Electromagnetic Theory

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Module - 3
Magnetostatics
Lecture - 26
Boundary Conditions

In the last lecture we introduced the concept of a vector potential, and we had seen that because of the fact that the divergence of B is always equal to 0, I can express B as a curl of a quantity which we call as the vector potential. This vector potential as we saw was not unique, but there was a choice in the sense that we could add to the vector potential the gradient of any scalar field, this was known as a gauge choice. And we had seen that one of the most popular gauges that we use is a gauge in which del dot of A works out to be equal 0.

Curl of A gives you the magnetic field which has a physical interpretation, since it is a vector field in addition to its curl one could specify its divergence and in this particular case the divergence of A was a meaningless quantity, and can be chosen as per our convenience and mathematical simplicity. We had also seen that vector potential is not merely a mathematical concept. There are actually experiments, which tells us that we can find physical realization of the vector potential in certain interference experiments. What we wish to do next is to talk about what boundary conditions need to be applied in the interface between two media and suppose, so this is something that we will be talking about today.

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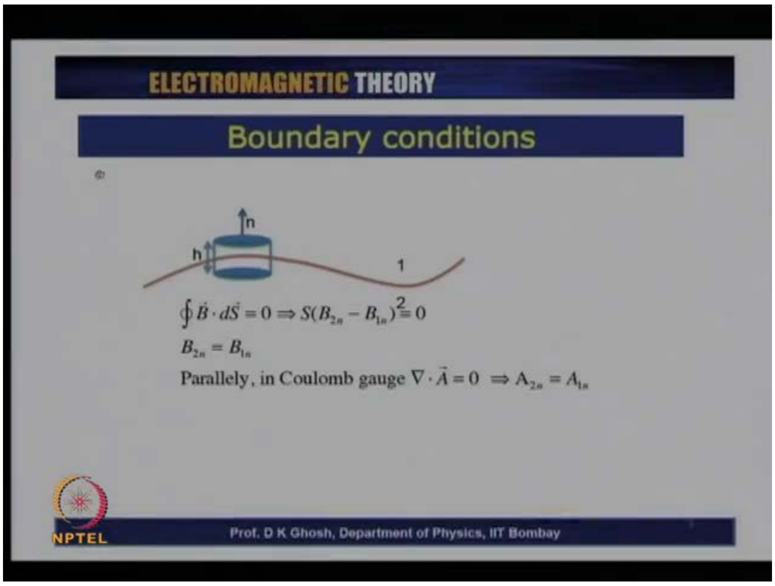

So, here for example, I have two media called 1 and 2 and first what I will do is to find out what is the relationship or how do the components of the magnetic field compare at the interface between the two medium. Now, for that what we do is to look at the Gauss's law which tells me integral B dot d S is equal to 0. Now, so what I have done here in this picture is to draw a Gaussian pill box and this Gaussian pill box is of a height h and this height is small compared to the other dimension namely the radius.

The n here is a outward normal from the, on the surface between the medium 1 and the medium 2. Now, so what happens is this that when you calculate the surface integral of the magnetic field B dot d S since h is very small compared to let us say the radius of the cap the contribution from the curved surfaces of the pill box is equal to 0. So, as a result I need to only worry about the contribution from the top surface and the bottom surface the two caps. So, here since B dotted with n and the normal n is outward.

Therefore, I get B dot n means B times S which is the, S is the area of this cap and when I put a minus sign here because the direction of the normal on this second face is opposite to the normal n there and since I have decided that is would be my direction of n since this is equal to S that is the surface area times B 2 n that is the normal component of the second surface minus the normal component on the first surface and that must be equal to 0. So, this tells me that the normal component of B is continuous.

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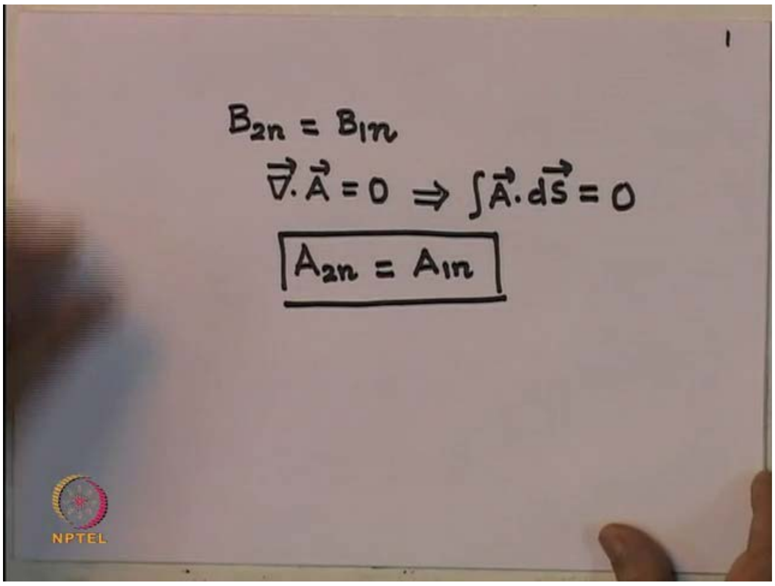

So, B 2 n is equal to B 1 n. The second relation that you want is for the, supposing I am working in Coulomb gauge. Now, we have seen that the Coulomb gauge means del dot of A is equal to 0 and just as del dot of B equal to 0 gave me integral B dot d S equal to 0 this will give me integral of A dot d S equal to 0. And just the way we had shown that integral B dot d S equal to 0 gives me B 1 n equal to B 2 n integral A dot d S equal to 0 will lead to the normal component of A namely A 2 n is equal to A 1 n. Having disposed of the normal component let us look at what happens to the tangential component. This is little tricky and let us refer to this picture here.

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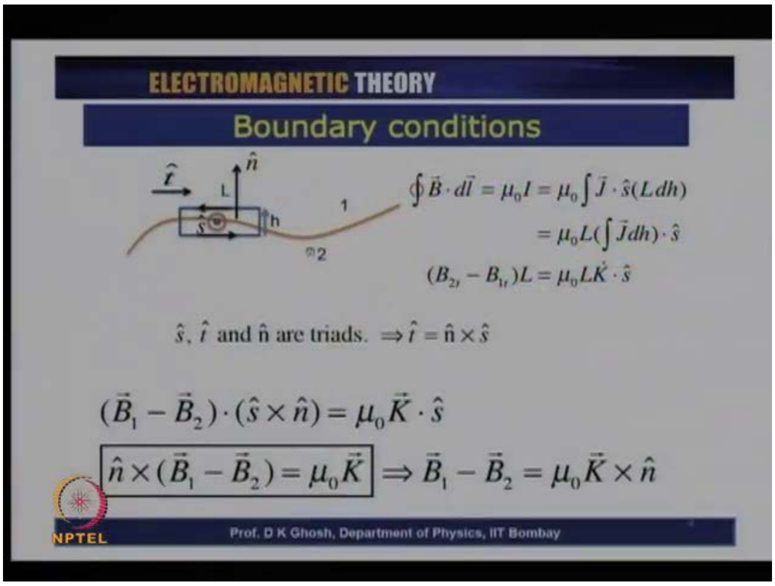

So, this is my interface between medium 1 and medium 2 and I am looking at what is the boundary condition first for the magnetic field of induction B and so what I do is this. I take a another an Amperian loop, I take an Amperian loop this way and the way I do it is this that this current you can see that I have provided a direction in which this current is flowing on the surface.

So, I take an Amperian loop which is consistent with this current which moves out from the plane of the paper on the surface. This is surface current. So, what we do is this, this Amperian loop I know that integral of B dot d l.

