Electromagnetic Theory
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Module - 3
Magnetostatics
Lecture - 25
Magnetic Vector Potential

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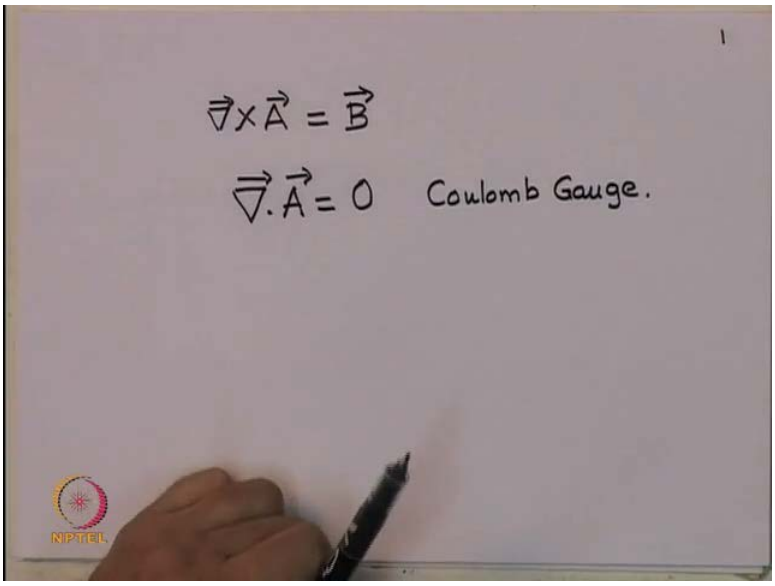

In the last lecture we had introduced the concept of a vector potential for the magnetic field, and it was defined through a relationship, which says del cross of A gives you the magnetic field B. What we also did is to say that because of the fact that curl of gradient of any scalar function is equal to 0. The vector potential is not totally completely defined by this equation. So, the curl of the vector potential is the magnetic field, which is of course, a physical quantity, but because it is a vector quantity, vector field. We still have the liberty of choosing what its divergence could be. We had seen that it is possible always to do so to have a proper choice such that del dot of A can have any value as we liked.

And in particular we had said that del dot of A equal to 0 is a very convenient gauge, which is known as the Coulomb gauge. So, what we will do today is to continue this discussion of the magnetic vector potential and calculate it in a few cases.

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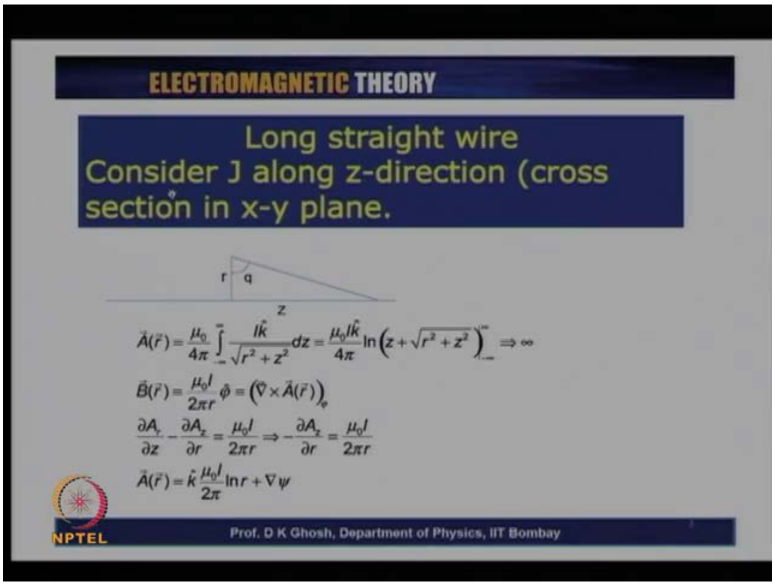

The first simple example that we want to do is to calculate the magnetic vector potential for a current carrying conductor, long current carrying conductor.

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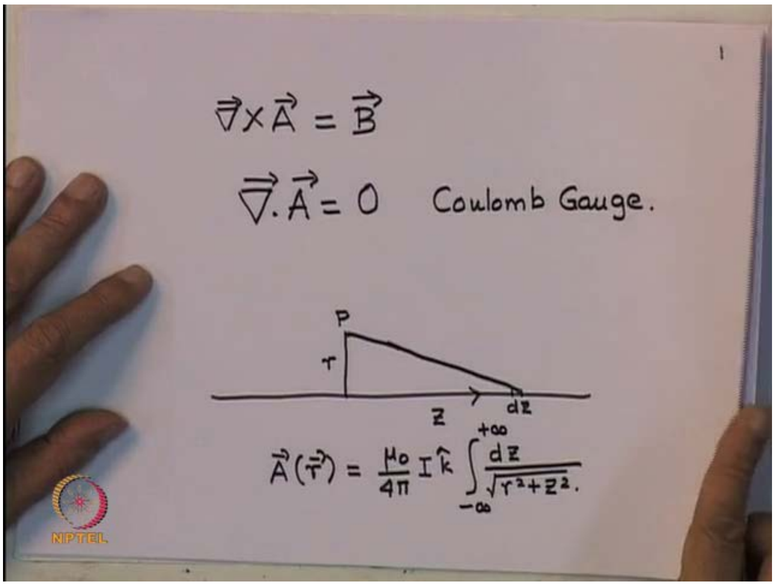

Let us take the magnetic the direction of the current to be along the Z direction. This is infinite and we had seen that there is a stronger relationship between the vector potential and the current density. I expect the vector potential to be in the direction of the current. So, let us look at, what the vector potential could be. So, for instance if I look at a distance r and sort of try to find out what is the vector potential there. Then this vector potential expression A of r is given by the constant mu 0 by 4 pi, if you remember that if I take a current element d l or d Z in this case here.

So, that this is the distance between the point P, where I want to calculate the vector potential and this current carrying conductor. So, this is simply given by because it is a line current. This is simply given by the current I multiplied by k d Z, I times d Z in the direction of k, gives me the current element on d Z divided by this distance, which is square root of r square plus Z square and because of the fact that my integrals d Z is from minus infinity to plus infinity. Now, this is what I expect it to be. Now, let us look at what it gives us.

