Electromagnetic Theory
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Module - 3
Magnetostatics
Lecture - 24
(a) Force between the Current Loops
(b) Magnetic Vector Potential

In the last lecture we had defined the magnetic field and we had seen that a source of the magnetic field is steady current and this is different from the way we produced electric field and that was by static charges. And we found that the magnetic field can exert a side wise force on a moving charge and this is called the Lawrence force. We had also done some calculation using the fundamental laws which determine how to calculate the magnetic field namely the Biot-Savart's law and the Ampere's law.

So, what we will do today is to use these laws to find out how much is the force exerted by one current carrying circuit on another and later we will define a potential corresponding to the magnetic field and we will also point out what the difference, what differences are there between the electric potential and the magnetic potential.

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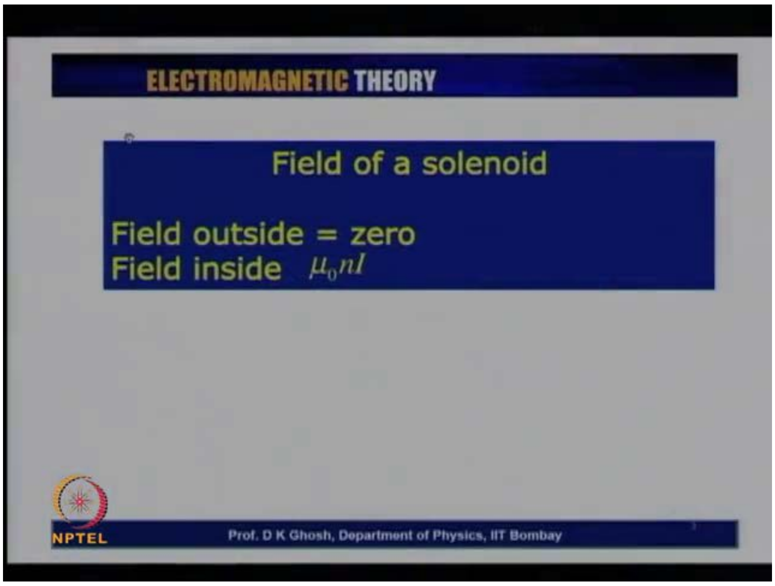

But before we do that let us quickly talk about two special cases. I will not be working this out, but they are rather trivial because most of them are done in school. The field due to a solenoid and we had seen this is rather easy to work out if you use the Ampere's law. The field inside in a solenoid if you neglect it is edge effect namely if you assume that the solenoid is infinitely long, is uniform and is given by mu 0 n I. It, its direction is along the axis of the cylinder and if you curl your hand like this and supposing this is the way the current is being flowing then of course, the direction in which you are finger points that thumb points that gives you the direction of the magnetic field.

So, the, notice this, this I will come back to this thing little later that the magnetic field inside a solenoid is uniform, but magnetic field outside the solenoid is 0. You could try to work this out the Ampere's law gets it in one line, but you could try to work this out using the Biot-Savart's law that is use, take a small ring of let us say width d z at a position z and find out what is the magnetic field due to this current element at a point on the axis which will be given by the same type of expression as we had calculated for, calculate for getting the expression for the magnetic field on the axis of a circular current carrying coil.

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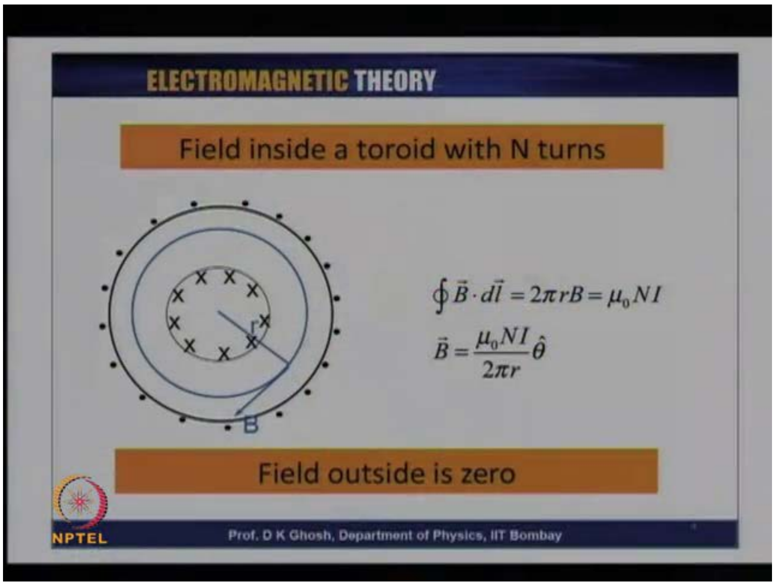

The other important geometry that one talks about is a toroid and as you know that a toroid looks very much like a donut and therefore, what you find here is current switch in this particular case is going in through this edge and is coming out from there and you can, what you can do is this that supposing you took a circle of radius let us say r by symmetry the magnitude of the magnetic field everywhere will be the same and the direction will be tangent to the circle. So, as a result the integral of B dot d l will turn out to be 2 pi r times B this is purely symmetry because d l's direction is tangential and the magnetic field is also parallel to the tangent.

So, this quantity is equal to mu 0 times the amount of current that is enclosed and that is obviously since if you assume that there are N terms of the loop the current loops then I get mu 0 times N times. I remember this N is the total number of turns that are there because each turn is enclosed. This is not quite the same expression as you get for the solenoid where you get an expression like mu 0 times small n which is the number of turns per unit length of the solenoid times the current. So, in this case the magnetic field is not uniform, it is given by mu 0 times the total number of turns times I divided by 2 pi r and its direction is of course, the azimuthal direction.

Now, once again the field outside will be equal to 0 because if you take a circular loop outside the toroid then of course, for every current loop that is going in there is a current loop which is going out. Therefore, the net current cancels out. Now, these expressions are going to be used very frequently and it is good to remember this.

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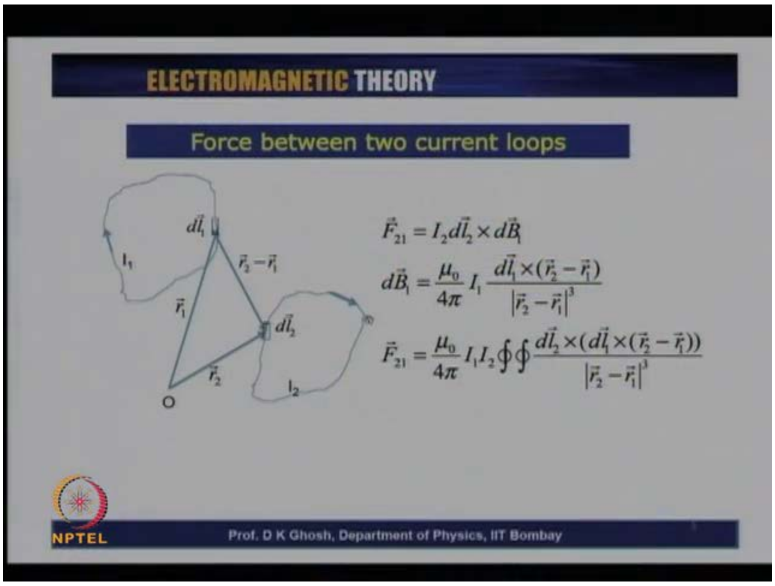

Now, let me now let me come to a slightly different aspect. We have seen that a magnetic field exerts force on a moving charge. Now, if you consider two circuits each carrying a steady current, the farmer carrying a current I 1 and this one carrying a current

I 2. Now, let us look at the magnetic field produced by this circuit one. Now, you already know how to calculate this because I have given you the Biot-Savart's law.

