Electromagnetic Theory
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Module - 3
Magnetostatics
Lecture - 23
Equation of Continuity

Up until now, we have been talking about phenomena of electrostatics. Electrostatics basically implies the effect of fields that has created by static electric charges. What we are going to do now is to talk about a different phenomena, and that is the production and the properties of the magnetic field, for which the source is a steady current that is a charge in motion.

Basically, there is one major difference between electrostatics, and what we will be calling as the field of magneto statics. We know that electric charges can be isolated; namely, I can have a single positive charge or a negative charge. However, it has never been possible to isolate a magnetic charge, namely either a magnetic north pole or a south pole individually. Whenever a magnetic north pole is there, accompanying it will be a magnetic south pole; that is the net magnetic charge is always 0. So, therefore, that is one of the major difference, what we say is magnetic mono poles do not exist.

However, I must tell you that there is really no theoretical reason, why magnetic monopoles do not exist. However, in spite of several attempts to isolate such things, it has not yet been possible. The source of a magnetic field is an electric current that is a steady current which is flowing, let us say in a wire or on any other conducting loop. Now, in such a field, the a moving electric charge experience is a side wise force. Remember that an electric field can exert a force on a charge, whether it is moving or not. But however, the magnetic field is characteristic in the sense that it only exerts force on a moving charge; and this force which is given by the expression v cross B.

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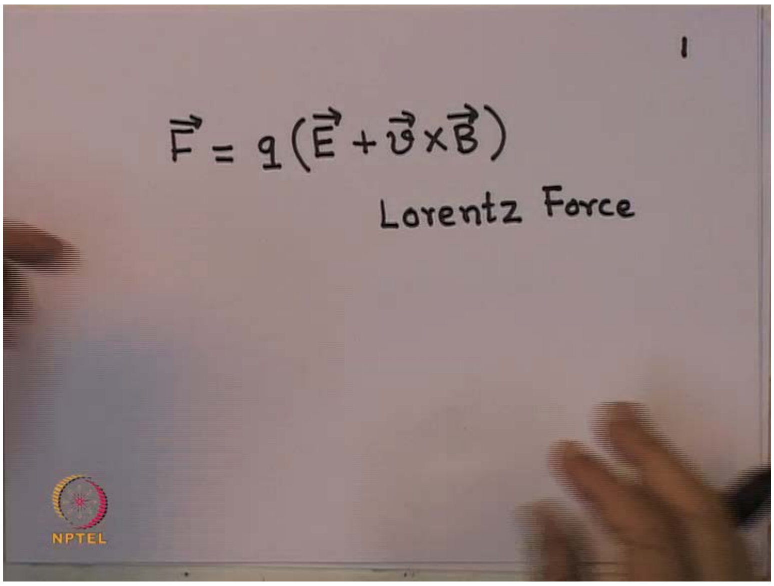

So, if have electric and magnetic field, I have the net force is given by q which is the charge, times the electric field which is the result of, or of the electric field, plus v cross B. I notice that this force due to the magnetic field is expressed is the cross product of the velocity with that magnetic field; which means, if the charges were static, it will only experience an electric field. But in order that it experiences a force in a magnetic field, the charge must be moving. This is known as the Lorentz force expression. So, what we have said is, the current is the source of the magnetic field.

Now, let us define current in a proper way. Current by itself is a scalar quantity. This is basically the amount of charge that is crossing the boundary of a surface s, of a volume v. So, whatever is the amount of charge that is crossing an unit area, per second, per unit time, that is what I call as the, a current.

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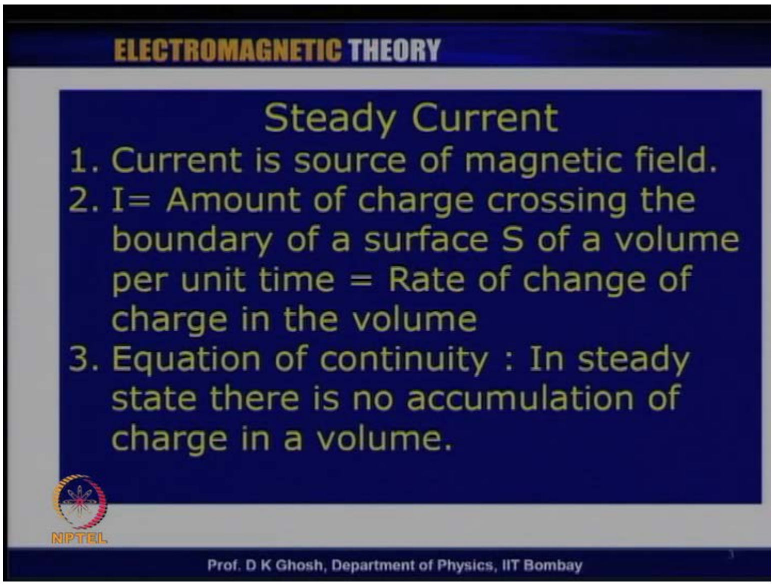