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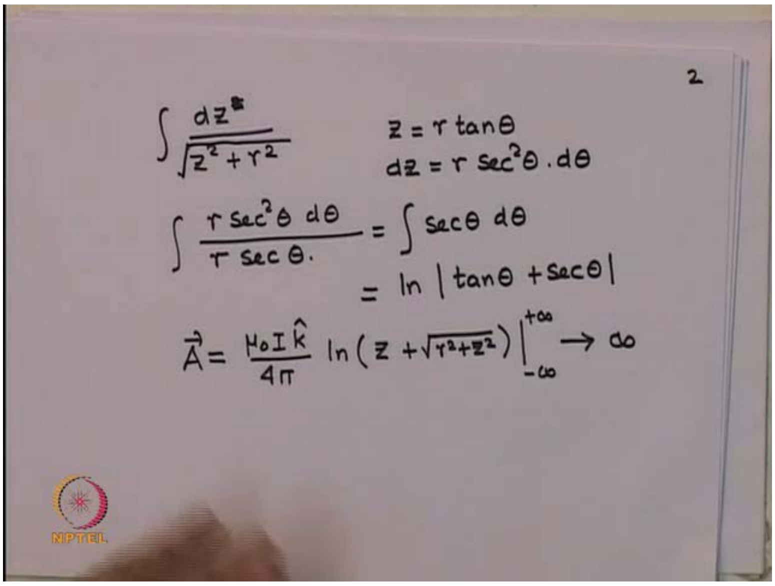

So, notice I have to calculate this integral which is d Z over Z square plus r square square root. Not a very difficult integral to do, if I simply use Z is equal to r tan theta. So, that d Z is r sec square theta d theta. So, this quantity becomes r sec square theta d theta divided by this is 1 plus tan square theta. That is sec square theta and so that is r sec theta. So, this is another simply an integral of sec theta, which we know is given by log of tan theta plus sec theta. If you put it back into this expression for A after putting in all the constants back. I get mu 0 I k by 4 pi. Logarithm of now, I have a tan theta plus sec theta and tan theta.

As you can see is Z by r and sec theta is the square root of r square plus Z square. So, this is given by Z plus root of r square plus Z square and this has to be evaluated from minus infinity to plus infinity and this goes to this result diverges. Now, this is this is done quiet give me the vector potential, if I calculate this straight way and the primary reason is that I have an infinite current element, a infinite current carrying conductor. However I can do something else.

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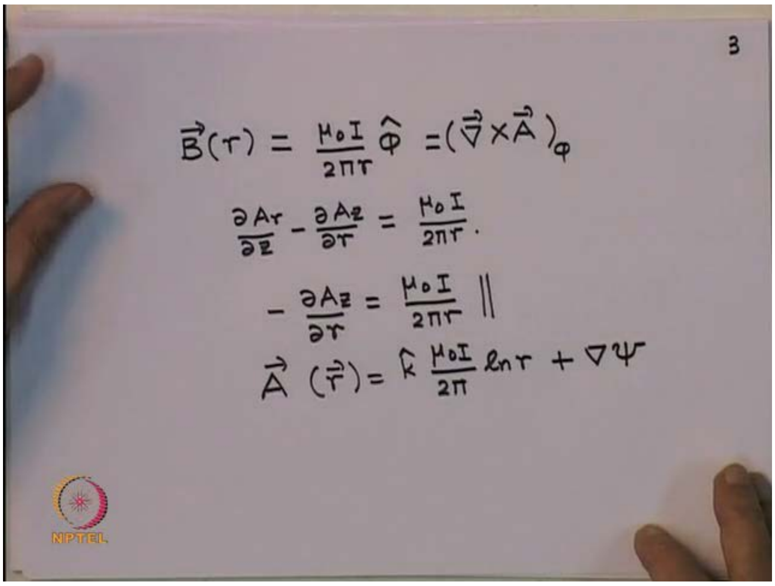

I can calculate the vector potential from the expression of the magnetic field itself. I know by Ampere's law, the field at the point r is given by mu 0 I by 2 pi r times well in the direction of the azimuthal direction. That is the direction pi and this quantity is del cross of A. So, what will do is this I notice this is only in the phi direction. So, if I use the cylindrical symmetry, then I know that I can simply get the phi direction from the the what the from the phi component of this. That is given by d A Z d A r by d Z minus d A Z by d r and this quantity is equal to mu 0 I by 2 pi r. Now, notice by symmetry by symmetry I do not expect a derivative with respect to Z to survive because I do not expect A r to have A Z dependence because Z is the symmetry axis. It is an infinite thing.

So, I expect for all values of Z A, given Z the value of the A r would be the same and that gives me that minus d A Z by A d r is equal to mu 0. I by 2 pi r and that tells me that the A is along the Z direction. So, A at the point r is along the Z direction. This can be integrated trivially to give me what is A Z. That gives me mu 0 I by 2 pi times logarithm of r. As you have said repeatedly this is not unique, but I can always add a gradient of an arbitrary scalar function there. So, this is the expression for the vector potential for a current carrying conductor. There is another trick which turns out to be very useful and that is that, we know that I can directly connect or relate the an integral of the vector potential with the flux and that is done this way.

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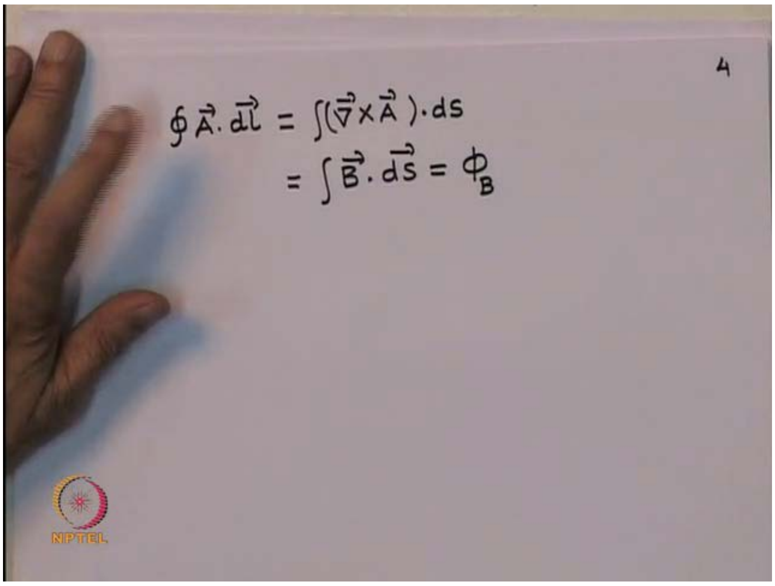

Supposing you have to calculate A dot d l that is the line integral of the vector potential along any closed loop. Now, I can use the Stoke's theorem to convert this into del cross A dot d s over a surface whose boundary is given by this curve. Del cross A we have said is B so this is nothing but B dot d s which is nothing but the magnetic flux fly. So, there are situations where it is more convenient to compute the magnetic flux. Then use the symmetry to find out what A could be and what A dot d l could be.