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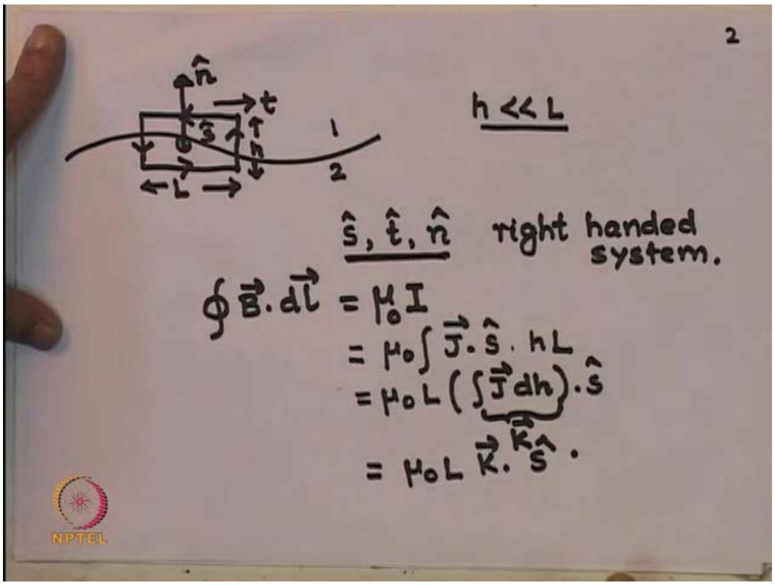

So, let me partly draw this picture here. So, we have taken Amperian loop. This height will be taken to be h and as before I will take h to be much less than the length L of the loop. Let me first fix the coordinate system. So, what we have said is that outward from the page of the paper is what I will call as vector S, the normal to the interface is by n and the if you look at a triode then of course, the tangential component will be just the other direction.

So, in other words what I find is S, t and n is a right handed system. So, let us look at this. So, what I am going to do is I am going to use the Ampere's law which tells me that integral of B dot d l is equal to mu 0 times of the current I that is flowing through. Now, in this case I am looking at a surface current. The reason for doing that is if you recall your electrostatics you notice that I had a discontinuity in the electric field when there were charges on the surface.

Now, in this case I had integral B dot d S equal to 0 that gave me that the normal component of the magnetic field was continuous. Now, integral B dot d l is not equal to 0. So, this will lead to a discontinuity the in the tangential component of the magnetic field and this tangential component will be related to the surface current density that is passing through. So, let us look at how does one do it?

So, mu 0 times I. Now, remember the current is flowing in the direction S so this will be mu 0. The, if current density is J so I get J dot S I need to multiply an area which is if this length is denoted by L this will be nothing but h times L. Now, I define a linear current density. This will be simply J times h. So, the idea is that if just as ordinary current density is something which passes through an area in this case a linear current density. A linear current is something which crosses a line. Therefore, this quantity will be written as mu 0 times well I will take L out and I will write this as integral J d h dotted with S, and this is my quantity which is the linear current density K this is an Ampere per meter.

So, this is mu 0 L K dotted with S. So, let us look at how this quantity the left hand side. Now, notice the left hand side is I have this side called as loop 1, this is loop 2. So, I have taken this as the direction of the Amperian loop. Therefore, since this is along minus t and we have said S t and n form a triad.

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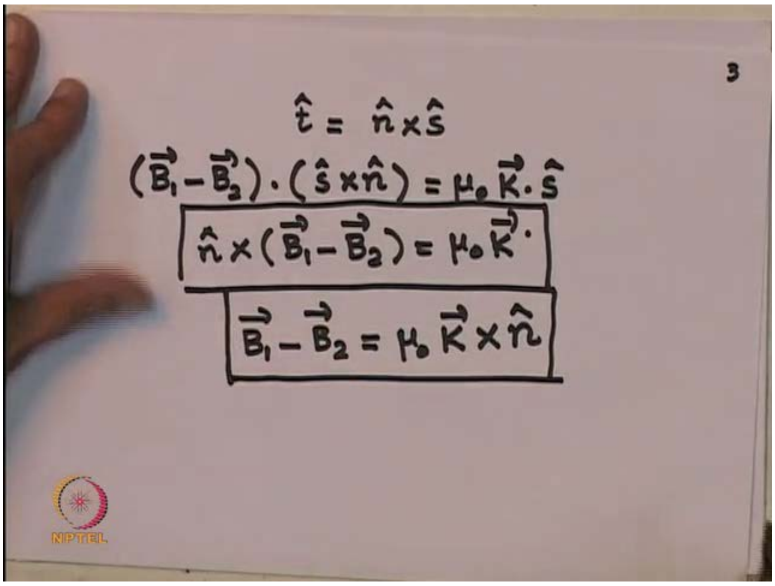

Since, S, t and n form a triad what I have is t is equal to n cross S. You notice that my direction of the Amperian loop was along minus t direction. This was along minus t direction. So, as a result what I get is the if I look at the left hand side I get B 1 minus B 2 dotted with minus t and minus t is nothing but S cross n and this is equal to mu 0 K dot S. Now, the left hand side I simplify by using A dot B cross c equal to B dot c cross A. So, this will be S dot n cross B 1 minus B 2 and this is also dotted with S. So, as a result I will get n cross B 1 minus B 2 is equal to mu 0 K. I can rewrite this.

Since, I already know that normal component of B are continuous I can write this as B 1 minus B 2 is equal to mu 0 k k cross l this is simply obtained by trial algebra because when you do A cross B cross c you get B A dot c minus c A dot c A dot B and I know That n dot n is equal to 1, but the n dot S and t become equal to 0. So, this is a complete statement of the boundary conditions that are applicable on B. What do you want to do now is to use this to find out what will be the corresponding boundary condition on the vector potential A? We had seen because of use of Coulomb gauge since del dot of A is equal to 0 the normal component of A was continuous. We will now derive the corresponding relation for the tangential component of A. So, here what we do is this.

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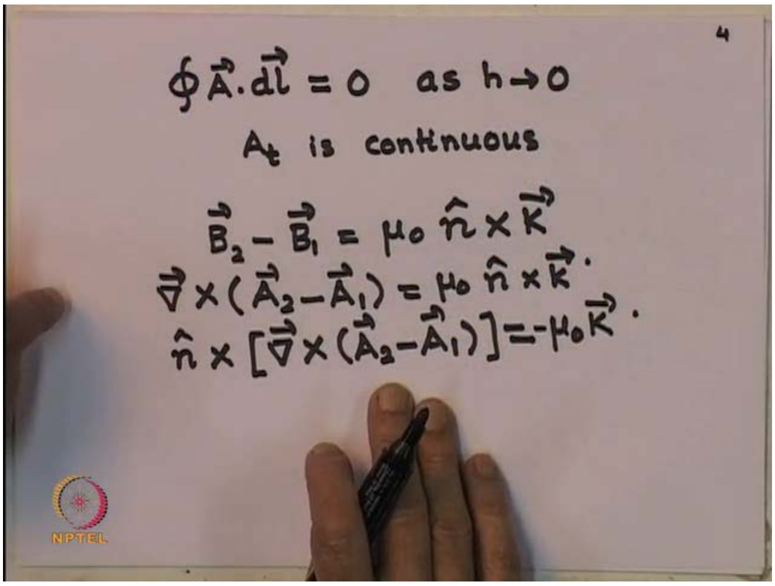

We are using the fact that A dot d l. This goes to 0 as h goes to 0. So, I know that A t is continues. Well, this actually is fairly straight forward because I know that A is curl of sorry B is curl of A. So, curl A dot d S which is same as B dot d S is equal to 0 and by using Stoke's theorem curl A dot d S is same as A dot d l the line integral. And since A dot d l is equal to 0 just as we had talked about B dot d l and obtained the from the Amperian loop here as I take h going to 0 this will give me A t the tangential component is continuous.