Now, this is an arbitrary circuit. Now, since this is an arbitrary circuit what we do is if I am calculating the magnetic field at a point. I have in this picture suppose I am interested in calculating the field due to this circuit which I call as circuit one it is carrying a current I 1 then with respect to this fixed origin I take a current element which is d l 1 and that is at the position r 1 with respect to this origin. And I am interested in calculating how much is the magnetic field at this position which is at a location r 2 vector with respect to the origin O.

So, you that from this current element this point P which I will come to this circuit later, but let us just talk about this point. This point P is at a position r 2 minus r 1. Now, what I have to do is this that in order to calculate the total magnetic field at this point due to this complete circuit I will write down F, well the notation I am coming to little later, the the force the magnetic field here will be given by the Biot-Savart's law.

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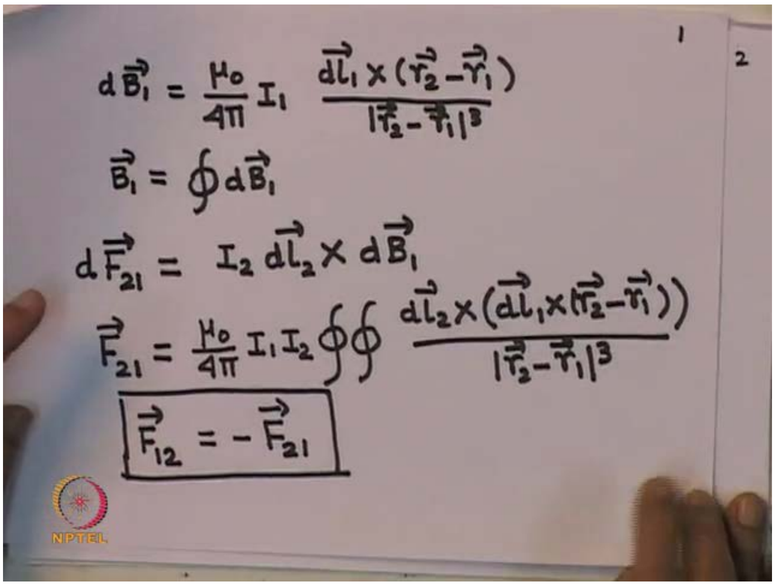

So, we that for instance the magnetic field due to the current element d l 1 is given by d B 1 let us say that is equal to mu 0 by 4 pi I 1 is the current and times d l 1 cross r 2 minus r 1 divided by r 2 minus r 1 cube. Now, in order to calculate the total magnetic field at the point P what I need to do is to find out what is B 1 at the point P which is simply the integral of this over the circuit, so, integral of d B 1.

Now, so, what we are trying to say is this that since this circuit creates a magnetic field at each point of the second circuit which is carrying a current I 2 and I know that the current is due to moving charges and we have learnt that a, if there is a magnetic field a charge will experiences a side wise force and that is the Lawrence force and as a result this current carrying conductor since there are moving charges in it will experience a force. So, what I do is this that if a take a current element here d l 2 then I know that the force.

Now, come to the notation force on circuit an element of circuit two due to an element of circuit one is given by I 2 cross I 2 times d l 2 cross d B 1. So, this is analogous to v cross d so d B 1 is the magnetic field due to the current element d l 1 and this is I d l 2 so this is acting on this. Now, what I need to do is this, what is my total? Let us call it d F 1 for convenience. Now, what is my total F 2 1? So, that total F 2 1 is the force exerted by the full B 1 on the full circuit therefore, I have one integral to find out the magnetic field, another integral over d l 2 because I am calculating force over everything.

So, this will be given by mu 0 by 4 pi I will get an I 1 from here and I 2 from there and a double integral I get d l 2 that is here cross now d l 1 cross, I will now substitute this which is r 2 minus r 1 divided by r 2 minus r 1 cube. Now, clearly this is a rather clumsy expression and only in cases of simple geometry you will be able to find out how much is it. But let us look at slightly different problem. So, this is in principle you can well if you know the current how it is distributed you know the position of the second circuit, in principle at least numerical it should be possible to calculate it.

Now, what we will do is this that we know that if a current, if a circuit one exerts a force on circuit two, circuit two since it is carrying a current that is also a source of the magnetic field which I will call as B 2 and this magnetic field will be exerting a force on the first circuit. Now, I expect that according to Newton’s third law action reaction principle that this force which I call as F 1 2, the notation is force on circuit one due to circuit two that must be simply equal and opposite to F 2 1. Now, this is of course, something which we expect.

Now, this is not clear from this expression that this is so. But we need to do a bit of an algebra to prove that this is so. Let us let us then write down so this was my expression for F 2 1 and what I will do is I will write down an expression purely by symmetry.

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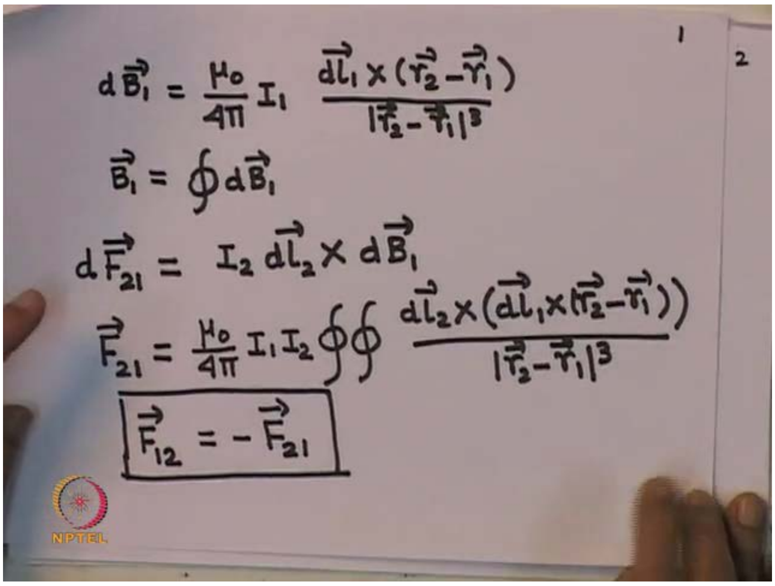

Because all that I need to do is to replace 1 interchange 1 and 2 so that I get mu 0 by 4 pi. I had I 1 into I 2 so I will still have I 2 into I 1 which is the same. I will have a double integral. Now, I had d l 2 cross d l 1 cross r 2 minus r 1. So, I will write this as d l 1 cross d l 2 cross r 1 minus r 2 instead of r 2 minus r 1 divided by since this is just a modulus, I do not really care how you write it. I could write it as r 1 minus r 2 whole cube or I could write r 2 minus r 1 modulus cube, does not matter here, this is the same.