So, in other words, this is, supposing I take a volume, which is let us say, defined by a boundary S. The rate, so if certain amount of charges flow into the volume, the amount of charge in that volume will increase. Now, in steady magnetic phenomena or in a steady state phenomena, I do not expect an accumulation of charge; and this gives rise to what is known as the equation of continuity. Now let us look at, what is equation continuity?

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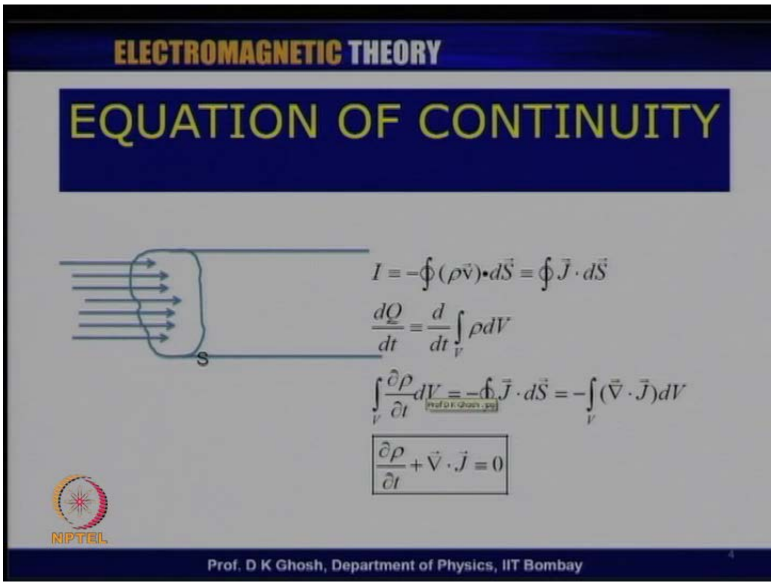

Now, firstly, let me assume that this is the volume, and this S is a surface which separates 2 parts of that volume, the volume to the left and the volume to the right. So, therefore, if electric charges are crossing this boundary, the, from the left I will expect it depletion of the amount of charge; and of course, to the right, there be an increase in the amount of charge.

Now, so therefore, what happens is this that, this current that I have got, which is, there is a minus sign wrong here, but which is the dot product of a quantity, which I call as the current density; and this is the current density which is crossing a surface and it is, that surface should be normal to the direction of the flow, j dot d S. So, therefore, the rate of change of the charge in that volume, which is d Q by d t is given by, well Q is nothing but the charge density integrated over the volume. So, therefore, this quantity, since this is the amount of charge that is increasing in the left part of the volume, that must be the decrease from this side, and which is nothing but the minus the j dot d S, which is minus of the current.

So, and I know that by using the divergence theorem, I can rewrite this as del dot divergence of j over the volume. So, therefore, I have got d rho by d t d V integral, is minus integral of del dot j d V; and if I bring it to one side, and use the fact that this is independent of what volume we take, it is true for any volume. What I get is this equation, which is d rho by d t, plus del dot of j, is equal to 0. This is known as the equation of continuity.

Basically, it tells me, talks about conservation of charge. So, if the left side of the volume is losing some charge because of the current flowing into, through the surface. So, I must have that increasing a corresponding amount of charge on the right hand side.