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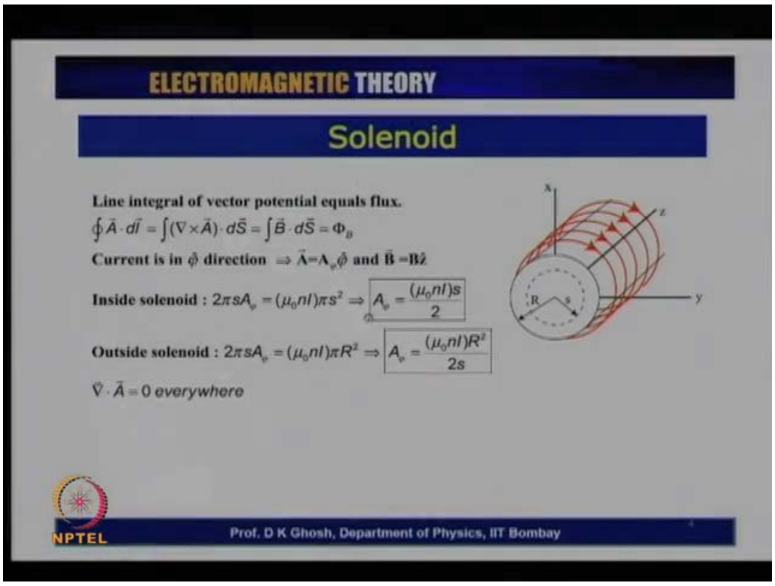

A typical example you can see it on this slide here, I am considering a solenoid and so basically the axis of the solenoid is along the Z direction. These are terms of the solenoid. Now, notice that if I take a circle of radius Z radius s such that this radius s is less than the radius of the solenoid. Then I have seen that inside a solenoid the magnetic field is constant. So, let us look at inside the solenoid.

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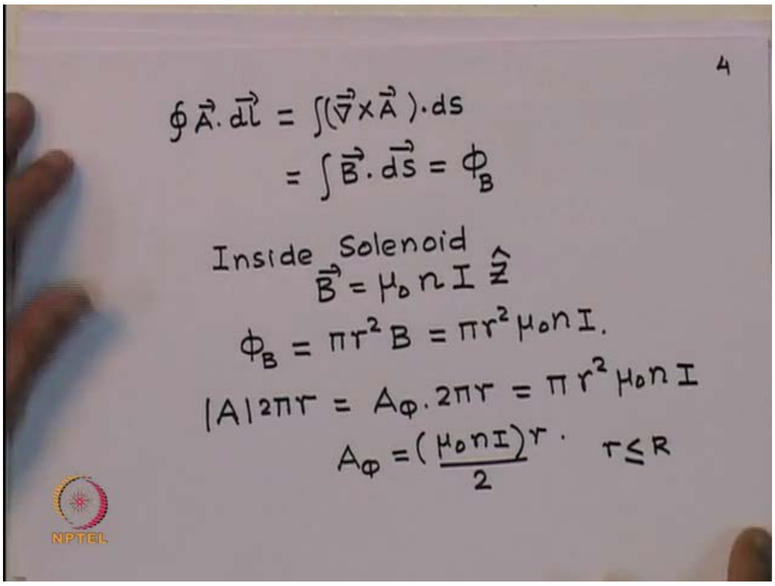

So, I expect inside the solenoid, the magnetic field B to be given by mu 0 n, which is the number of turns per unit length times the current and it is along the Z direction. So, if I take a circle of radius small r since, the magnetic field is constant the flux phi through that surface will be given by pi r square times field B, which is pi r square mu 0 n I, but since, the loop is cylindrical, a loop is circular. Since, the loop is circular and symmetry tells me there is no reason why the vector potential should depend upon should vary from point to point on the loop. So, A dot d l should be A times 2 pi r.

Notice that I have not talked about the direction of the current because a direction of the vector potential because I have said that the direction of the vector potential will be along the direction of the current. In this case the current is in azimuthal direction therefore, this is actually a phi and that is is equal to pi r square times mu 0 n I. This tells me that the vector potential which has only the phi component after cancelling out 2 pi 1 of the r something is like that is mu 0 n I r divided by 2. This picture had that radius as s, but let me since, I have used r. So, let me say this is r is less than or equal to r. Now, notice one thing that outside the solenoid the field is 0, but if I take a loop outside the solenoid.

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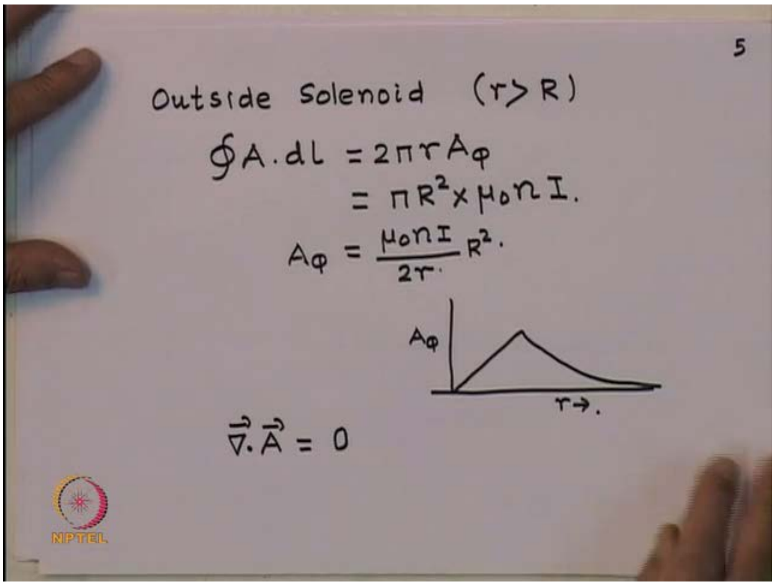

My A dot d l which is still given by outside solenoid r is greater than capital R, which is the radius of the solenoid and A dot d l is given by 2 pi r times A, which is you have seen is along a phi. Now, this is the flux. Now, the flux there is no contribution to the flux from outside the solenoid, because the field there is 0. Therefore, what I get is pi r square, but this time capital R square times the magnetic field which is mu 0 n I. That tells me that a phi is given by mu 0 n I by 2 r and times of course, r square. So, notice that the magnetic vector potential with distance r is falling as 1 over r and inside the solenoid. We had seen that it was proportional to r.

So, typically the vector potential a phi would do this that is inside I will have a linear increase. Then of course, there will be a decrease like that and of course, A only depends upon phi. You can check that del dot of A since, there is just a phi component. You can compute del dot of A by looking up the expression for the divergence in the cylindrical quardinate. You can show this to be is equal to 0. So, we are working in Coulomb gauge. Now, this fact that the vector potential has a non-zero value outside the solenoid as well as inside the solenoid, unlike the magnetic field which was 0 outside, but had a finite value inside has been of great use in proving the physical reality of the vector potential itself. This has been done by, this is actually useful or these are more a quantum mechanical experiments.