Now, what I need to show is this is equal and opposite to the expression that I had for the previous one. So, let me then try to prove this. So, let me come back to the numerator, the quantity inside the integrand of the first expression. So, I have d l 2 cross d l 1 cross r 2 minus r 1 divided by r 2 minus r 1 cube. And I have already told you this denominator I do not need to change actually, but on the numerator I will use what is known as the a cross b cross c formula.

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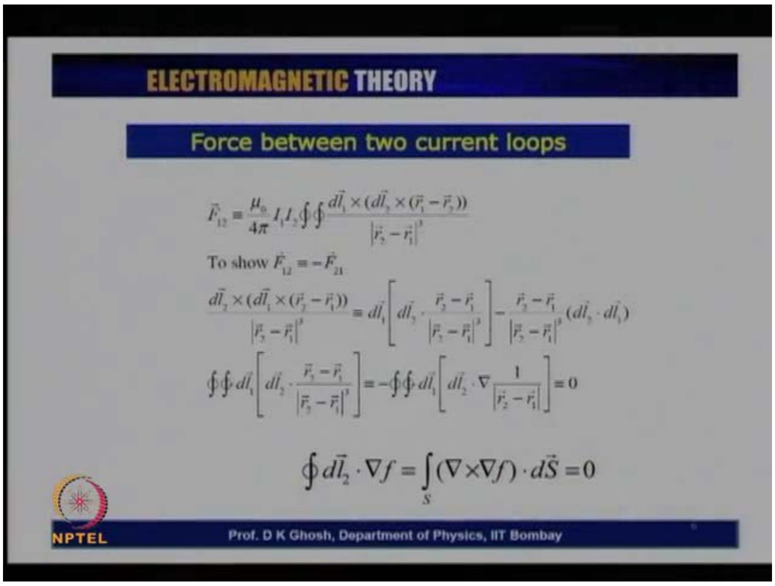

Now, what is a cross b cross c formula? The a cross b cross c formula as you know is b a dot c so I will write it as d l 1 times d l 2 dot r 2 minus r 1 by r 2 minus r 1 cube minus c which I have I have taken c to be r 2 minus r 1 by r 2 minus r 1 cube. So, I will write it as r 2 minus r 1 by r 2 minus r 1 cube and the other one is d l 2 dot d l 1. Now, notice this, the. So, this is my d l 2 cross this. Now, I do not have to do much to this one because you notice that this term is minus r 2 minus r 1 by r 2 minus r 1 cube which is same symmetric term because you d l 2 dot d l 1 is same as d l 1 dot d l 2 etcetera. And r 2 minus r 1 with a minus sign is same as r 1 minus r 2, but I need to do something about this term and you recall that there is an integral there.

So, therefore, I take double integral and I say d l 1 times d l 2 dotted with r 2 minus r 1 divided by r 2 minus r 1 cube. Now, notice this r 2 minus r 1 by r 2 minus r 1 cube is nothing but gradient of 1 over r 2 minus r 1, but because the gradient will give me a minus sign so I write this as minus integral double integral d l 1 times d l 2 dotted with grad 1 over r 2 minus r 1.

So, so basically what I have got is this. Now, I have now notice that I have done this integral so let me write it down, it becomes slightly clumsy. So, I have an integral to calculate which is d l 2 dotted with gradient of 1 over r 2 minus r 1. After I calculate this I will then do an integral over l 1 and then of course, pick up a minus sign, but you notice this that this quantity is a line integral of a gradient. Therefore, I can write this as by using the Stoke's theorem as a surface integral of the curl of the same quantity and of, as you know this is the integral is your d S.

As you know the curl of a gradient is always 0 so this expression is identically 0. So, that tells you that the F 1 2 and F 2 1 are oppositely directive. So, as I told you that this expression that I have given is rather complicated and unless you know the type of current distribution you have you will not be normally in a position to calculate the force due to one circuit over the other.

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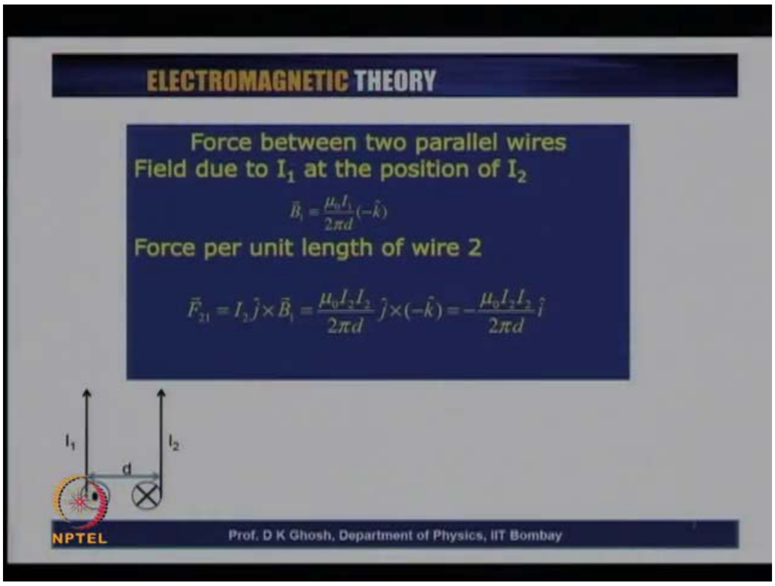

But let us just for illustration let us talk about a simple case. Supposing, I have two wires which are parallel.

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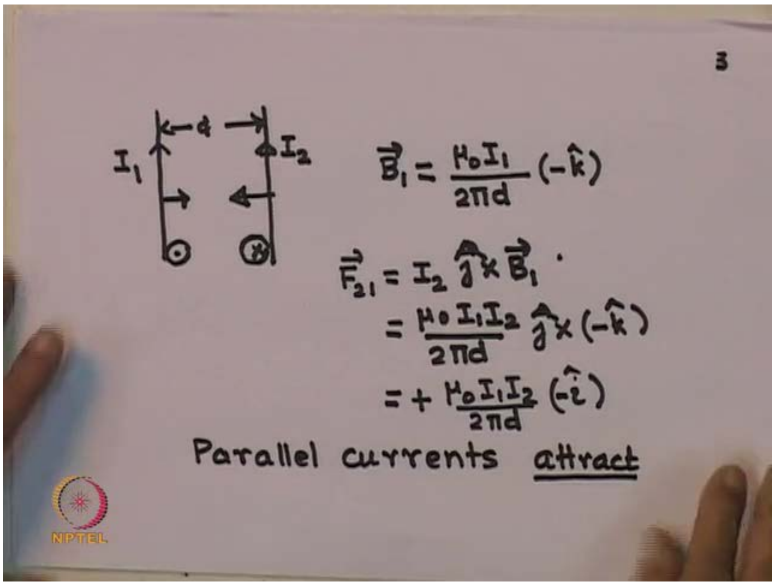

So, I have a I 1 carried by this wire and an I 2 two long wires I 2 carried by that one and these are let us separated by a distance d. Now, notice this that the there is a. Now, since this wire is to the right of the first wire, this wire which is carrying a current creates a magnetic field assuming that these two wires are on the plane of the paper; this creates a magnetic field which is in to the plane of the paper at the location of the straight wire number two. Parallelly the second wire creates a magnetic field at the location of the first one which is out of the plane of a paper.