So, what we have proved essentially is that though the tangential component of B has a discontinuity because of current density on the surface, the surface current density the potential A the vector potential A both its tangential and the normal component are continuous. In other words the vector potential A at the interface is a continuous quantity, does not suffer any discontinuity.

However, we are now going to show that though that is so the derivative, the normal derivative of the vector potential suffers a discontinuity and this arises from the discontinuity that the tangential component of d has if there are surface currents and that we had seen that B 2 minus B 1 can be written as is equal to mu 0 n cross K, this is what we have just now proved. And now we write B 2 and B 1 in terms of the curl of the vector potential. So, this is curl of A 2 minus A 1 this equal to mu 0 n cross K. So, notice what you could do is you could multiply n cross on both sides so that I will get n cross del cross A 2 minus A 1 equal to mu 0 times n cross n cross K.

Now, I know that A cross B cross c is B A dot c minus c A dot B and that turns out to be equal to simply minus K so it is minus mu 0 K. Now, from this it is fairly simple to show that the normal derivative of A suffers a discontinuity. I am not going to explicitly prove it, but you could take this as an exercise. See, basically what you do is this. We had shown that t n and S they are a system of right handed coordinate system they are like Cartesian coordinates. Therefore, you can treat them like your x y z and x find this out. Now, when we expand this out you will find that the derivative with respect to n the, this is very similar to you have derivative with respect to x y z the definitions are very similar will turn out to have a discontinuity.

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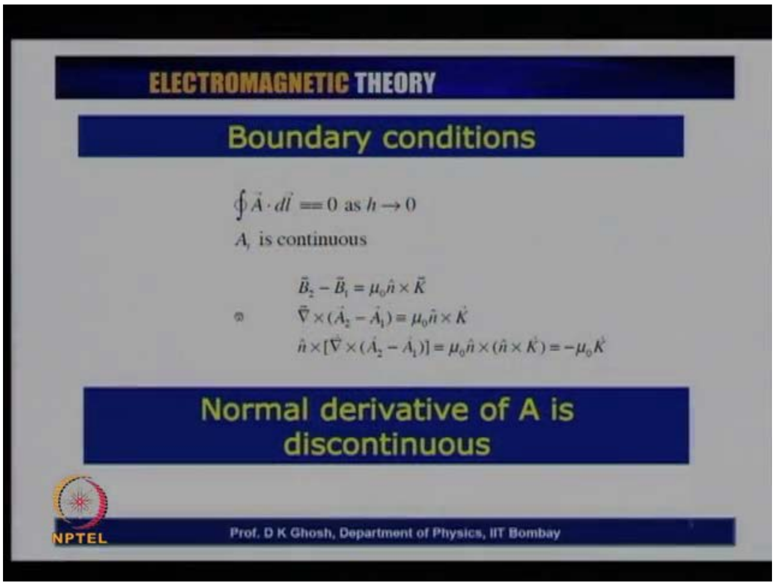

So, what we say is that the normal derivative of A has a discontinuity. Remember, that the vector potential A both the normal component and the tangential components are continuous. Now, I am going to do something which is rather unusual. We had derived a vector potential and we had seen that for a magnetic field, because del cross of B is mu 0 J, I cannot define a scalar potential unlike in case of electrostatics.

However, excepting where I have sources supposing, I have a current distribution, but I am looking for solution or looking for the magnetic field in space other than where there is a current distribution. Now, then at the point of observation my current density J is equal to 0. Now, if the current density J is 0 at such points I must have del cross of B is equal to 0.

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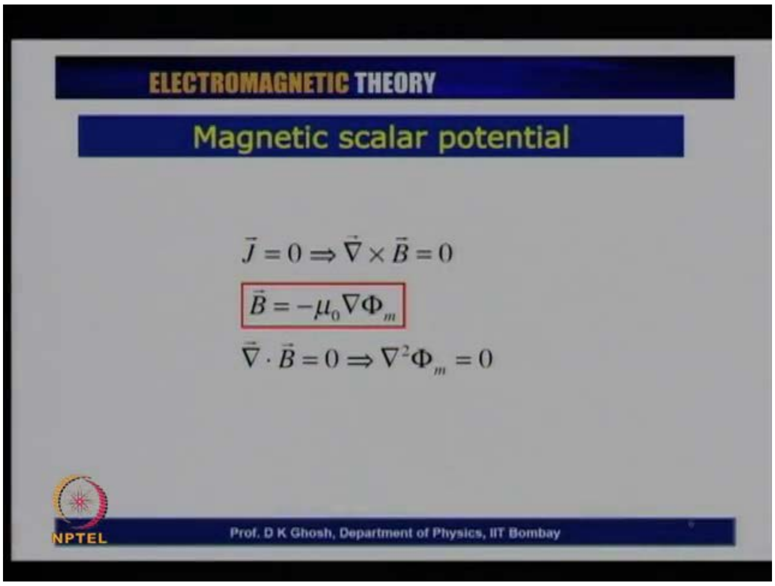

Now, if del cross of B is equal to 0.

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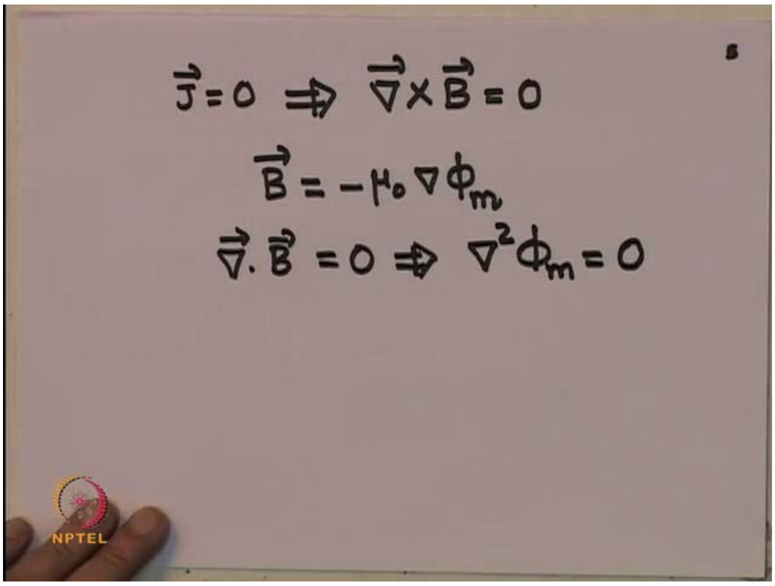

So, J equal to 0 implies del cross of B which is equal to mu 0 J that is equal to 0. So, that enables me to drive, define that B can be written as a gradient of a quantity which I call as a potential for dimensional reason I put in a mu 0 there and minus I retain to have similarity with the electrostatics where we have defined the electric field as minus gradient of the potential. Therefore, del dot of B equal to 0 the Gauss's law implies that the scalar potential satisfies the Laplace's equation del square phi m equal to 0. What we will do is to look at this is a useful thing in certain circumstances and what we will do is to talk about couple of examples of how to calculate magnetic scalar potential.

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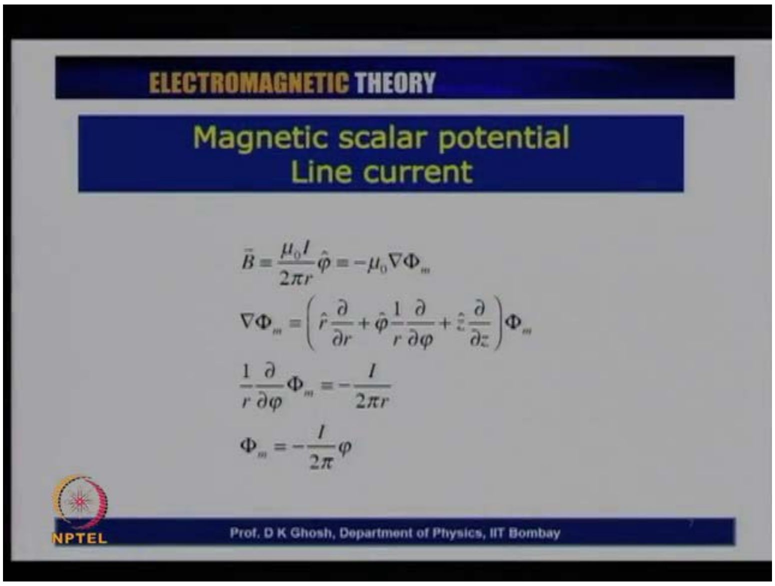

So, let me take first a line current and we of course, know that if I had a line current for a line current the magnetic field.