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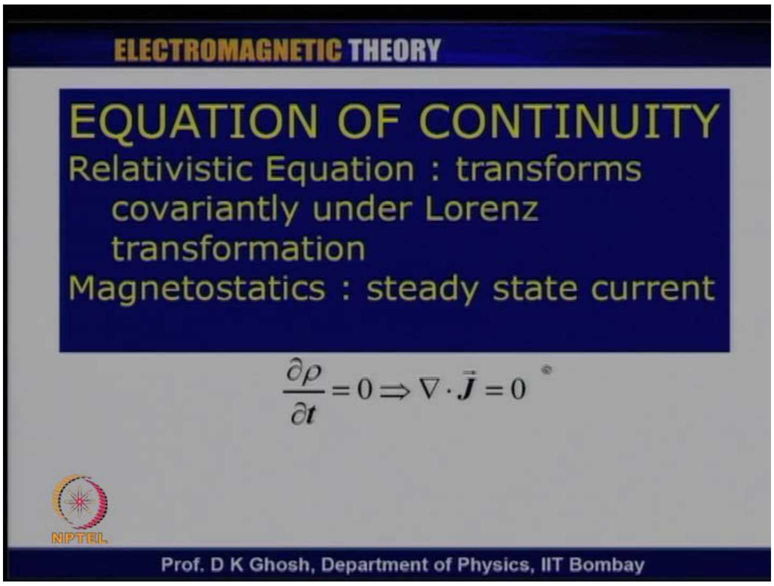

Since, we will be interested in, primarily magnetostatics, namely my currents are steady. So, therefore, I must have d rho by d t equal to 0, because there is no accumulation of charge in the volume. And as a result, the corresponding equation then becomes del dot of j equal to 0. So, this is the steady state form of the equation of continuity.

One point, which of course, will not be explaining in any detail, is to tell you that this equation is also relativistically valid; that is if you use, what is known as Lorenz contraction, then you will find this equation will plans from coherently under it. So, this is just to tell you that the equation, this equation is lot more rigorous and powerful than we thought it was.

While doing electrostatics, we have been familiar with the basic law which is the Coulomb's law; which essentially told you that the force between charges is given by an inverse square law, namely q 1 q 2 divided by r square, and there was a multiplicative constant, dimensional constant 1 over 4 pi epsilon 0.

Now, if I have, had a charge distribution, what I would do is, that electric field due to this charge distribution at a point, I build up by finding out what is the electric field due to a volume element containing the charge in that distribution, at that point p, where I am looking for the electric field. Now, since we have said that the source of magnetic field is a current distribution, so what we do is, we will look at, when we look at a circuit, we look at individual current element; that is a small length of the current wire let us say, carrying current; and, I define the direction of the length element, along the direction in which the current happens to be flowing at a particular point on that circuit.

Now, if this d l, so I times d l is the length element regarded as a vector in the direction of the current. So, I times d l is my current element.

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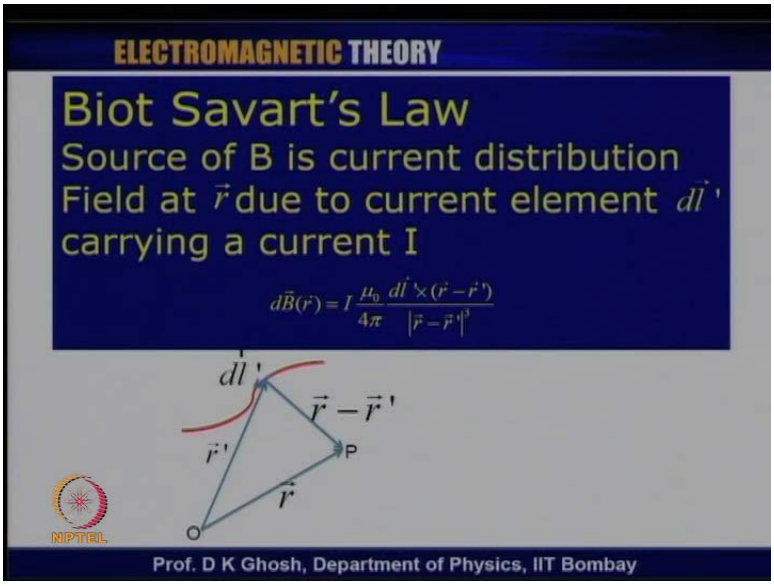

Now, what we do is this, that we calculate the magnetic field at a point p, this picture illustrates this; that this red thing is a current carrying wire, and here I have shown a small section of it, and this is the direction in which the current is being carried. And I am interested in calculating the magnetic field due to this current carrying system at this point p. Now, similar to Coulomb's law, we have a law here, which is kwon as the Biot Savart's law.