Now, let us just calculate the write down the magnetic field due to this current so the magnetic field let us call it B 1 which is the magnetic field at the location of the second circuit, but created by the first current which I know is simply mu 0 I by 2 pi r is the general formula. So, I will write it is a mu 0 I 1 divided by 2 pi d and since we have seen that the direction of the magnetic field is in the minus k direction so I will write as a minus k. Now, what I want is this. How much force does this magnetic field exert on a unit length of this other wire?

So, that is again F 2 1 so that is equal to simply I times or I 2 times because its carrying this times j cross B and if I write down the magnetic field expression so I will write it as mu 0 I 1 I 2, this is actually B 1 I 1 will come from there divided by 2 pi d j is the direction in which the current is being carried. Therefore, this is j cross minus k. They are all unit vectors this is the unit vector j cross minus k.

So, I know that j cross k is I. Therefore, this is minus mu 0 I 1 I 2 by 2 pi d. Let me write it as along the minus I direction. So, you notice this that this wire is experiencing a force in this direction. In other words two wires carrying currents in the parallel direction attract each other, this if you did the same calculation here this will of course, be like that. So, parallel currents attract. Now, this of course, is different from the way you have learnt for similar charges. So, we have parallel currents attracting each other.

Now, before I go over to a discussion of potential since last time we introduced the magnetic field and I made some comments that magnetic field and electric field are different manifestations of the same physical process. Now, what you want to do is this. I want to sort of convince you that it is indeed so. I get some simple examples and I said that well suppose I have a let say a charge which is static and it it exerts a an electric force. Now, this will exert an electric force on another charge now, whether that charge is moving or not. On the other hand if I have a steady current such a steady current will not exert a force or if the charge, the test charge is static.

Now, what I want to tell you is this that I could always supposing I have a moving charge. Now, if I have a moving charge I know that there is a magnetic force on the moving charge. Now, suppose I went to the frame of reference of that charge itself. So, I have a static charge. Now, what happens? The force that I experience is real; however, now I will experience only an electric force, not a magnetic force.

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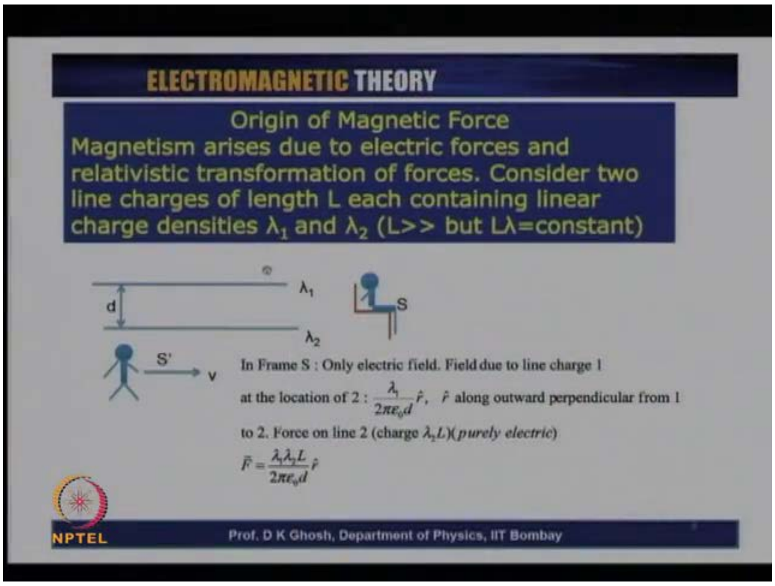

So, in order, this is what I am trying to show you here, in this picture. So, let me take two straight wires parallel wires, the, these are not carrying any current, these are just have a charge density lambda 1 and lambda 2 and I have two observers here. So, one of the observers is sitting in the laboratory. Now, this I will call as the frame S. You will do this in more detail in a relativity course, but let us look at what we are trying to say. So, in this frame which is my frame S, there is only an electric force between these two charges because everything is static here. And this electric force is what I could do is calculate the electric field due to this charge distribution which I know is lambda 1 divided by 2 pi epsilon 0 d and the direction of this magnetic field, this electric field is outward radial.

Assuming of course, the charge lambda 1 is positive and vector r is in a perpendicular direction. Now, as a result the, if I now take a length L of the second wire which also has now a charge lambda 2 times L because lambda 2 is the charge density of the second wire. The, this second wire will experience a force, a length L of the second wire will experience a force which is this field multiplied by the charge that is there. So, that is lambda 1 lambda 2 L divided by 2 pi epsilon 0 d r. So, let me just write it down here.

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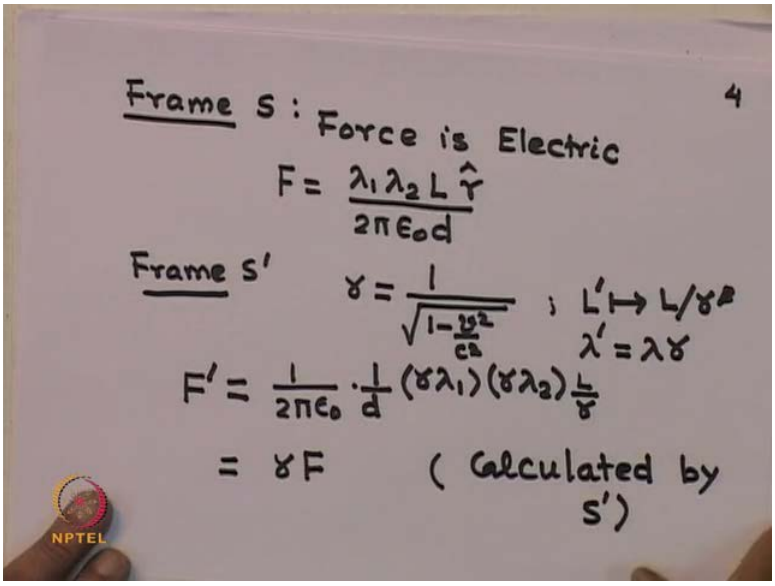

So, we will say frame S force between the two wires is line charges is only electric and the force is given by lambda 1 lambda 2 L. Remember, L is the length that I have taken as the second wire, 2 pi epsilon 0 d and according to this it is in the radial direction from the first wire. Now, let us try to look at the same thing from the point of view of a second observer who is moving along the positive x direction with a velocity v and I call it S prime. Let us do some calculation in the frame S prime.

Now, notice in the frame S prime because the observer is moving with a velocity v along the x direction all lengths along that direction gets contracted. So, as a result L and the factor by which it gets contracted is usually written as gamma which is equal to 1, 1 over 1 minus by v square c square. Now, since v square by c square is a quantity which is less than 1, this gamma is a greater than 1 and as a result every length every length along that direction gets contracted by this factors.

So, L will become L by L by gamma. So, L prime let us call it and that will become L by gamma. So, as a result the electric force that is calculated by the observer S prime is 1 over 2 pi epsilon 0. Now, remember that the distance d is along the transverse direction and since my observer is moving along the x direction there is no contraction or expansion of this length. And so only thing that I have is the contraction of the length, but contraction of the length gives rise to an increased charge density because charge density is nothing but the amount of charge per unit length.