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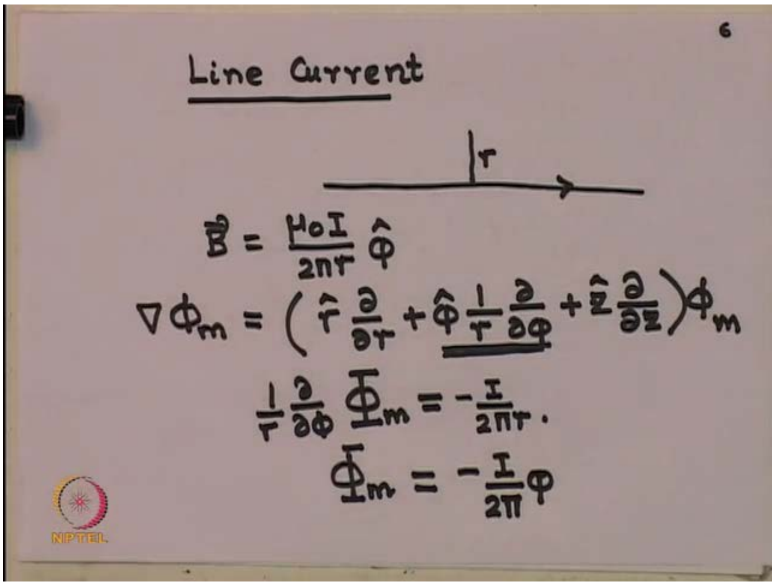

So, if I have a infinite current at a distance r the magnetic field is given by mu 0 I by 2 pi r, r is the distance from the line and I know its direction is Azimuthal direction. We had seen that what you have to do is to hold the wire with your thumb pointing in the direction of the current and then of course, the way your fingers curl they will give you the direction of the magnetic field there. Now, since the I have a cylindrical symmetry let me look at what does gradient look like in cylindrical coordinate system. So, gradient if you look at cylindrical coordinate system is r d by d r plus unit vector phi 1 over r d by d phi and of course, unit vector z times d by d z of phi.

So, since the gradient of phi is B and B is only in the direction phi so I do not have to worry about the dependence of phi on r and z, I only need dependence on phi because that is the only term which gives me term proportional to the unit vector phi. So, as a result what I have is 1 over r d by d phi of phi n that is equal to if you compare with this expression remember I had in my definition, I had a minus sign therefore, this quantity must be equal to minus I by 2 pi r, r and r will cancel out and this gives me this scalar potential to be equal to minus I by 2 pi times the angle phi itself.

Notice, one thing that this potential is not really unique because every time you circle the origin the the you know your angle phi increases by 2 pi. That tells us that the the curl does not have, the vector potential does not have certain amount of liberty that we take. So, that is about line current.

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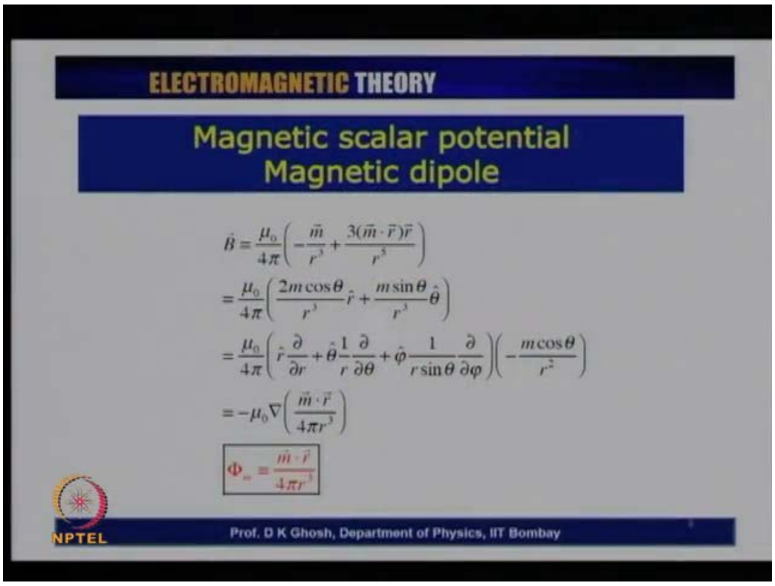

Little more complicated, but let me talk about a magnetic dipole. Remember, that we had talked about that a current, a small current carrying loop is an example of a magnetic dipole and the magnitude of the dipole is simply the area of the loop and the direction is outward normal and the outward normal of course, is defined by how you circle the current loop. So, we had already derived an expression for the magnetic field for a current loop.

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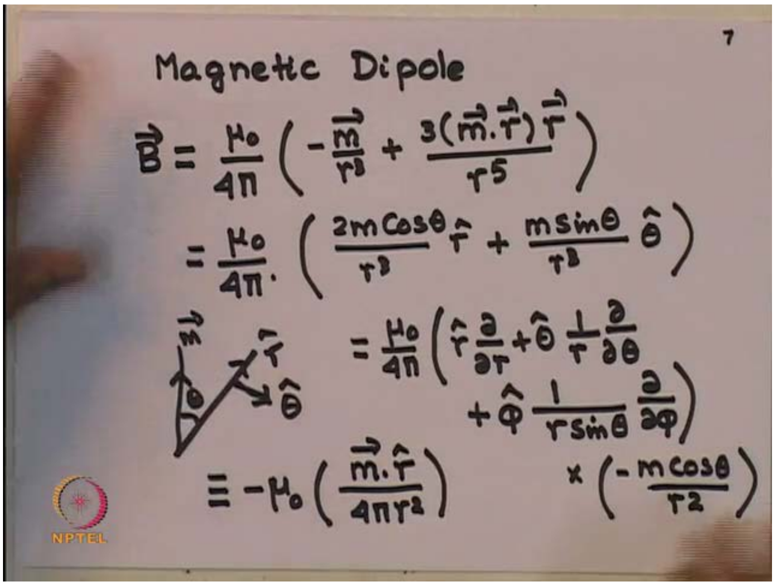

So, let me write it as a magnetic dipole. The magnetic field corresponding to a circular current carrying loop in the coordinate field representation was written in one of the earlier lectures to be minus m by r cube plus 3 m dot r r divided by r to the power 5. Actually all of them are inversely proportional to r to the power 3, but there are two r's in the numerator here. So, let us try to find out how to write this. So, this is mu 0 by 4 pi. Now, what I am going to do is this that I am going to rewrite this, I am going to rewrite this in the r theta representation.

Now, that is fairly straight forward. I take the magnetic movement m the angle between the magnetic movement m and the radial vector I take as theta. So, that this you can do it as a fairly simple exercise. So, basically what I do is I have a magnetic movement m there and a vector r there. So, this angle is theta and I know that the remember that this is the angle, angle is measured from the direction of the magnetic movement which is taken to be along the z axis. As a result the (( )) vector theta is along this direction.

