 Z. Outside the sphere it is a little more complicated. I get M by 3. So, I get a R cube because r less is r. So, I get a R cube divided by r square times cos theta.

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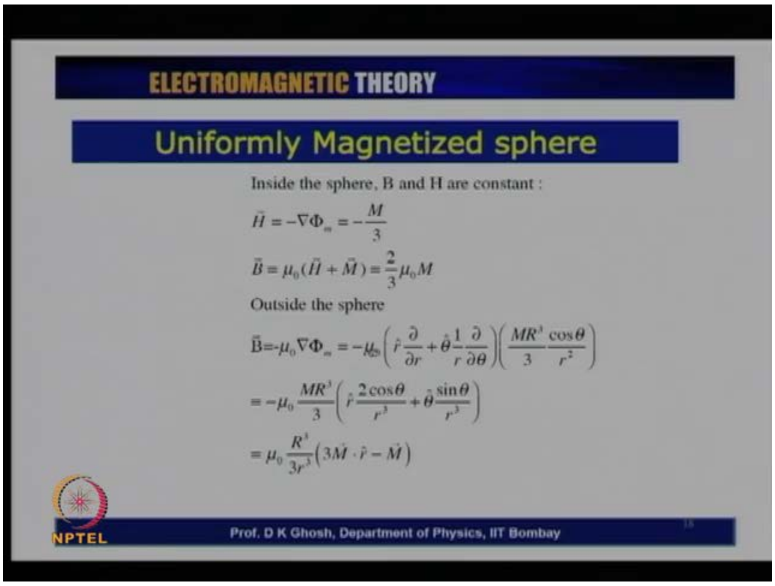

So, what you have to do now is to obtain take the gradient of these expressions to find out how much are the various magnetic fields.

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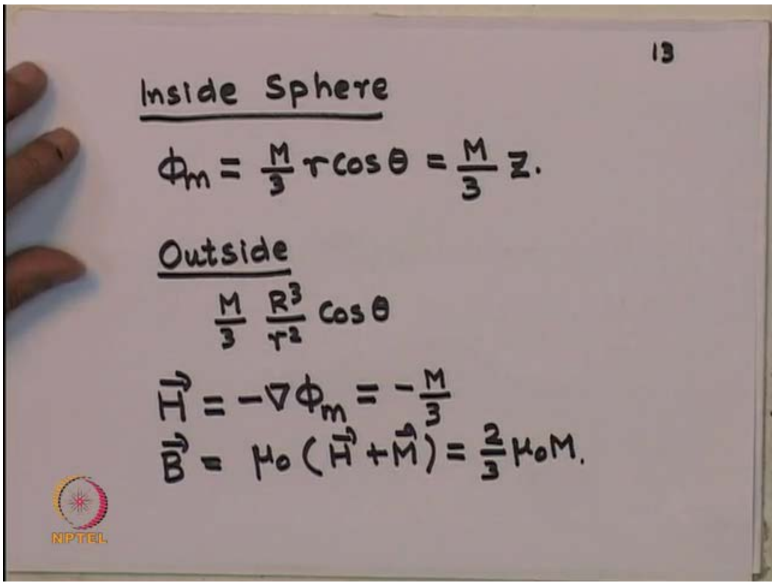

So, remember I defined that scalar potential of H is given by minus grad phi is my H field. So, since the grad is nothing but d by d z in this case so that is simply minus M by 3, this is inside and the corresponding B field is given by mu 0 into H plus M which is equal to 2 by 3 mu 0 M.

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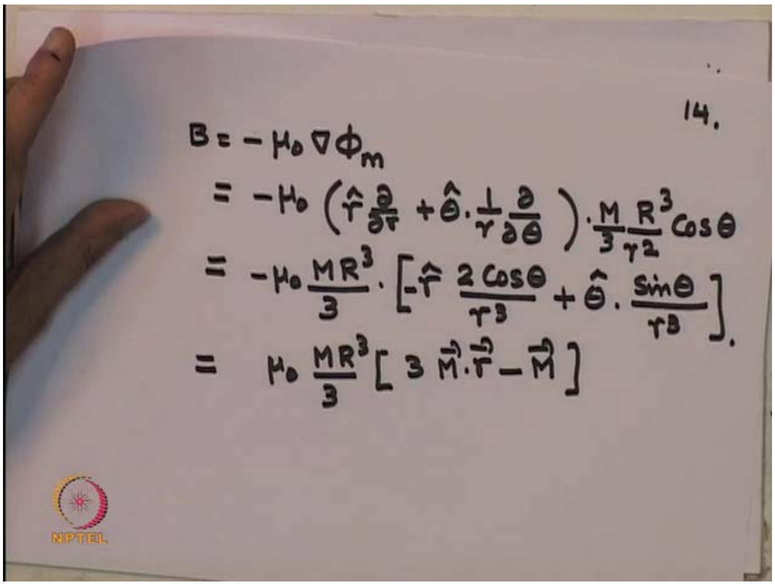