So, as a result my charge density which was lambda will now become lambda prime which will be nothing but lambda times gamma. So, this will be gamma lambda 1 gamma lambda 2 instead of lambda 1 and lambda 2 I have got gamma lambda 1 lambda 2. The length L of the second wire which second charge line charge whose length was L which we have considered is going to become L by gamma. So, this quantity you notice is one of the gammas will go away and if you recall my original expression was this so this is nothing but gamma times F. Now, this is the calculated electric force, please understand this.

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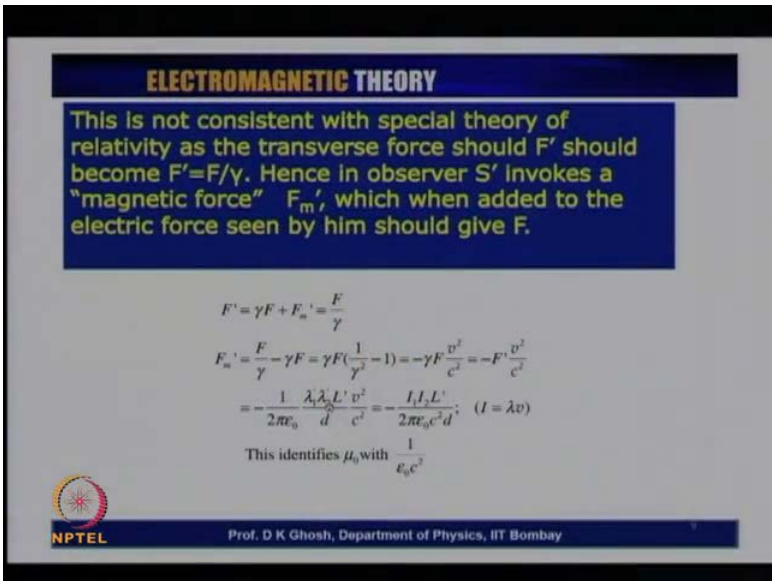

Now, however this is not consistent. This is not consistent with special theory of relativity because according to special theory of relativity the force the transverse force F prime that is actually there should become F by gamma. If there is a there is a force F transverse for F in frame S, the frame S prime should experience F by gamma and indeed the force that is that our observer S prime will measure should be F by gamma, but he calculates gamma F. So, what he does is this that he realizes that in addition to the electric force that he has calculated because of the fact that in his frame of reference there are moving charges.

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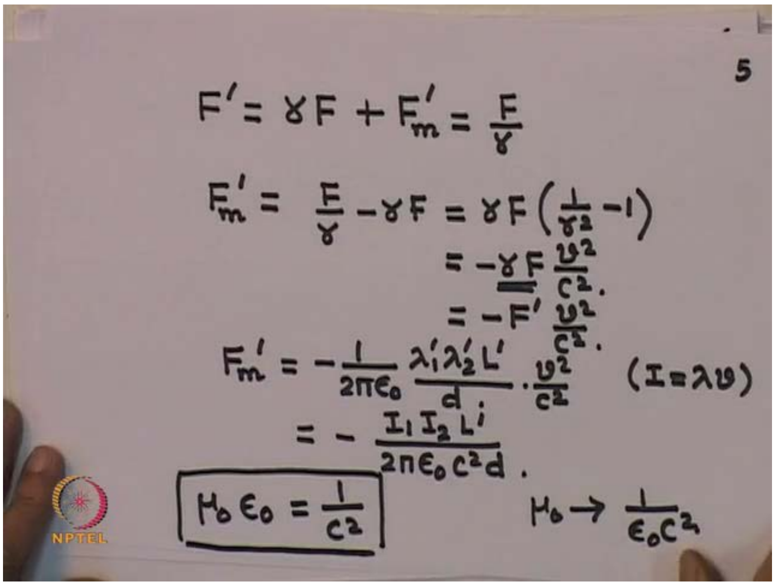

He invokes the presence of a magnetic force and he says that well the force F prime in his frame of reference is gamma times F which he as calculated be the electric force plus a term which you will call as F m prime which is the magnetic force and this plus this together in order that it is consistent with the special theory of relativity, this should be F divided by gamma. So, what he does is this, he will invoke a magnetic force F m prime in his frame of reference which is F by gamma minus gamma F.

Let us take gamma F to be common so that I get 1 over gamma square minus 1 and you remember that gamma was 1 over square root of 1 minus v square by c square square root. So, this is nothing but minus gamma F into v square by c square. And this quantity which is since gamma times F is the F prime that he calculates so this is equal to minus F prime v square by c square. So, this is the force that he calculates and therefore, F m prime according to him should be 1 over 2 pi epsilon 0 lambda 1 prime lambda 2 prime L prime divided by d and of course, v square by c square. If you recall that the current is nothing but lambda times v.

Therefore, this expression becomes minus I 1 I 2 that takes care of the two v square L prime divided by 2 pi epsilon 0 c square times d. Now, notice this that we had already derived what is the magnetic force between two parallel wires carrying current I 1 and I 2 and we had seen this expression should have been I 1 I 2 times length divided by mu 0 d 2 pi d. Therefore, this factor mu 0 can be identified with epsilon 0, 1 over epsilon 0 c square. So, mu 0 epsilon 0 becoming is equal to 1 over c square is actually one of the fundamental relations of electromagnetism.

So, we have seen that the forces that you calculate will be consistent with theory of relativity if the moving observer also invokes a magnetic force. This was essentially to point out that electricity and magnetism are different manifestations of the same physical phenomena. Sometimes you find this, sometimes you find that, sometimes you find both. So, it has got to do with what frame of reference you are looking for. With this let me now go over through a, the discussion of magnetic potential.

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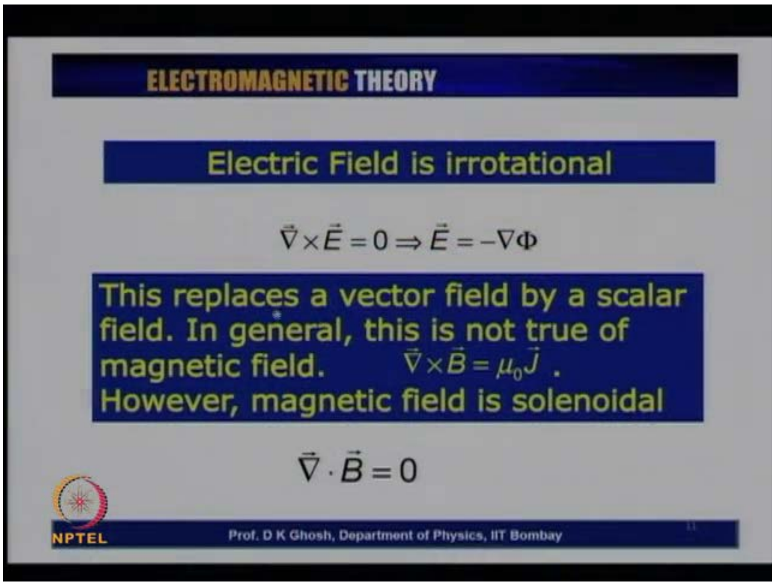

Now remember that the electric field was irrotational as a result the del cross of E was equal to 0, that enabled us to define a scalar potential which is which was minus gradient of phi. Now, the advantage of this was that the electric field which is the vector field was replaced by a scalar field and that is of course, a great advantage because it is much easier to deal with a scalar than with a vector. So, if I had a charge distribution I could calculate by using super position principle what is the total potential and then differentiate it. However, this is not really true for the magnetic field and that is because the del cross of B is not equal to 0, but is equal to mu 0 times J. However, there is one point here that the magnetic field is solenoidal namely del dot of B is equal to 0. So, let us write down those relationship.