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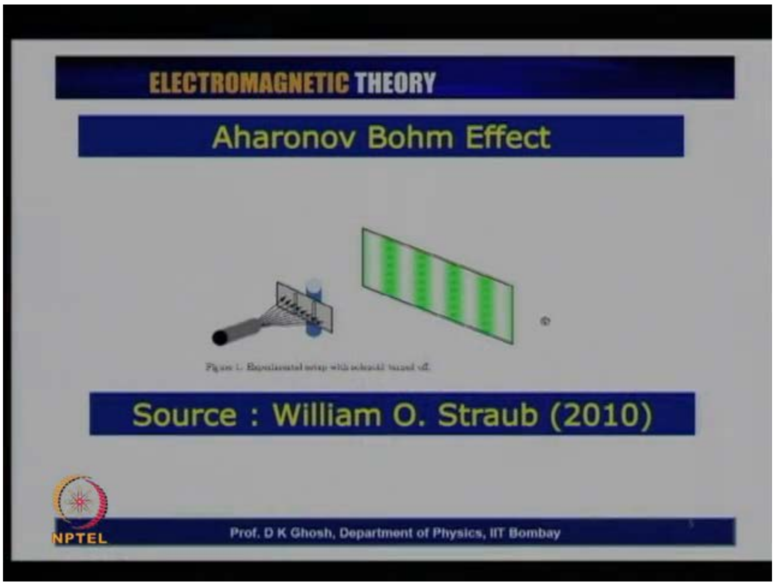

But I would point this out here to tell you that the vector potential very much has a realistic origin and this is done by an experiment, which goes by the name Aharonov Bohm effect. Now, the experiment is very simple. You recall you are Young’s double slit experiment. So, basically I shine well in this case instead of light, I shine a source of electrons that is a electron beam is shown upon a Young’s double slit type of experiment. The Young’s double slit experiment is not just restricted to light, it can be done by electrons or any other beam that if you like.

Now, notice that what is done in this experiment is, that if this is of course, the beam of electrons which is shown on a double slit and just outside that just outside that means between the screen and the slits I have put in a rather small size solenoid. Now, in initially solenoid is there, but there is no magnetic field in it. That is I am not passing a current through the solenoid. I just do the standard double slit experiment with the beam of electrons. We know that interference pattern will be seen on the screen, that is because there is a face difference between the electron wave coming from one of the slit and the other. That just like an optics gives me a interference pattern.

Now, what we do next is this. That we switch on a magnetic field. That is we switch on the current inside the solenoid. Now, and let the electron beam pass through this. Now, remember because of the fact that the solenoid is extremely small, most of the beam it as a very small cross section most of the beam actually passes outside the solenoid. So, in principle in the limit of a extremely small solenoid, I do not expect this to affect the interference pattern because when I have a beam of electrons passing through a magnetic field it is of course, subject to a force, but in this case very insignificant amount of the electron beam passes through that solenoid. But however, what one notice is that there is a phase change, that is the interference pattern changes.

Now, in this course I will not be able to give you a realistic explanation of why this happens because it is actually quantum mechanical in origin. One can relate that the fact that the region in which the electron beam is passing through, in that region the vector potential is not equal to 0. This has an effect on the electron waves phase. This phase difference shows up in the pattern that you actually see, on the screen when you switch the magnetic field. So, this is something which is good to keep in mind that what we had thought to be a mathematical artefact, actually has a reasonable physical effect which one can actually demonstrated. But I warn you these are difficult experiments done in with quantum waves we continue with our examples of the vector potential.

So, let us calculate the vector potential corresponding to a situation where the magnetic field is constant. This is a very useful thing because very often in an experiment, you deal with constant magnetic field.

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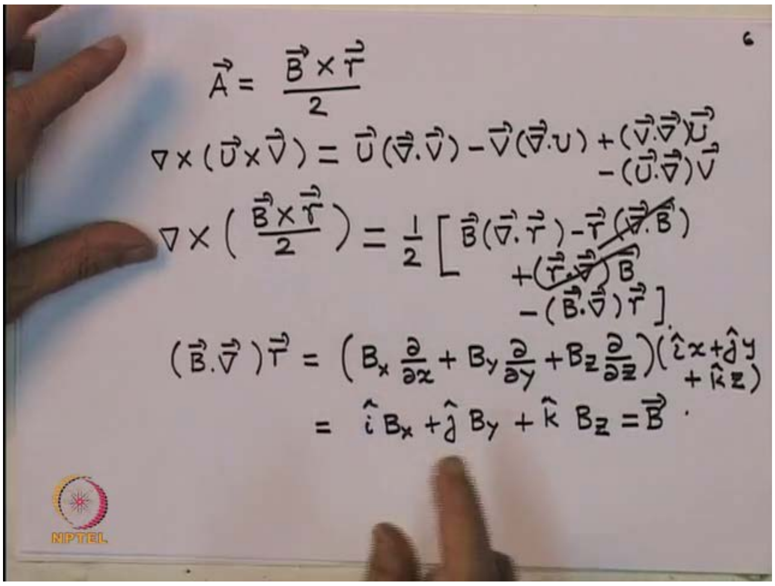

So, let me write down constant magnetic field to be B. At this moment, I am not talking about a direction. So, I claim that the vector potential A is given by this expression B cross r by 2. This is of course, as we have said repeatedly that this is one of the possibilities. So, notice that I know this is a vector algebra. So, del cross of U cross V, where U and V are two vector fields is given by U del dot V minus V del dot U plus V dot del U minus U dot del V. This I am not going to derive it, but just be careful these are all vector expressions.

So, remember when you say del of V it actually means three things and gives rise to you know del of V x del of V y del of V z. Then that vector is to be dotted and so because of that each one of these is actually a vector. So, notice that I am claiming that A is B cross r by 2. In other words I am saying that if you take del cross e, it should give me B. So, let us check why? So del cross B cross r by 2. Notice there is a half of course, in this case identify U with B and r with V. So, what I get is B times del dot r minus r times del dot B plus r dot del of vector B minus B dot del time of vector r. Now, notice we are talking about a constant magnetic field B, therefore that gives me del dot of B equal to 0. So, this term goes away.

I know del dot of r which is divergence of r must be equal to 3. Similarly, r dot del B because B is a constant that is equal to 0. So, I need to calculate what is this term. So, this is also equal to 0. So, I need to calculate what is B dot del so far. So, remember that the way to understand this is B dot del is to be written as B x d by d x plus B y d by d y plus B z d by d z and vector r is of course, i x plus j y plus k z so this is equal to d by d x of this quantity gives me simply i, because only d by d x of x is important. So, that gives me i times b x similarly, d by d y gives me b y so that is j times b y and of course, k times B z, which is nothing but vector B itself. Del dot of r divergence of r is equal to 3.