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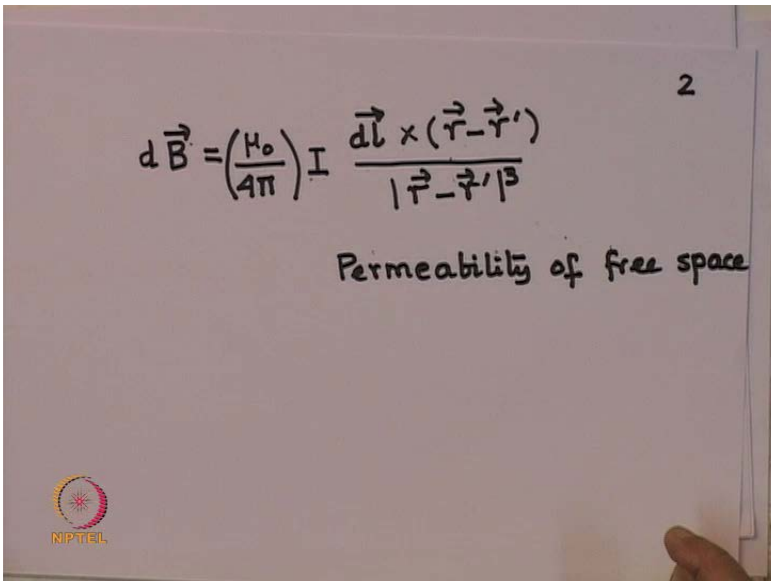

And Biot Savart's law tells you that if you take a current element I d l, the magnetic field d B due to that current element is given by a first, like 1 over 4 pi epsilon 0, I have a constant here, which I will come back to, which is mu 0 by 4 pi. Now, current element; since, I is a scalar, I will write it separately, I d l. Now, the d l cross, the position of the point where I am looking for the magnetic field with respect to the position of this d l. So, if you refer back to this picture again, you find that if I am looking for the magnetic field at a point p which is, with respect to my arbitrary origin at a vector position r, and we, the current element is at a position r prime vector. So, with respect to that current element the point p is at r minus r prime.

So, coming back to this formula again, I have, d l cross r minus r prime, which is the vector distance from that current element, and divided by the distance r minus r prime cube; and you can see that this is also essentially an inverse square law, because there is a power in the top and a power in the bottom.

This constant mu 0 is known as the permeability of the vacuum or permeability of the free space. So, at this moment, we will leave at that, that this is the constant which appears in the Biot Savart's law, just as a constant 1 over 4 pi epsilon 0 , with epsilon, epsilon 0, being known as the permittivity of the free space came in. Now, this has something to do with the magnetic properties of the materials; if there is material then the mu 0 will change, but we will come to it at a later stage. So, this is the Biot Savart's law.

So, when you have a current carrying circuit, and you are interested in finding out what is the magnetic field at that point, at a particular point, then you split your current distribution into a large number of infinitesimal current element, and the Biot Savart's law gives you the magnetic field and, at that point, and essentially what you have to do is to sum it up, namely integrate it.

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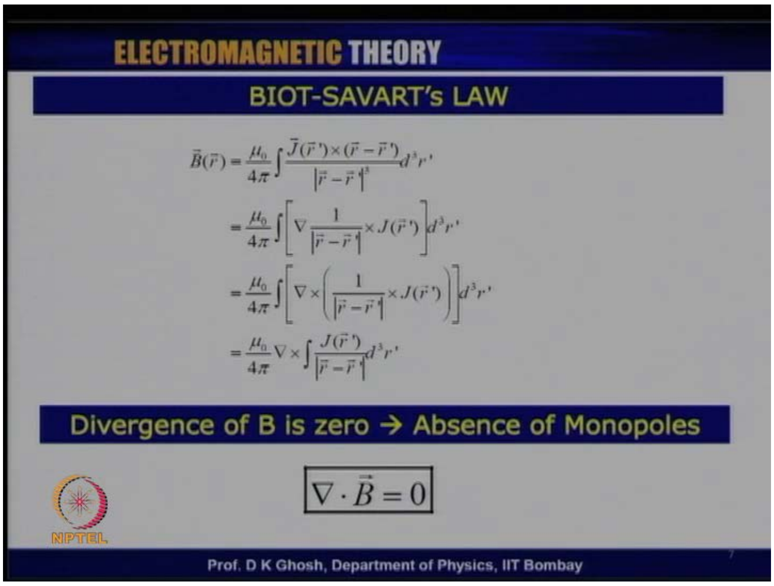