You have to realize that we always define that direction of unit vector as the direction in which the quantity increases just as outward direction of r the direction in which r increases is the direction of radial vector. Similarly, the direction in which theta will increase will be the direction of this theta vector just as we define unit vectors along x y and z as the directions in which x y and z increase. So, if you look at that and you know this angle then you can rewrite this fairly straight forward exercise as 2 m cos theta divided by r cube unit vector r plus m sin theta divided by r cube again and unit vector theta.

Now, I want this quantity to be written as minus mu times gradient of a quantity. So, I claim that this can be trivially shown that this is mu 0 by 4 pi. I am simply writing down now what is the expression for gradient in this spherical polar system and that is r d by d r plus unit vector theta 1 over r d by d theta and just for completeness I will also write down unit vector phi 1 over r sin theta d by d phi. Now, notice since my magnetic field does not have any phi dependence it tells me that I could choose the vector potential or a potential here to only depend upon theta and phi.

So, that this term will always give me 0 and it is fairly straight forward to see that if this quantity operates on minus m cos theta divided by r square you get the expression which is in the previous step. Now, you can see why. This is d by d r so I get minus m cos theta and minus 2 by r cube. So, that this gives me 2 m cos theta by r cube unit vector r. Unit vector theta is 1 over r times d by d theta, d by d theta of cos theta is minus sin theta there is a already a minus there that gives me plus sin theta and of course, you have 1 over r that takes care of this r square and giving me r cube.

Therefore, this is the expression from where I need to derive my potential. Now, I want this this quantity then is identical to minus mu 0, I already have a mu 0 there. There is a 4 pi there so that I must observe inside this minus I have taken care of. So, I must write it as a 4 pi r square times m cos theta. I will write this as m dotted with unit vector r.

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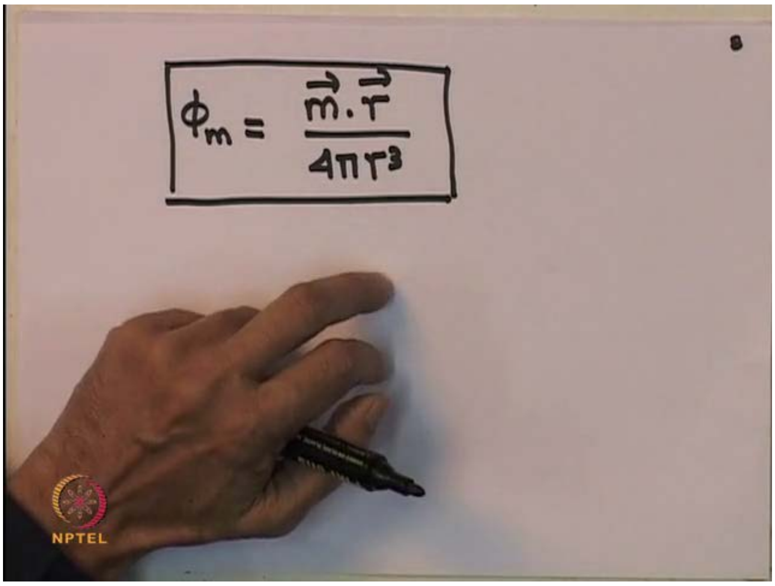

Alternatively, I could simply write the vector potential as equal to m dotted with full vector instead of the unit vector r divided by 4 pi times r cube. So, this is my vector potential corresponding to a magnetic dipole. Now, notice notice this, this is now I want to write it in a slightly different way. Now, here is my current loop, this is the current loop on the screen.

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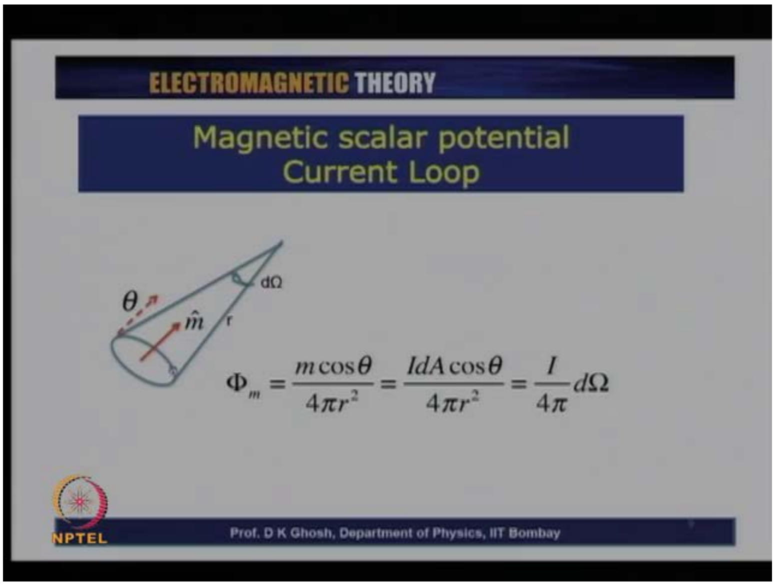

This is the direction of the vector m and what I have done is this is the observation point. Now, suppose this observation point makes an angle, makes a solid angle d omega here.

Now, my expression for the vector potential is m cos theta by 4 pi r square. Now, I know that how to write down the area the projection of the area in terms of the solid angle so I write this as I times d A that is r. So, I have a 4 pi r square is I times d A that is my magnetic movement. Therefore, this quantity can be written as I by 4 pi because you have this d A by r square, d A cos theta by r square is nothing but the solid angle d omega.

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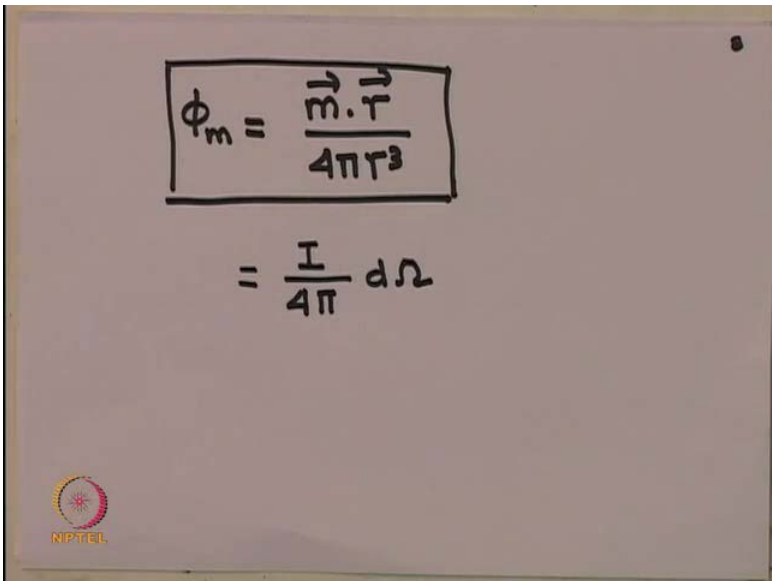

So, as a result this phi m can also be written as equal to I by 4 pi, I is the current passing in that loop times d omega. Now, this is an useful formula to have and let us look at what what is the problem?