So, that gives me 3 B. this is the minus B. So, that gives me 2 B and of course, there is a factor of 2 here. So, that tells me that del cross B cross r by 2 is the indeed the vector B, as it what to be.

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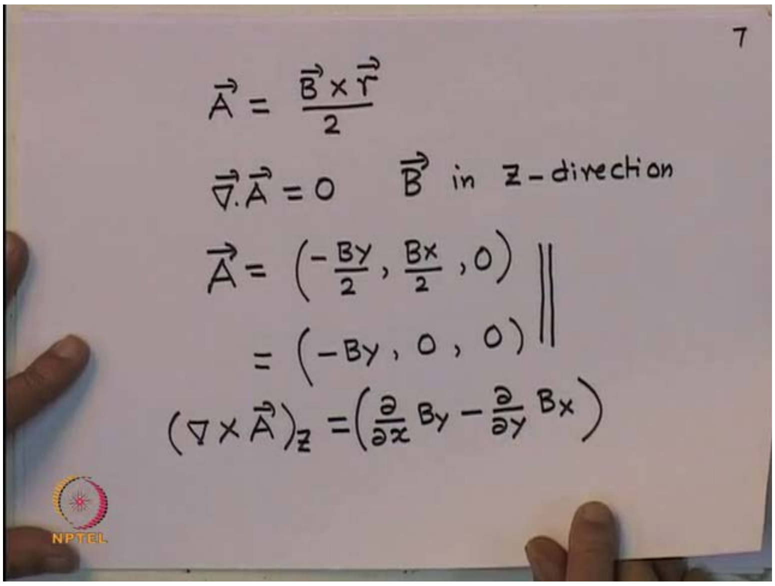

So, this tells me that an expression for the vector potential corresponding to a constant magnetic field is given by B cross r by 2. This is a rather important expression to remember. Now, let us let us look at a Coulomb gauge that del dot of A equal to 0. So, in Coulomb gauge del dot of A is equal to 0. Suppose B is in z direction. Let me take B in z direction. You can immediately see I can write the vector A which is B cross r, so since it is B cross r components now, you can check that this is given by one of the possible expression is for example, minus B y by 2 B x by 2 comma 0, that is the x component of the vector potential is proportional to y, y component is proportional x.

You can check that del dot of A is equal to 0 and del cross of a will give you vector B 1 when calculated. But this is not the only expression possible. As we had seen for instance, you could do this. You could be, B y 0. 0 remember we are only interested in del cross A s z component because I know that I wanted the magnetic field to be in the z direction. So, this is nothing but d by d x of B y minus d by d y of B x. So, take the first expression d by d x of B y b y is B x by 2. So, that gives me B by 2, d by d y of B x, B x is minus d y by 2.

So, that gives me another B by 2 so that is B. Alternatively I have a minus B times y, this is 0 and minus minus plus d by d y of that is B. So, either of these expressions are many other possible expressions can be done, can be shown to correspond to the constant magnetic field. This is as I pointed out this a rather important relations. Let me take vector potential corresponding to a current sheet. We know that if you have a current sheet you, can basically it gives me linear current because there is no cross section actually. You take a cross section put it to be equal to let it go to 0. So, what I get is a linear current density, which is measured in ampere per meter.

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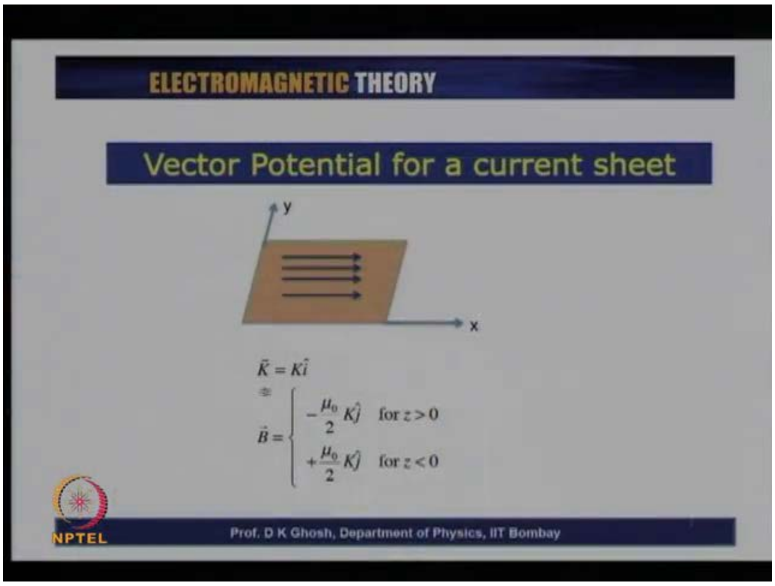

So, I have this linear current density as k times i. The magnetic field corresponding to such a current density can be easily computed using the Ampere’s law. So, that gives you that the magnetic field is constant both above and below it points as you can see because I am if I point my thumb along the direction of the current that direction, in which the my fingers curl that gives me the direction of the magnetic field. So, in this particular case that what I notice is this, that if this is my x direction and this is than the minus y direction.

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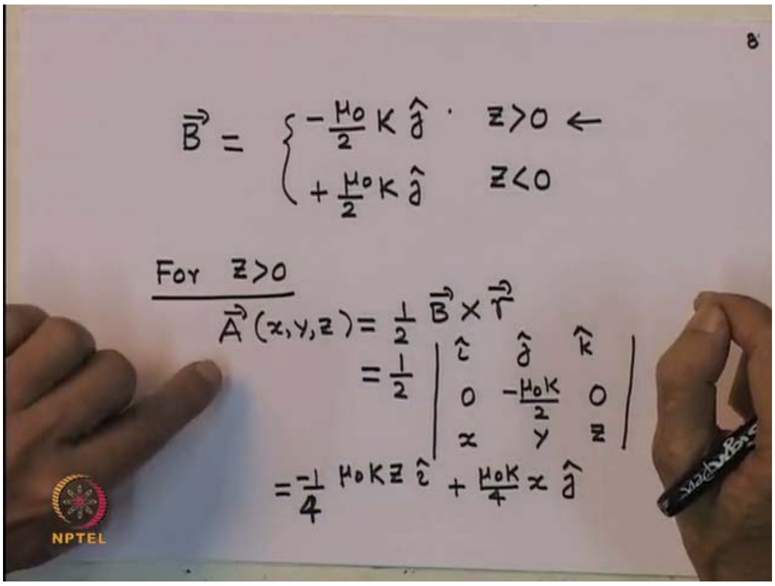

So, for the expression for the magnetic field for a current sheet is given by minus mu 0 by 2 linear current k along with z j direction is plus mu 0 by 2 k along the j direction. This is above the plane z greater than 0 and this is below the plane for z less than 0. Now, I need to calculate the magnetic vector potential corresponding to this. So, let let me do one of them only. So, let me just calculate it for z greater than 0. Now, notice this field is constant and just know we have seen that the vector.