Now, what I will do is this, that I will try to get a slightly difference form, of this Biot Savart's law by doing a little of algebra. So, let me, let me look at what we are looking for. So, let us look at this. So, this is my I d l cross r minus r prime by r minus r prime cube, is my structure. Now, remember, what I had, supposing I wanted instead to write it in the form of a current density. So, remember I had, I times d l. So, I is a current and d l is of course, a length; and I know that what I could do, is to write this I as j dot d s, where j is the current density. So, therefore, that area element which goes with the current density to define my current and the length d l, taken together, gives me a volume.

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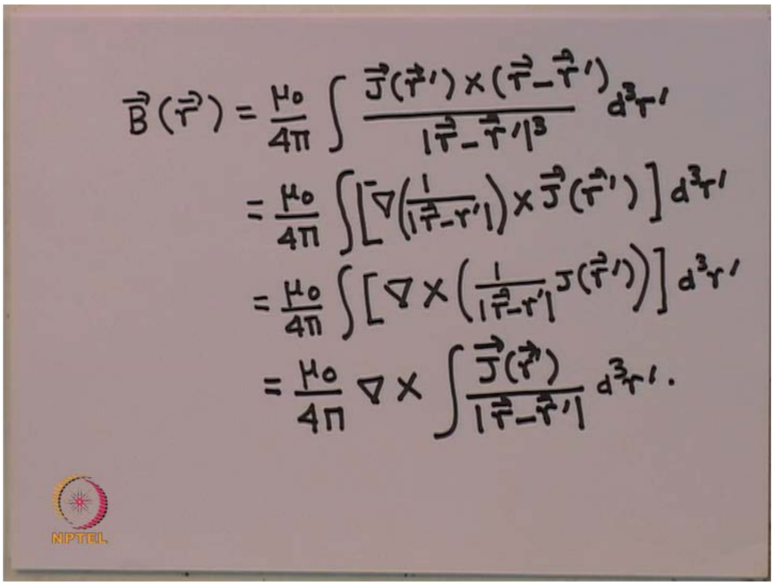

So, therefore, my magnetic field at the point p can be expressed as the, let us say at the point r is given by mu 0 by 4 pi integral, now remember this r is a fixed position with respect to our arbitrary origin, this is the point at which I am determining the magnetic field, now this is not a quantity which is varying, but when I have quantity is belonging to the source, I am going to write it as given by r prime vector. So, current density at r prime cross r minus r prime divided by r minus r prime cube and d cube r prime. So, this is my magnetic field expression.

Now, I am going to do a bit of a algebraic manipulation. Now, recall that gradient of 1 over r minus r prime, is minus r minus r prime by r minus r prime cube. So, what I am going to do is this, I am going to write this expression in a slightly different way. I will write it as a mu 0 by 4 pi; I have a minus sign if I write it as a gradient of 1 over r minus r prime. Now, in order to take care of that minus sign, I am going to interchange the order of the cross product which is j of r prime d cube r prime.

Now, so what I am going to do is this, I am going to rewrite this. So, I have got gradient 1 over r minus r prime cross j r prime, I will write it as is equal to mu 0 by 4 pi integral del cross, I will come back to the explanation of this, of 1 over r minus r prime, times j of r prime. Now, let us see why? Remember that 1 over r minus r prime is a scalar. So, if I have a del cross of a scalar times a vector, this is equivalent to the scalar times del cross of that vector, which in this case is 0, because this gradient operator is with respect to variable r del, which of course, we have said in this particular cases is not varying. So, del cross of j r prime is 0. So, I have got del cross scalar times a vector, is a scalar times del cross of vector which is 0, plus gradient of the scalar cross product vector. So, this is the result that I have got. So, I have written it this way, and this integrated with d cube r prime.