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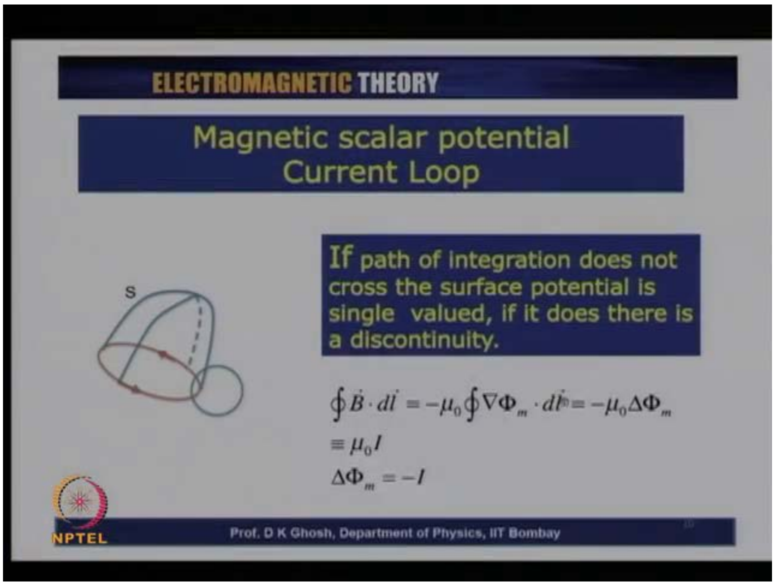

Now, suppose I have a loop here. Now, notice that if this loop is encircled by a path which crosses this loop like for example, this one then the according to Ampere's law if I am going to integrate on this path then I will have a discontinuity because integral of B dot d l is equal to mu 0 times the current enclosed and in this path there is a current enclosed. But on the other hand suppose I make this line a current loop as a rim of let us say a surface S.

Now, as long as I go from one point on the surface to any other point for example, these paths along this path without crossing the loop then my potential will be single valued. Now, if it does across for example, this path then there will be a discontinuity. Now, this will enable me to write down what will happen to the potential.

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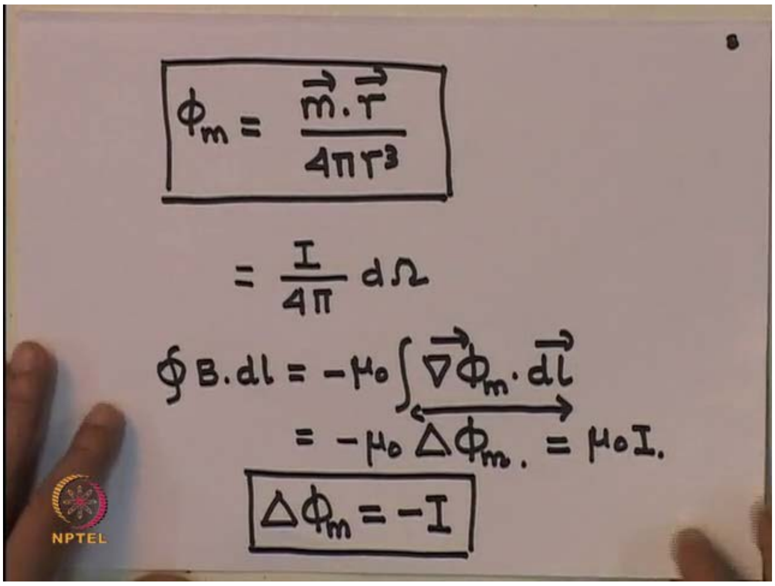

Now, notice integral B dot d l. Let us do it on the screen on the paper, integral B dot d l is equal to minus mu 0 del phi m dot del and by fundamental theorem of calculus this quantity here is nothing but the change in the potential. So, it is minus phi 0. So, I will use this sign delta to show this is the amount of change and this must be equal to mu 0 I, if this is intersecting a current loop. So therefore, the discontinuity delta phi m is going to be simply given by minus I. Now, let us look at this then.

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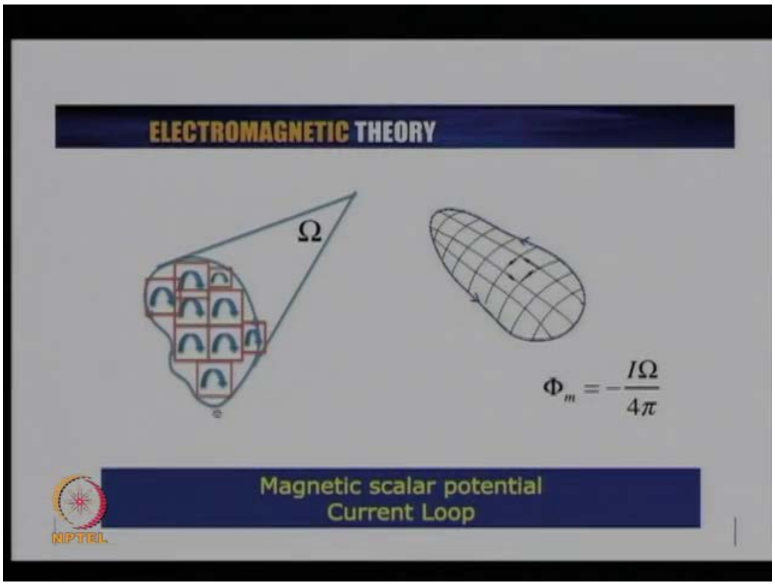

So, what we have done is this that here I have a current loop, arbitrary current loop. Now, and I am looking at what is the scalar potential at a point like this. So, firstly let let me take this loop, I have shown it here on this picture. Now, I can divide this current loop into large number of small loop each one of them behaves like a magnetic dipole.

Now, you can see that if you have adjacent loops then of course, the direction of currents will cancel out from the adjacent loop leaving me only the outer boundary. Now, this is of course, a type of networking that we have done earlier. Therefore, the contribution of these entire loop comes from the solid angle that this entire loop makes here because only the outside things on the boundary will contribute and therefore, keeping in view that my discontinuity is supposed to be minus I if I make one loop which means if I change this, if I go around it the solid angle increases by 4 pi.

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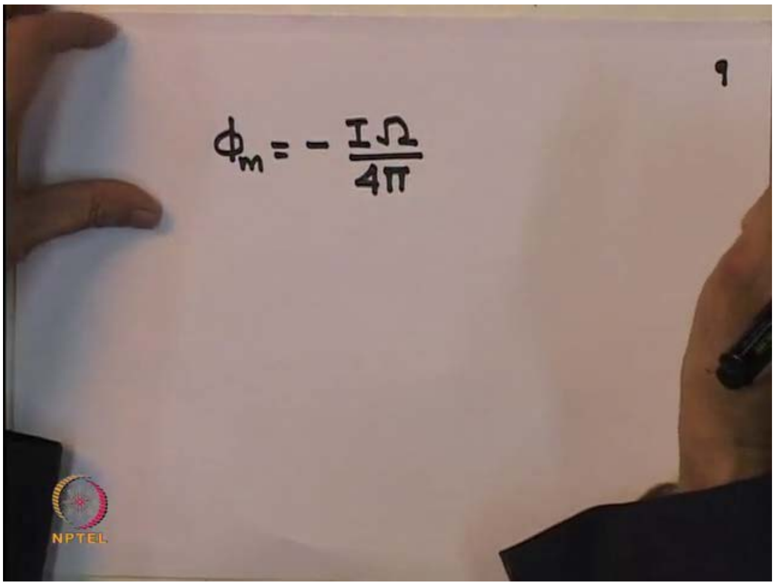

Therefore, my discontinuity should be minus I and taking care of that it tells me my expression for the scalar potential phi of m should be given by minus I omega by 4 pi. Having done that let us switch over to a slightly different subject.