Potential corresponding to a constant is given by half B cross r. Therefore, A is equal to A above x y z is given by half of B cross r. Since, I have cross product I write it in terms of a determinant which is i j k. I need components of B in the next row, components of B is only along the y direction. So, it is 0 minus mu 0 k by 2 0 and of course, component of r which are x y and z. This is rather easy to calculate, this is equal to half i times you can see minus z mu 0 z k by 2. So, it is mu 0 k by 2 which makes this 2 into 1 by 4 mu 0 k this times z times i and the the there is no j components, but there is a k component, that is is equal to plus this is a minus sign, there by determinant plus mu 0 k by 4, this times an x, this time an x and a j.

You can easily check that corresponding to z less than 0, the expression will be all most identical with these minus sign become a plus and this plus sign becoming minus. Therefore, one of the things that you notice is the following, that if you are looking at any component normal or the tendencial component of A, that is actually continuous across the boundary, because both these expressions they are continuous across the boundary.

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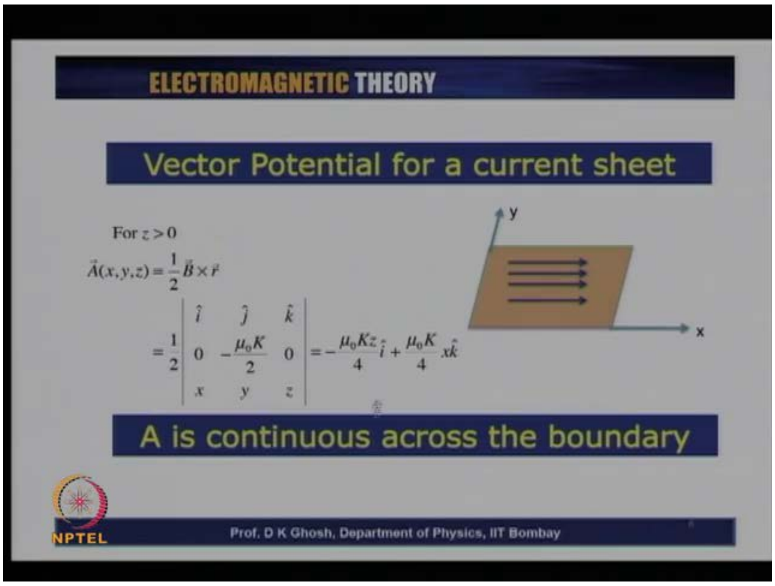

My next example is going to be a circular current loop, is already tricky, but is also lot more important, because of its utility.

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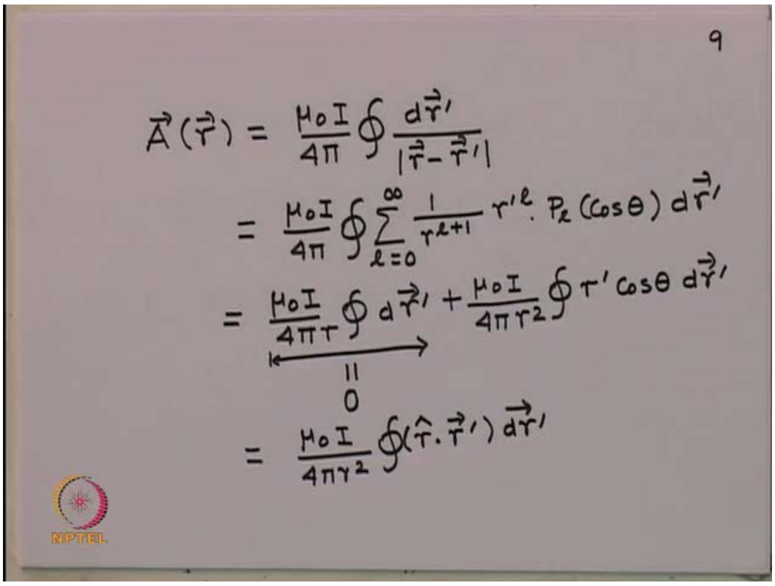

So, let us return back to the expression of the vector potential at A point r, at a position r which is given by mu 0 i by 4 pi integral of d r prime. Remember according to our usual convention, the primed quantities are the source that is where the current is flowing.

Unprimed quantity namely r is the position at which I am calculating the vector potential or magnetic field or whatever you have. So, I have a 1 over r minus r prime. Now, what I am going to do is to assume, that my current distribution has a smaller dimension compared to the distance where I am trying to calculate the magnetic field or the vector potential. So, that I will assume this r is greater than r prime.

So, what I will do is while doing electrostatics we had seen 1 over r minus r prime had an expansion, which is mu 0 i by 4 pi. Loop integral remains since, r is large I take this as sum over l is equal to 0 to infinity. I will pull out 1 over r to the power l plus 1 and when I have r prime raised to the power l and p l cos theta. This expansion in associated Legendre polynomial is something which we had done earlier. Now, what I am going to do is this, that let me retain only some low lowest powers of this. So, for example, I can write retain l is equal to 0 term first, mu 0 by 4 pi. If I take l is equal to 0, this is r prime raised to 0 is 1 and you have got an 1 over r there.

So, let it come out and a loop integral of d r prime. I need a d r prime back here. The second term is mu 0 i by 4 pi l is equal to 1 is r square loop integral of r prime to the power l, which is just r prime p l cos theta or p 1 cos theta is just is equal to r cos r prime cos theta and d r prime. Notice this term is integral of a vector in a closed loop. So, it is equal to 0. So, the first leading term that we need to calculate is this term which is mu 0 i by 4 pi r square, integral of remember that theta is the angle between r prime vector and r vector, but I have just r prime cos theta there. So, what I will do is to write this as the angle between unit vector r with r prime vector and d r prime. So, this is the formal expression and I need to calculate this.