The expression for the outside field, the outside there are no magnetic material. So, I can directly write down B is equal to mu 0 minus mu 0 grad phi. Now, this is minus mu 0. Remember, that grad is given by r d by d r plus unit vector theta 1 over r d by d theta and the expression was M by 3 R cube by r square cos theta. So, this differentiation we can easily do. So, I get minus mu 0. Let me pull out the constants M R cube by 3.

So, I will be left with unit vector r and d by d r. So, I get 2 cos theta divided by r cube plus unit vector theta sin theta divided by r cube again. Now, you can combine them and express in the coordinate free form as mu 0 M R cube by 3 3, this should have been a minus sign because one differentiation was there minus M R cube M dot r minus M. This is an expression that we had seen earlier.

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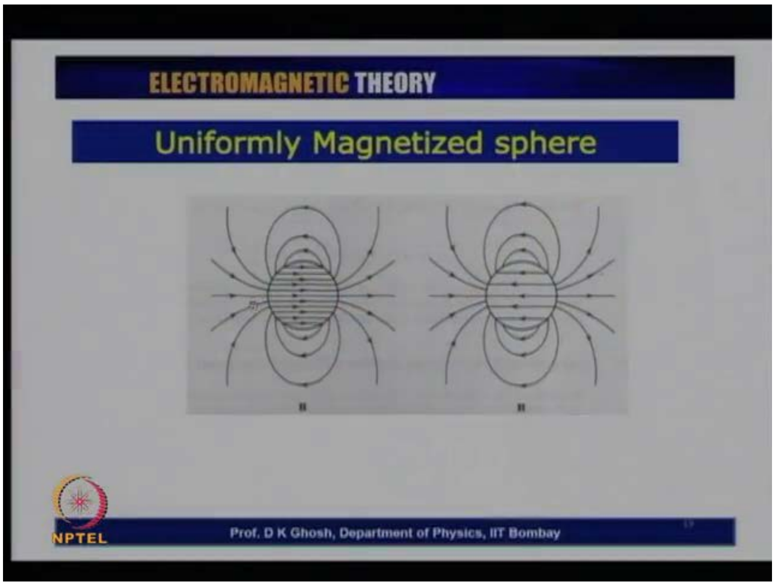

So, if you now combine these results you find that the B field and the H field look like this. Notice, that the fields as you have seen are constant inside we had seen that B is given by two-thirds mu 0 M. But of course, outside we know that they will form close loops and so this is the way the B and H sort of behavior.

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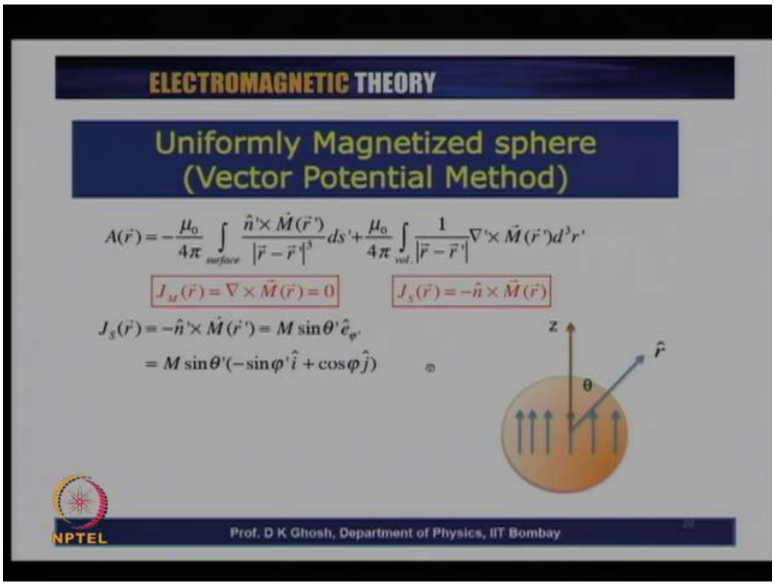

In the next lecture I will use the vector potential method of deriving the same things and what we do there is to take this expression for the vector potential and we realize once again because of the fact that del cross of M is equal to 0. I am left with only a surface quantity to work out and we will see that vector potential method can also be used to obtain an expression for the magnetic field in this case.