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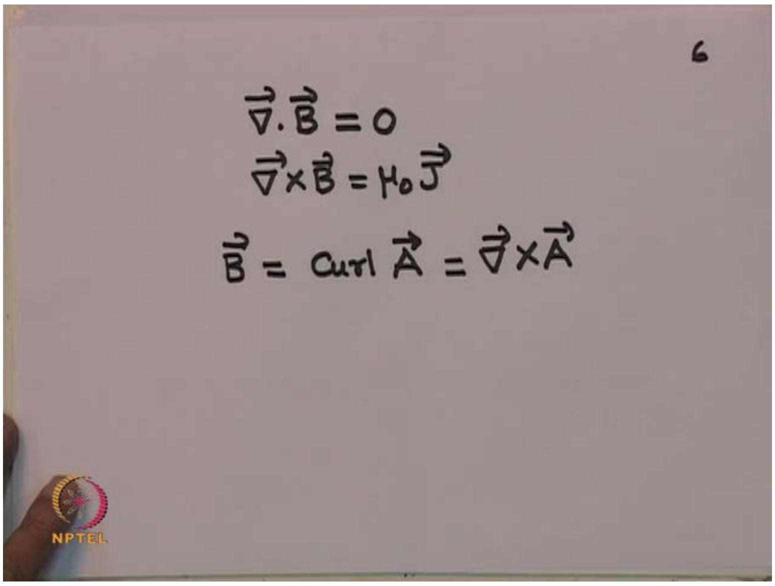

That del dot B equal to 0 this is of course, we have seen is nothing but, Gauss's law and del cross B according to Ampere's law was equal to mu 0 times the current density J. Now, the question is this that the can I get a potential at all? The answer is yes. The, if you look at the following fact that since divergence of B is equal to 0 I should be able to write B as curl of something because di curl is always 0. So, what I do is this. I define B as equal to curl or rather I define a vector potential A by saying that curl of A which is same as del cross of A gives me the magnetic field.

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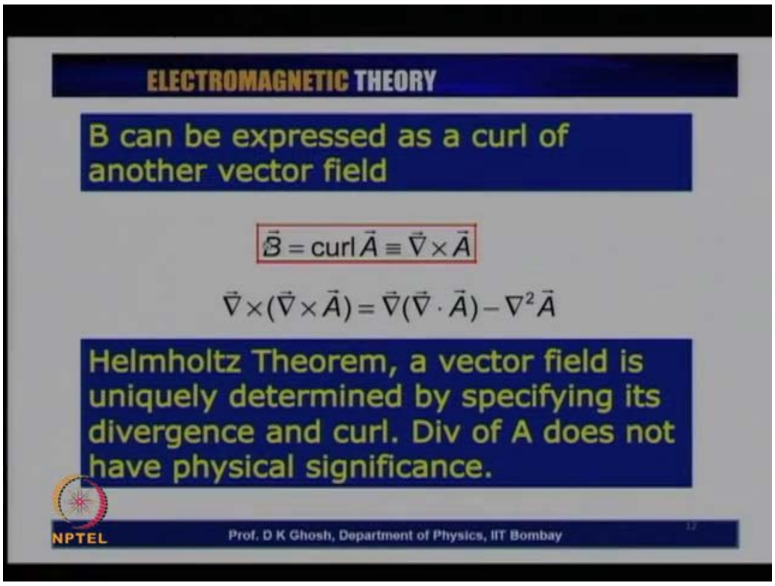

Now, notice that del dot of B is equal to 0 del cross of B you recall our Helmholtz theorem which told us that any vector field in order that it is uniquely specified I can specify it by providing both its curl and its divergence, the divergence. So, let us look at what we know.

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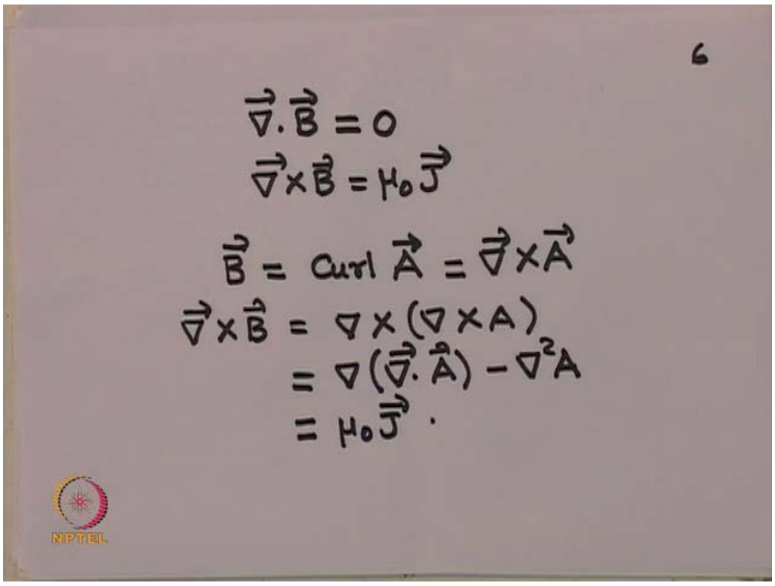

Now, since curl of B curl of A is B. Now let us look at what will it give me for let us say curl of B. So, curl of B is same as curl of curl of A that is equal to del of del dot of A minus del square A and according to my Ampere's law this quantity should be equal to mu 0 J. So, notice this that curl of A has a physical significance because curl of A is a magnetic field, but divergence of A does not have a magnetic, a physical significance.

So, as a result what I can do is to choose the divergence of that corresponding to whatever curl of A is that is the given magnetic field. So, I could choose it as per my convenience so that my work becomes simple. So, one of the ways to do it is to choose this curl divergence of A to B equal to 0.

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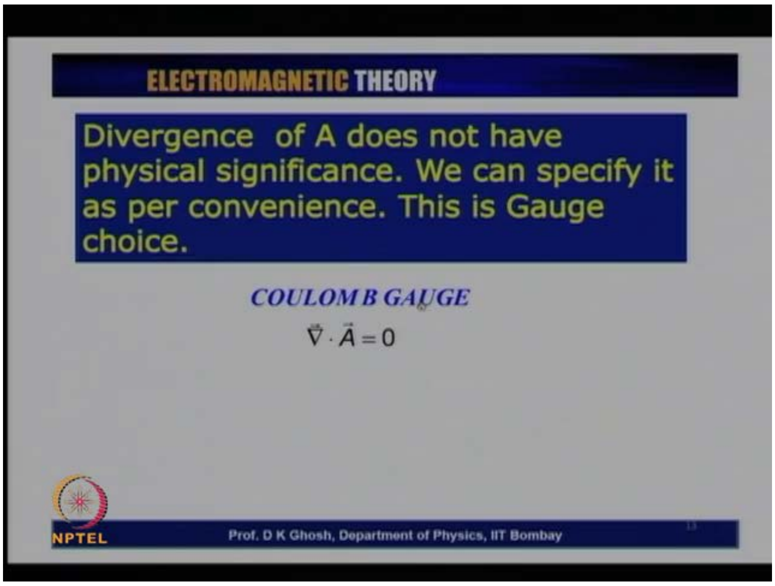

Since, it does not have a physical significance we just specify it as per our convenience and this is known as a Gauge choice. This Gauge that I talked about that choose del dot of A equal to 0 which tells me that is that is called a Coulomb Gauge.