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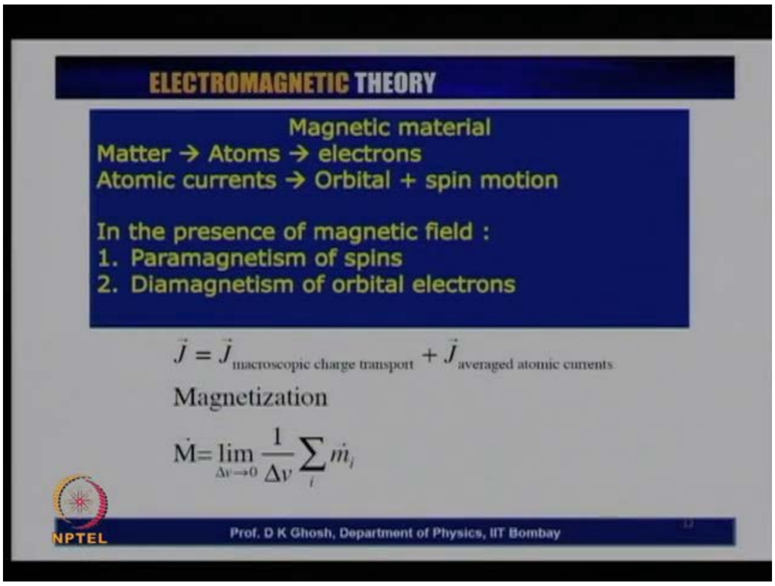

If we recall when we discussed electrostatics we had seen that if I have a medium and it is subjected to an electric field, it polarizes the medium. Now, in our case we have something very similar, excepting I do not have an electric field, but I have magnetic field and we know magnetic field will exert a force on moving charges, but there are moving charges inside matter, there are atoms. So, matter consists of atoms, atoms sub electrons. Now, the electrons provide me, the orbital motion of the electrons provide me atomic currents.

Now, the other thing is that as we know that each electron has an intrinsic magnetic moment because of its spin. Now, so as a result the atomic currents if you like have two parts. One is because of the orbital motion of the electrons and the other one is not strictly a quantum mechanical language, but other one is because of the spin motion. That is a rather classical picture, but this will do for our purpose. Now, what happens in the presence of magnetic field?

So, there are two major effects we will be talking about. One is something which we will be talking in a forth coming lecture, but you must have already learnt about it from school that there are materials which are known as diamagnets and what the diamagnets do is if there is a changing flux it opposes the change in the flux. As a result, this leads to the magnetization or the current which will oppose the direction of the, in which the current is flowing.

There are materials which are known as paramagnetic where the microscopic charge transport will be, the additional thing will be in same direction. So, what I do is this that my total current consists of two parts. One is the type of current if you like conduction current which we have been talking about. That is what I am calling as the macroscopic charge transport that is the bodily motion of the electrons that takes place for example, in a metal. The other one is the atomic currents, but of course, the atomic currents are too tiny to talk about individually. So, what we do is to do an average atomic currents and I define a quantity called magnetization.

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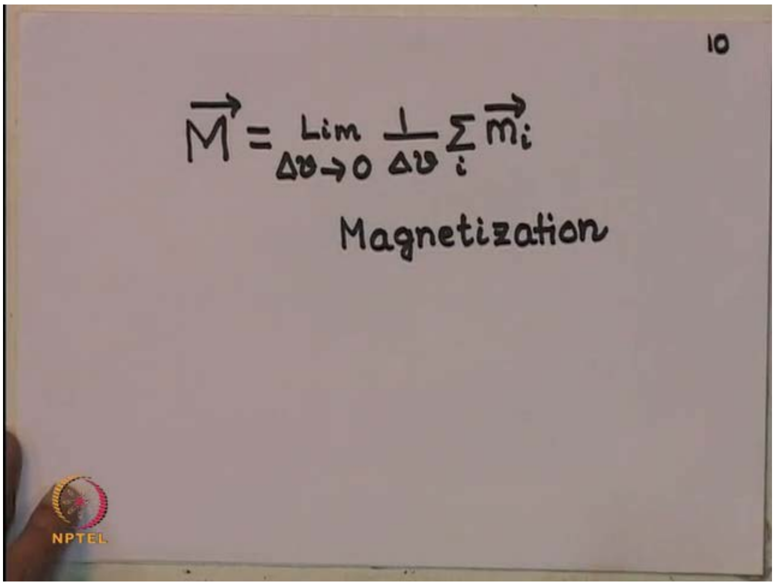

The magnetization is defined as I take a collection of material and find out how much is the net magnetic movement. I vectorially add the magnetic movement of each tiny dipole and find out how much is the magnetic movement per unit volume and of course, I give it the usual way in which I define by taking the volume to be small. So, delta v going to 0 sum over 1 over delta v sum over i m i.

So, this is simply the magnetic movement per unit volume and this is given a name magnetization. This is analogues to what we talked about as polarization in our electrostatics lectures. Now, what we are going to do next is to calculate the vector potential of a magnetic material. So, we had seen the total contribution will be because of two things. One is that I have the charge transport and of course, I have the average microscopic currents.

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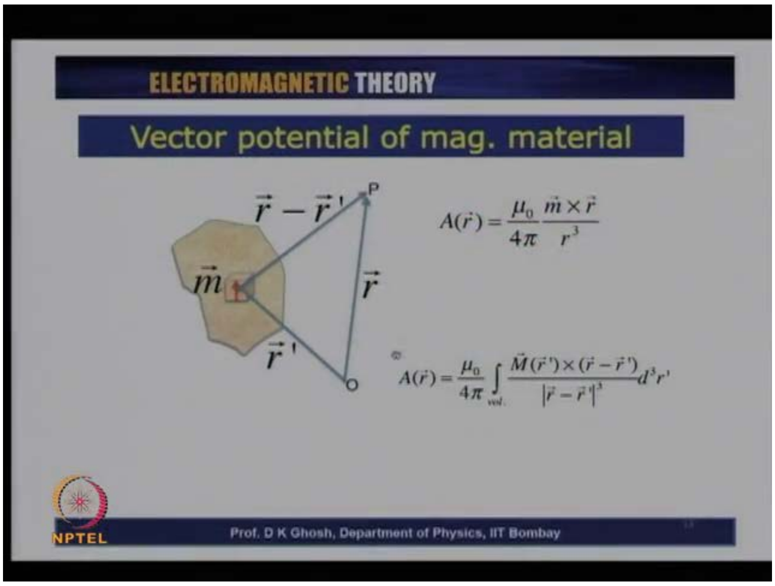

So, let us look at this picture here. In this picture I have shown at a position r prime a magnetic movement m and I am looking at the field at the point P. Now, recall that for a magnetic movement m the vector potential at a position r taken from the position of the magnetic movement the tiny dipole is given by mu 0 by 4 pi m cross r by cube. This is the expression which we have derived some time back on the vector potential of a magnetic dipole.

Now, what we have is I have a collection of magnetic dipoles. Therefore, this A of r is given by so I take the location of the magnetic movement at r prime and I am looking at what is the field at the point P. So, with respect to this position it is at position r minus r prime.

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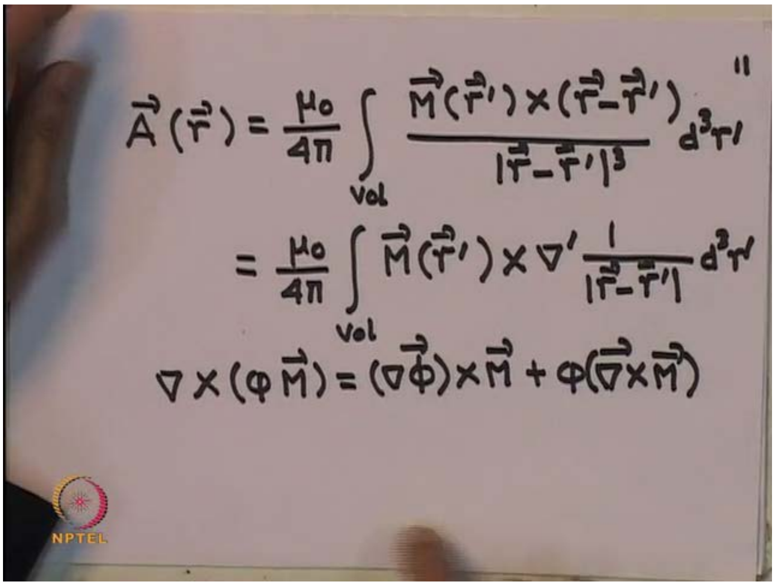

So as a result by simply summing over I get vector potential at the position r is given by mu 0 by 4 pi. I will have to integrate overall volume. Magnetic movement, this is actually the magnetization because I take a volume d v, volume d v times the magnetization give me, gives me the magnetic movement at the position r prime cross r minus r prime divided by r minus r prime cube d cube r prime. And very analogous to the expression that I wrote down. So, this is M r prime, d cube r prime is my, this is magnetization, this is, this into this is my magnetic movement and rest of it is exactly what we had done earlier.