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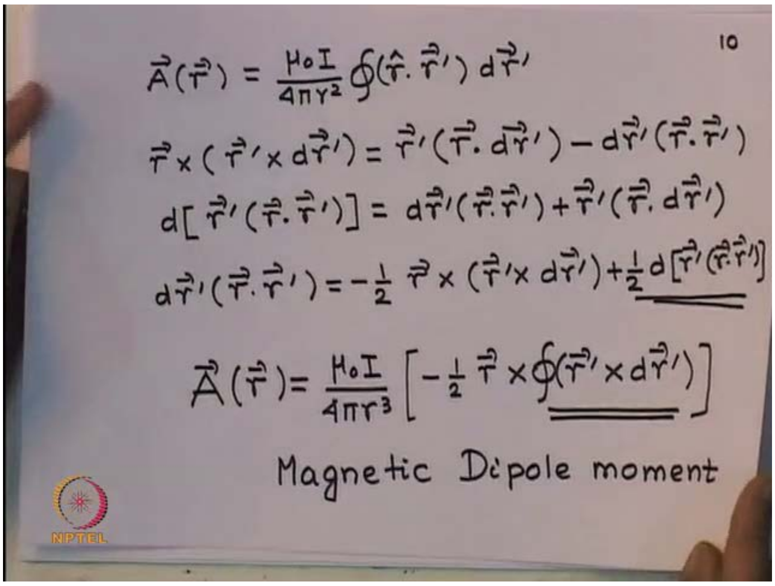

So, let me write it again. A little cumbersome mathematics, but on the other fairly straight forward mu 0 i by 4 pi r square loop integral of r dot r prime d cube d r prime. Now, what I am going to do is this. I am going to express this loop integral in a particular fashion and this particular fashion is to express it as a subjective measures or difference of two quantities. One of them is a perfect differential and I know it is a loop integral of a perfect differential it is is equal to 0 because it just returns back. As you know the finite integrals only depend upon the end points, so to do that I notice this.

If I look at r cross r cross r prime cross d r prime, I will alert one thing here, that you see this is a d r prime vector in other words this is a change in the r prime. How, that is that is a small change in the vector r prime. Now, this you use the standard B A dot C minus C A dot B which is the standard expansion for A cross B cross C.

So, that is is equal to r prime vector multiplied by A dot C which is r dotted with d r prime minus d r prime vector and r dotted with r prime. Now, I know that if I take a differential d of r prime times r dotted with r prime, I would get remember the vector r is a fixed vector. So, what I am going to get is d r prime times r dotted with r prime plus r prime times r dotted with d r prime because the expansion the differentiation is only with on r prime. Now, if you combine these two expressions in other words, I am going to replace this r prime r dotted with d r prime through this this minus, that. So, what you get is that d r prime times r dotted with r prime is equal to minus a half. You check one of the term comes twice and that gives a factor of 2 r cross r prime cross d r prime plus half of this differential r prime r dotted with r prime.

Now, remember that I have to put this inside that integral and this that will make this term by this. So, A of r for the circular current loop to the order of expansion that we have been making that is retain the l is equal to one term only is mu 0 i by 4 pi. Now, remember that I had a unit vector there. Now, I could write it as by bringing in another r in the denominator as vector r and then I can use this expression. So, I get 4 pi r cube times minus a half r cross r is a constant vector. So, it comes out and the loop integral is simply r prime cross d r prime.

So, this is this is an expression for the vector potential, but but notice what is this. This quantity here along with the factor half is nothing but an area it is the area of the loop circular loop. Now, if this is the area of the circular loop that multiplied with that current is in the direction of that vector area is what we define as the magnetic dipole movement. So, the magnetic dipole movement of a closed circuit is given by the current multiplied by the area vector. That is given by this expression there half of this is the area and therefore, if you look at this expression.

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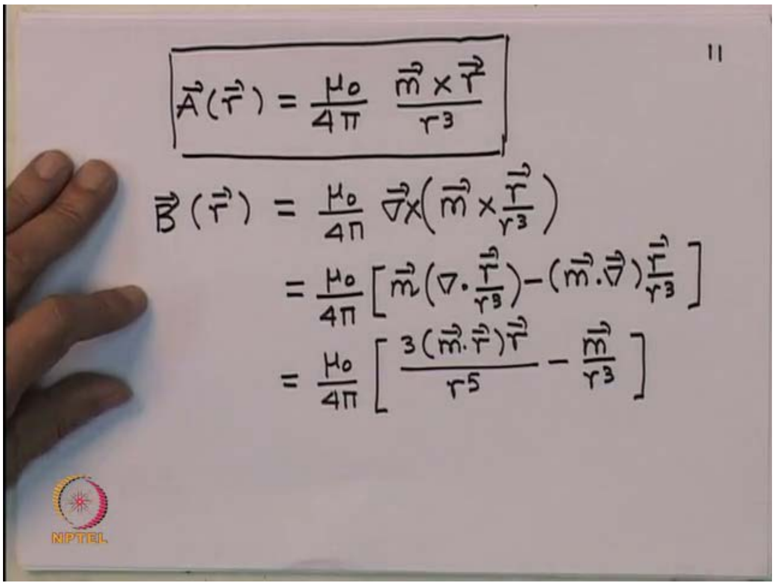

You will get a of r is given by mu 0 by 4 pi, i is taken care of in the definition of the magnetic movement and I get m cross r that minus sign I have taken care of by reversing the sign of the order of the cross product, I get m cross r divided by r cube. Now, this is the general expression for the vector potential for a small current loop. Small I say because I have done expansion in spherical harmonics and retained the first order term only. Now, I can now calculate from this, the corresponding expression for the magnetic field.

Now, recall that earlier also we had done, we had calculated the magnetic field due to a circular current distribution, but we were only able to calculate the magnetic field on the axis of the circular coil. So, what you are going to do is this since to the order of approximation which you have used, this is a general expression. So, B of r due to the magnetic dipole is given by mu 0 by 4 pi del cross m cross, I will take the second vector as r by r cube sorry del cross, that I will use the following remember that vector m does not depend upon position r. Therefore, there is no differentiation etcetera comes for the vector m. But this sort of acts like a scalar it is not really a scalar, but it is a vector. But it acts like a multiplying factor for the del operator.

So, this gives me m times del dot of r by r cube minus m dot del r by r cube. I leave to you as an exercise because we are several times done this calculation, divergence of a scalar times a vector, this you can easily calculate. This gradient calculation something which I did some time back, that it is to be what is meant by gradient of a vector that is I have to take component wise. and do it. You can show that this gives me mu 0 by 4 pi 3 times m dot r r divided by r to the power 5. r to the power 5 because there is a del dot of 1 over r cube being taken and there is a notice there are 2 r there that is a 1 over r cube as again as before and that is minus m by r cube.

This is the rather well known expression. This is the coordinate free form for the expression of the magnetic vector potential, magnetic field due to a current loop. Having done this, I next go over to a discussion of the boundary conditions which are applicable both for the magnetic field and for the magnetic vector potential. Remember we did similar things, with the electric field. But this time I have slightly different equation therefore, I expect some changes.