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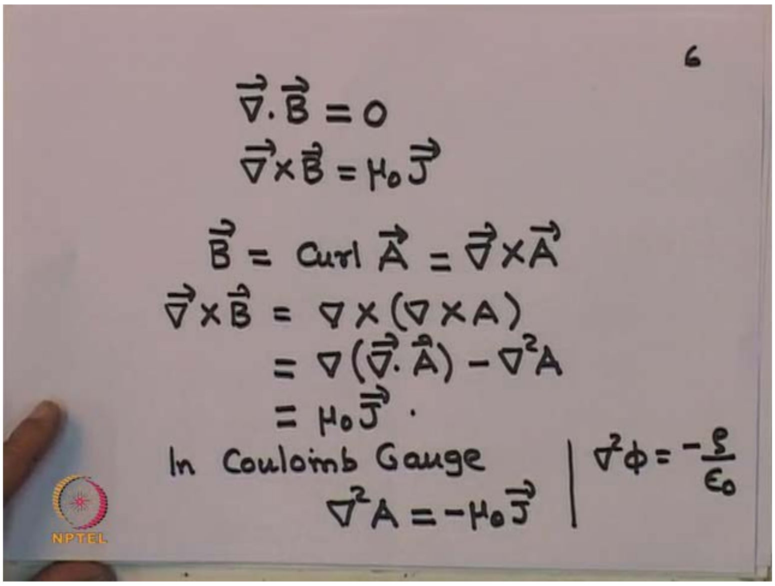

So, in Coulomb Gauge I must have del square of A is equal to minus mu 0 J. Therefore, the vector potential A in this Gauge satisfies a Poisson's equation. I know the current distribution and del square of A equal to something is nothing but giving me the, this is nothing but a Poisson's equation. Recall this is to be compared with del square of phi is equal to minus rho by epsilon 0 parallel to that. Of course, this is actually three equations.

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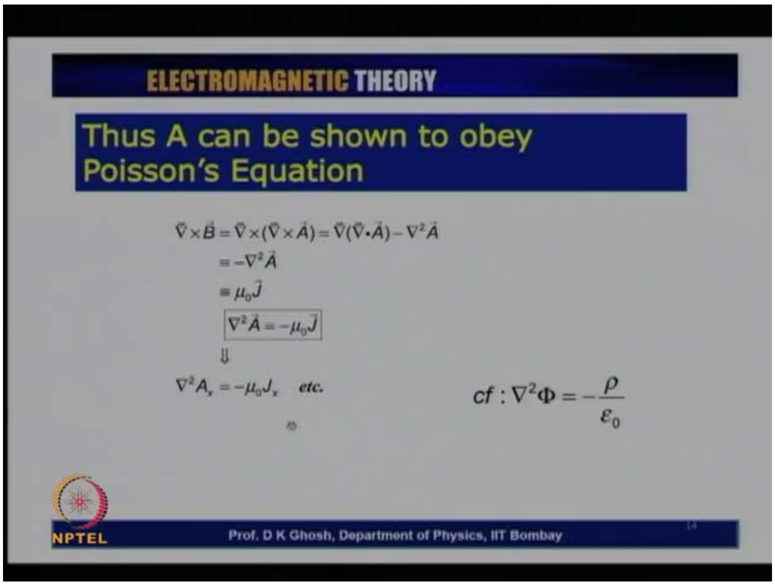

This stands for del square A x equal to minus mu 0 J x, del square A y equal to minus mu 0 J y and del square A z equal to minus mu 0 J z, but since a solution to Poisson's equation can always be found we are perfectly okay in saying this.

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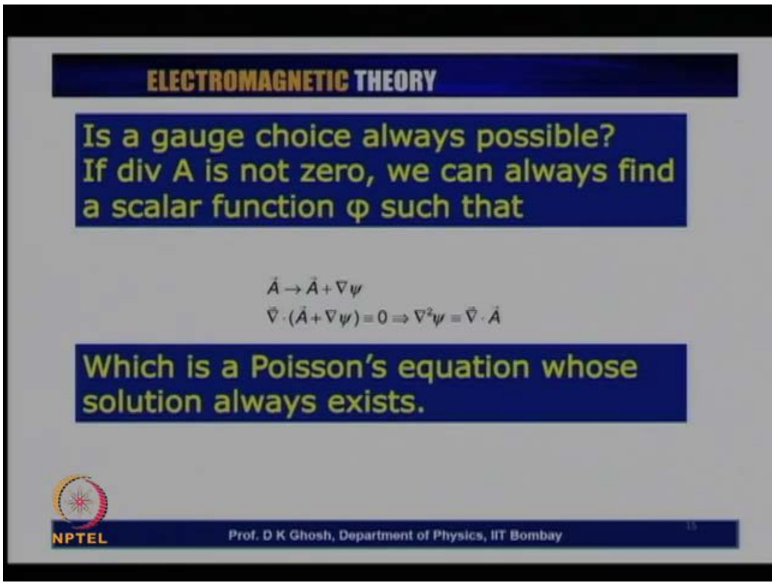

So, what we are trying to say is this. Can we always have a Gauge choice? That is the question. Now, that tells me that suppose my divergence of A is not 0, that is we are not in Coulomb Gauge.

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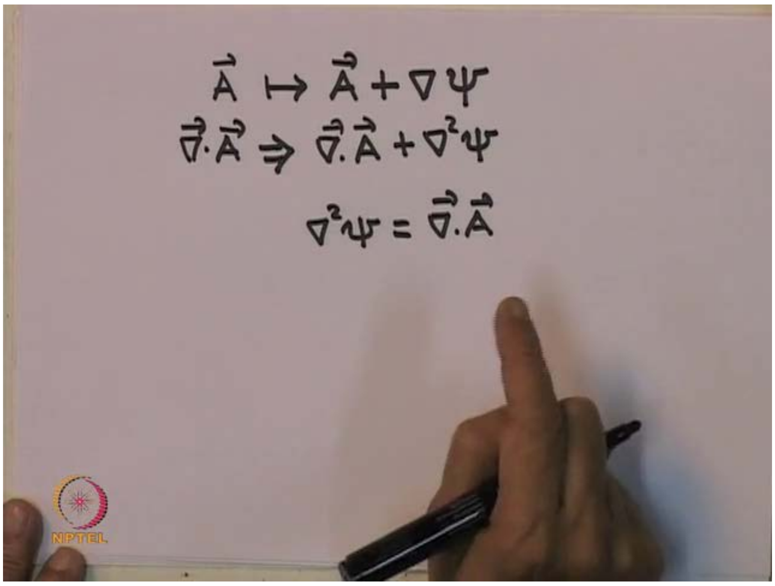

Now, question is this that supposing I write A and let this A go to A plus some gradient of a scalar function. Then you notice my del dot of A can be written as or del dot of A goes to my old del dot of A plus del square sin. So, in other words if you have started with an A for which del dot of A is not equal to 0. I could find a size such that the new del dot del dot of A becomes equal to 0 provided I know how to solve this equation del square sin is equal to del dot A.