Now, of course, this del cross can be taken out; that is because this has nothing to do with the integration variable. So, I will write this as mu 0 by 4 pi, del cross of integral of j r prime, divided by r minus r prime, d cube r prime. Now, this is a rather important form in which the Biot Savart's law is expressed. So, this is a expressed in terms of a current density distribution in principle.

Now, since, the magnetic field is written as a curl of a quantity, and we know that divergence of a curl is always 0. So, this tells you the magnetic field at the point B is such that, its divergence is equal to 0. So, that is del dot of B is equal to 0. So, the magnetic field is such that, its divergence is equal to 0, now this is the actually a consequence of the fact that there are no magnetic monopoles. How do I say that? Recall that, in electrostatics the corresponding equation was del dot of E was equal to rho by epsilon 0, where rho was the charged density. So, as a result, since, in this case I do not have magnetic charges, free magnetic charges, my, that is I do not have monopoles, the magnetic field is such that its divergence is 0.

Now, we have said several times that if you have a vector field, a vector field is uniquely specified, by specifying its divergence and curl; actually, there is something else to be done, that is specify also the normal component of the field at some surface, but let us leave it out that. So, basically what we are saying is, that a vector field can have the divergence, and it can also have a different curl; different vectors field can have same divergence, but different values of the current. So, in order to specify it completely, I am going to now calculate, how much is the curl of the vector v?

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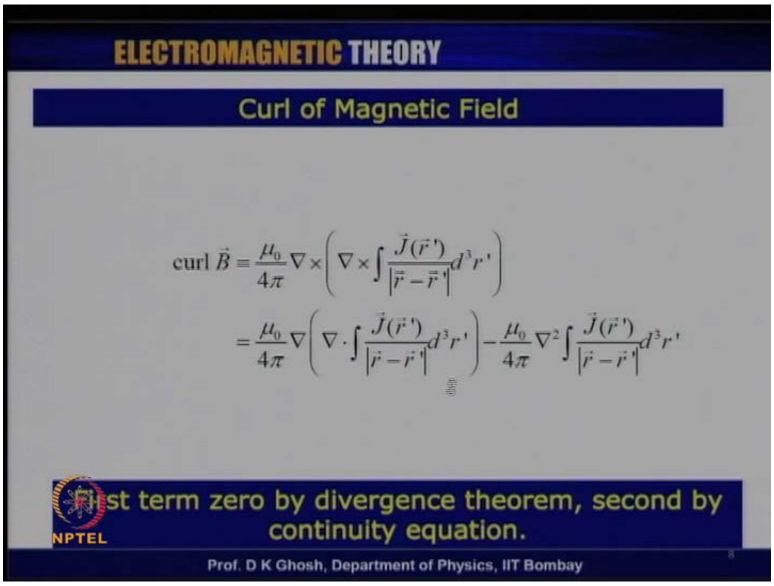

So, let us look at, what is the curl? Now, curl of B; now this expression, now is, what has been obtained, written by me, sometime back, as the Biot Savart's law expression. So, I got mu 0 by 4 pi del cross of del cross of this quantity.

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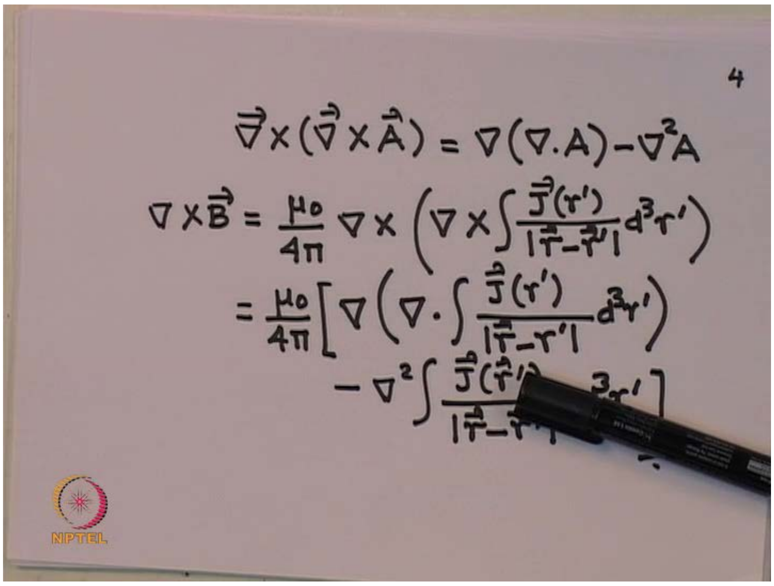

Now, we have several times seen that a del cross, del cross of a vector field is given by gradient of the divergence of that field, del of del dot of A, minus del square A. So, my curl of B, since B itself is written as a curl of something. So, I got mu 0 over 4 pi, curl of, this is my Biot Savart’s law’s expression, curl of j r prime by r minus r prime, d cube r prime.