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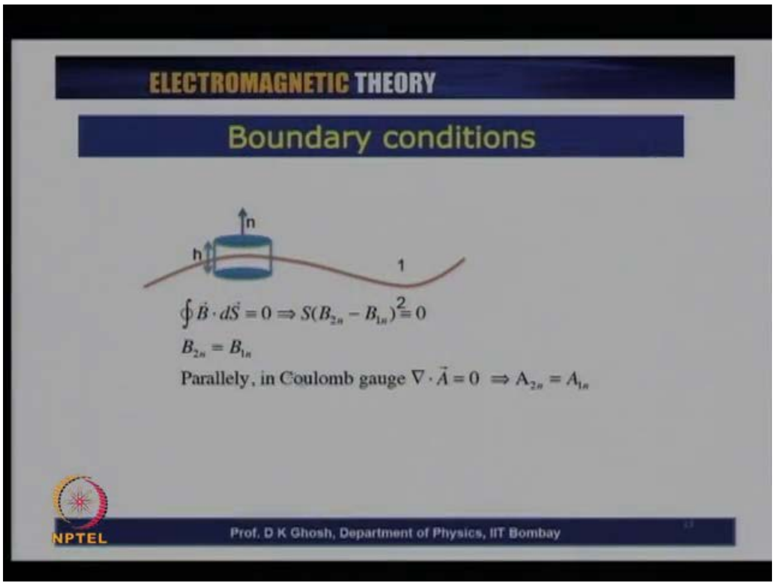

So, let us look at the condition for the boundary conditions on the magnetic field itself. So, this is inter face arbitrary interface between the medium 1 and the medium 2. Now, I know that del dot B equal to 0.

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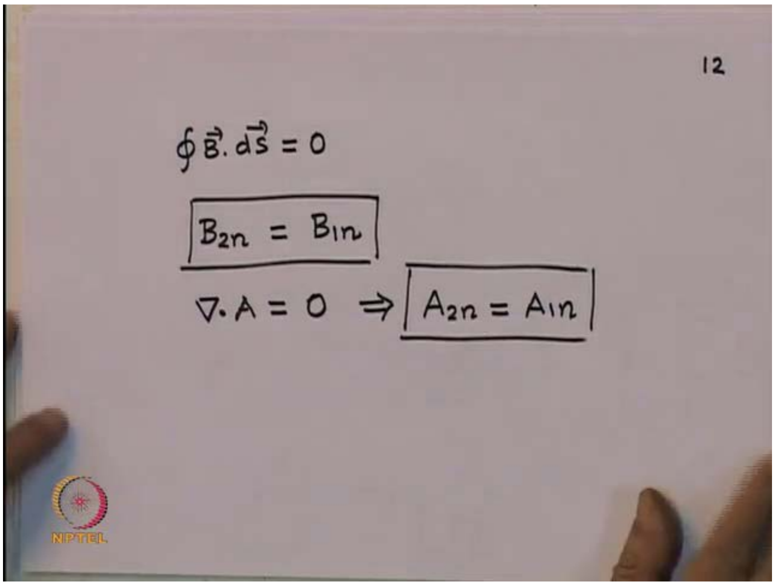

Alternatively, I have B dot d s equal to 0 over any close surface. Now, what we doing is this, that if I have, if I have this figure, I take a Gaussian pill box. I take a Gaussian pill box of height h which I will take it as negligible. This is negligible means this is small compare to let us say the radius of the pill box, the outward normal on this surface is that way and on that surface is this way. So, the B dot d s there is no contribution from the sides because I have taken h to go to 0. If I take capital S to be area of these end caps, then B dot this 2 is for the medium and not to be taken as a square.

So, I get area times B 2 n and minus B 1 n minus because the direction of the normal is, I have to take one direction for the normal, but here the outward normal would have been that way. So, B 2 n minus B 1 n is equal to 0. Cancelling out the S I have B 2 n is equal to B 1 n. So, let me write it down. The normal component of the magnetic field is continuous. Now, remember that in Coulomb gauge I had del dot of A is equal to 0, which is identical to this expression there. That integral B dot d s equal to 0. I can have integral A dot d s equal to 0, using the divergence theorem. So, so just the same way, I can write down A 2 n is equal to A 1 n. In other words the normal component of the magnetic field as well as the normal component of the vector potential are continues across a boundary. Now, the tangential component possess a little more tricky situation, but let me go through this.

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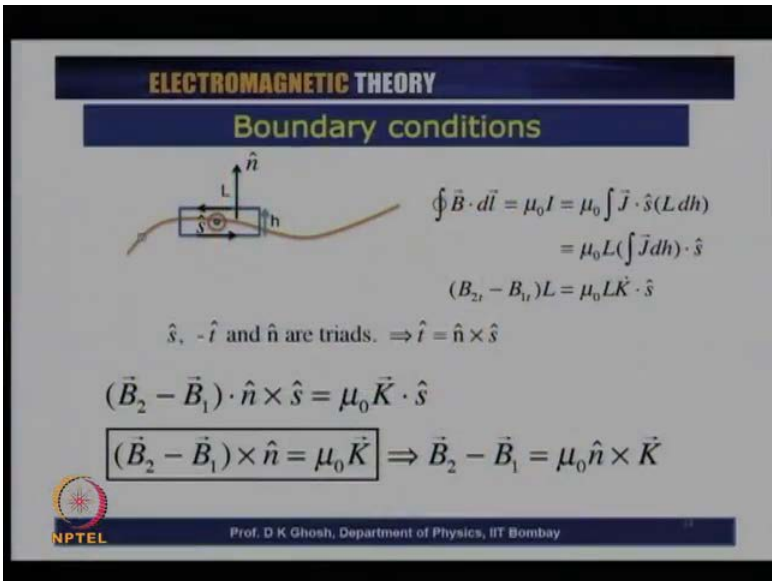

What I do is, I have this boundary and I take a rectangular loop of height h which will be taken to be small compare to these distances. What I will do is, to calculate what is integral of B dot d l. We know that integral of B dot d l is mu 0 i. The current i is J dot s and the surface area is l d h. So, what I will do is this, that you see in this picture I have shown a direction which is S, unit vector S which is which corresponds to this loop. That is, if I take the loop in the direction of my thumb, then the direction in which the finger points for the thumb points is the direction of this normal.

This is not to be confused with the direction of the normal to the surface in any case. Look at this that, then I get B 2 t minus B 1 t into L is equal to mu 0 L K dot s. Now, what this is telling us, which will do repeat next time is, that the tangential component of the magnetic field has a discontinuity, has a discontinuity when there is a current, surface current across the two media. This is very similar to what we did earlier, for the electric field case and we found that the normal component of the electric field had a discontinuity. When I had a charge density on the surface, we will return back to these boundary conditions in the next lecture.