So, what does it mean? It tells me that I start with an A whose del dot is not equal to 0. Then I seek a psi such that del square of psi is del dot of A which we have said is not equal to 0. Now, if you solve this equation infinitely solution of this can always be found because this is nothing but a Poisson's equation. Then plug this in into the first equation. So, that gives me now a new A for which the del dot of A is equal to 0. In other words this type of a Gauge choice is always possible.

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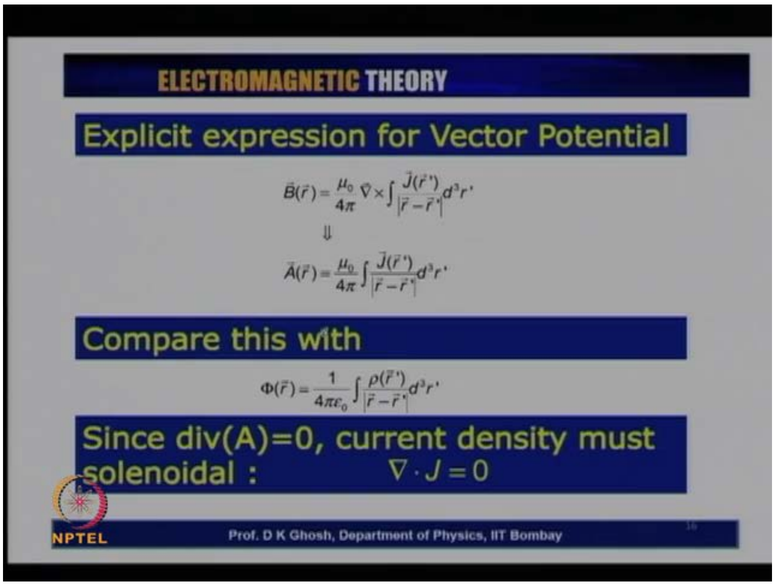

Now, recall that when we wrote down the Biot-Savart's law.

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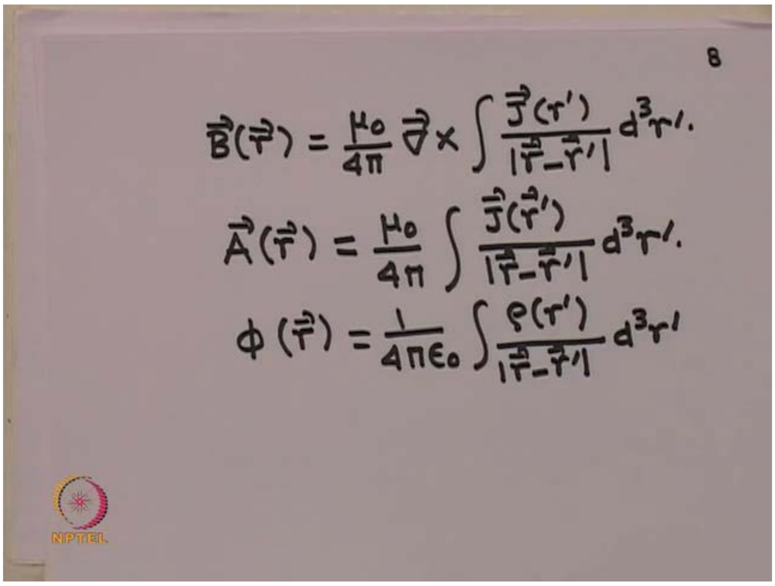

One of the forms in which we wrote down Biot-Savart's law was B of r, was B of r is written as mu 0 by 4 pi del cross of integral of J J of r prime divided by r minus r prime and of course, the integration variable is d cube r prime. r prime is the integration variable. So, if you compare this expression then you realize that gives me, it gives a formal expression for the vector potential A of r is equal to mu 0 by 4 pi times integral of J r prime divided by r minus r prime d cube r prime.

You could compare this with the expression for the scalar potential that we had in case of electric field which was given by 1 over 4 pi epsilon 0, integral the charge density rho r prime divided by r minus r prime d cube r prime. So, notice that in the gauge in which we are working del dot of A must be is equal to 0. So, that tells me of course, the current density is solenoidal which is obvious because I had a steady current. So, this is the, this summarizes the properties of vector potential.

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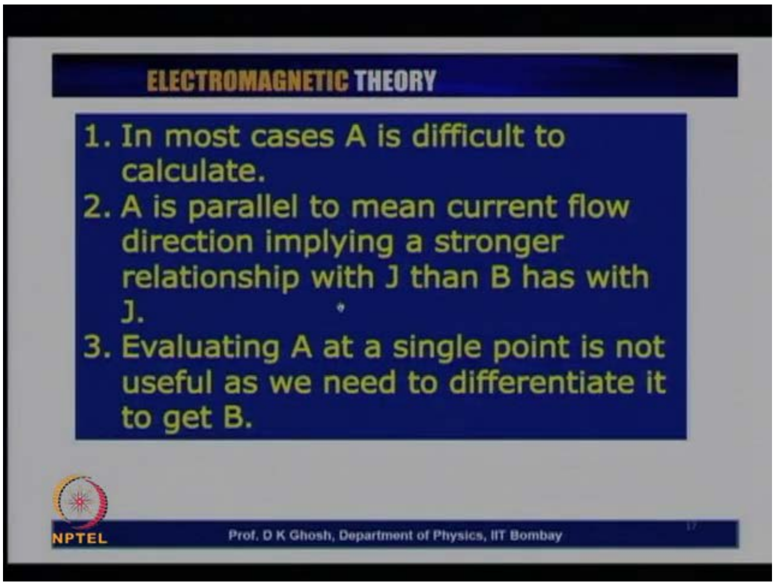

So, first thing you realize is this unlike an electric scalar potential, the vector potential is it is being a vector is you have to calculate three quantities and its somewhat difficult to calculate. The other thing is that if you look at this expression that you gave that A of r is is given by this expression. You notice A is parallel to the direction of current flow which tells me there is a stronger relationship between the vector potential and the current than B has with the current density because this is not a direct relationship. The other thing is in order to calculate the magnetic field which is physically meaningful you do not calculate the vector potential at a single point because B is obtained from A by means of a differentiation. So, you need to calculate the magnetic the vector potential at various points.

Now, this is of course, very similar to the way we did scalar potential if you want the electric field from the scalar potential you need to take its gradient. Therefore, you again need to take the scalar field. So, what we have done today is to define a potential analogous to the way we defined a potential for the electric field. The, it is analogous, but different in the sense that this potential is a vector and the curl of this vector gives me the physically meaningful quantity namely the magnetic field.

We have also seen that we have a choice which you called as the Gauge choice in determining this magnetic field and this vector potential which in turn determines the magnetic field. The question then arises is this is there a physical meaning to this vector potential? You recall that in case of an electric potential we had given a physical meaning in terms of the work that is done in bringing a unit charge etcetera. For very long time people had assumed that the vector potential is simply a mathematical artifact having no more content or having no content different from that what magnetic field has.

It, however there are several experiments that have been done, some very classic experiments that have been done which have proved otherwise that vector potential has indeed a physical meaning. In the next lecture I will be bringing this aspect of the vector potential into account and we will also calculate the vector potential for few standard geometries.