Now, I want to write this in a slightly different way. If you recall when we did the electrostatics we had very similar situation and what we did is to resolve it into two parts. One which we call as the bound volume charge and another we call as the bound surface charge. The question that we are asking is, is it possible to have a similar separation in case of the vector potential? The answer is yes and let see how. So, what we have done is this. The first thing that we do is, we will write this as mu 0 by 4 pi integral over the volume M of r prime cross. Now, notice r minus r prime by r minus r prime cube is negative gradient of 1 over r minus r prime, but I am going to take care of that negative sign since my position r here is a fixed position I am going to take care of the negative sign by taking the gradient with respect to r prime and that I am denoting by a prime there and I will write it as 1 over r minus r prime d cube r prime.

Now, I am going to be using several vector identities now, but the first identity that I will use is this that if I have a del cross of a scalar times a vector, this is grad phi cross M plus phi times del cross M. Notice, in this expression here I have got M cross gradient. So, this is very similar to what I have here excepting that here the grad comes on this side, M comes on that side. So, that is can be taking care of a by by very minus sign minus sign. So, if you now use this relationship and plug it into this expression I will get a quantity which is well I have a del prime so del prime cross M by r minus r prime and the minus sign that I have talked about. So, if you do that then what you get is the following. So, I am using this expression to write this del phi cross M as this quantity minus that quantity.

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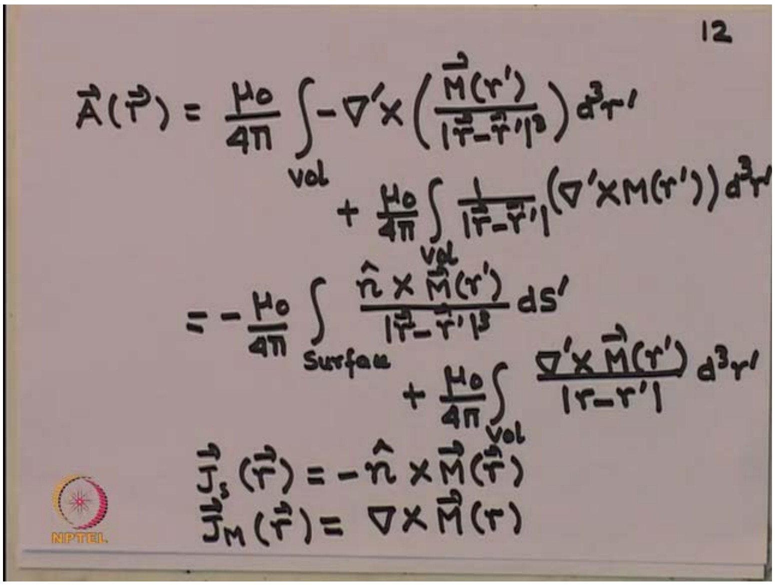

So, that is A of r is equal to mu 0 by 4 pi integral over volume minus del prime cross M of r prime by r minus r prime cube d cube r prime plus mu 0 by 4 pi integral over volume again 1 over r minus r prime that is the scalar times del prime cross M of r prime d cube r prime.

Now, let us look at this. How can, how do I simplify this? Firstly, you notice this expression here del prime cross this quantity d cube r prime. Now, by argument very similar to what we had in case of Stoke’s theorem excepting that Stoke’s theorem usually was used to reduce the curl of a quantity the surface integral to a line integral. In this case because it is a volume integral we will reduce it only to a surface integral. So, this will give me minus sign is already there minus mu 0 by 4 pi integral over surface of n cross M of r prime divided by r minus r prime cube d S prime, prime because I have always putting prime over that and the next one I simply write it the way it was mu 0 by 4 pi integral over volume del prime cross M r prime over r minus r prime d cube r prime.

You notice that this is, there is a surface integral, this is taking the role, this is taking the role of a current, a surface current. So, I can define a surface current J S of r prime or I have now changing it over to r as minus n cross M of r and this is my volume current density which is simply del cross. So, J of if you like magnetization current density J M of r is equal to del cross of M of r.

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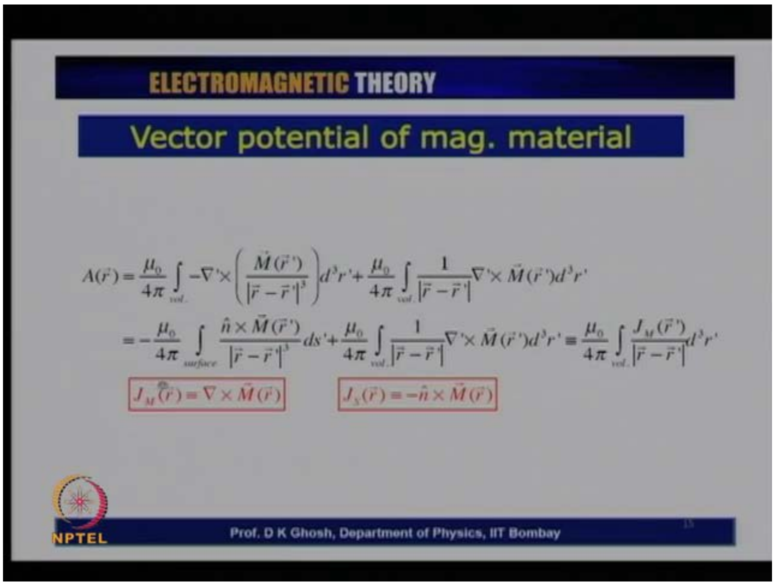

Come back to this screen here. This gives you that there is a volume current density due to magnetization and there is a surface current density due to magnetization.

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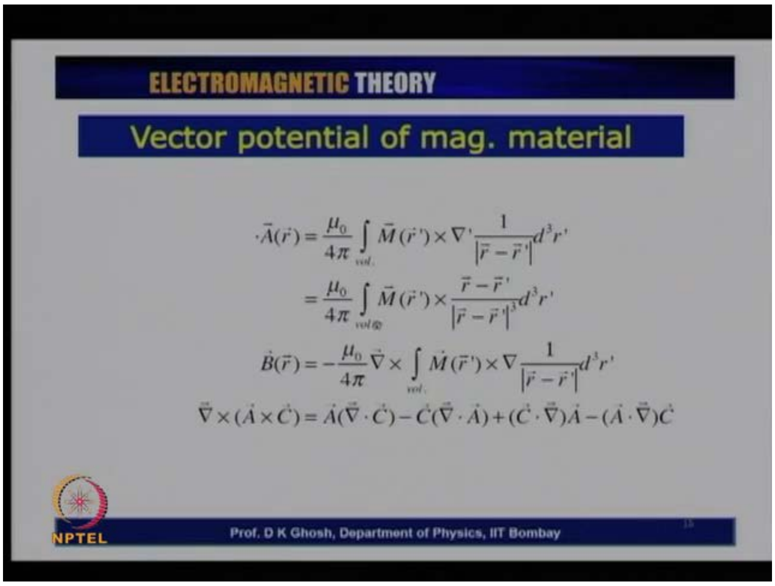

Now, what we are going to do next is to use this expression for the vector potential and try to see how does it affect or how does it give me the magnetic field, how does it change the magnetic field due to the magnetized magnetization of the material? We will continue with this next time.