And, I am going to use this formula that I wrote down, that is del cross del cross of this quantity is, the constant mu 0 by 4 pi, gradient of del dot of this quantity, j of r prime by r minus r prime, d cube r prime, minus del square of j r prime divided by r minus r prime; this looks like a rather horrible expression, but let us try to look at, simplify this. So, first what I am going to do is this, I am going to look at the first term that is del, del of del dot of this quantity.

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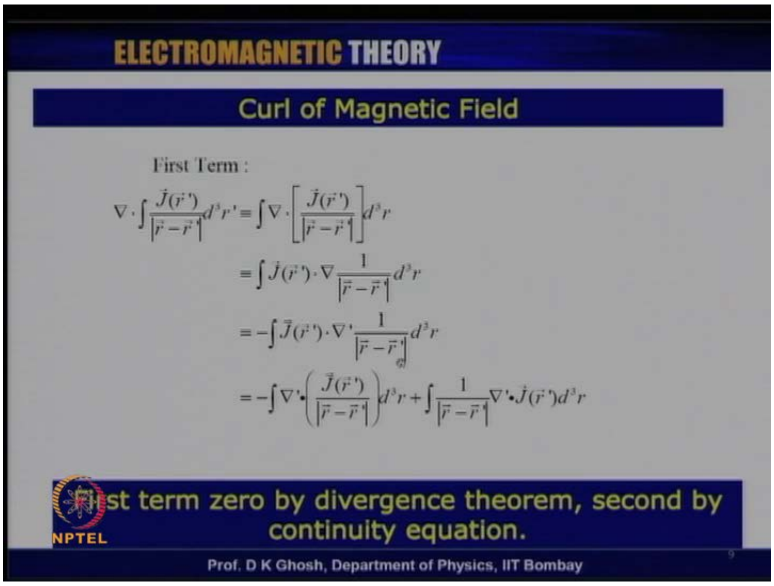

Now, look at this; first, before I calculate the del, let us look at what is this quantity here, divergence j of r prime by r minus r prime, ok. So, what I am doing first is this; since, I know that the quantities inside the variable of integration is r prime, and the gradient in which it is with respect to r, I can simply take this del dot inside. So, I get integral del dot j r prime by r minus r prime, d cube r prime. Now, so this is again divergence of a vector times a scalar; scalar is 1 over r minus r prime.

So, the same way, we write it as the vector dotted with gradient of another r minus r prime, that is because scalar times del dot of j prime, j, which is the function of r prime is 0; so I write this. And now, what I do is this; since, the gradient acts on a function of r minus r prime, I will use a minus sign and make that gradient a gradient prime, meaning there by that it acts on the variable r prime, and not on r; and that is, as long as it is a function of r minus r prime, I can do that by simply adding a minus sign. So, I got this.

Now, this quantity, now everything here is with respect to the prime coordinate. Now, what I am going to do is, that write this as del dot j or r prime by r minus r prime plus, there is a minus sign and hence there is a plus, 1 over r minus r prime, del prime dot j d cube r. So, this is a simplification that we have achieved of the first term of that.

Now, let us look at, what does it mean? Firstly, this term, which is an integration of a divergence of a quantity, integrated over d cube r prime, this integration variables should be d cube r prime, and not d cube r. So, I can convert this using the divergence theorem, to this quantity, dotted with d s prime; and as we have done several times, I can take the surface to infinite distances and make this integral vanish. So, I will be left with simply this term; and we have seen that we are dealing with steady state magnetic phenomena, and I talked earlier about the continuity relation and we had said that for the steady state phenomena del dot of j must be 0. So, therefore, this is also 0.

So, therefore, from the expression that I had written down here, of the curl of B, the first term vanishes, and I am simply left with the second term. So, let me look at what the second term is…

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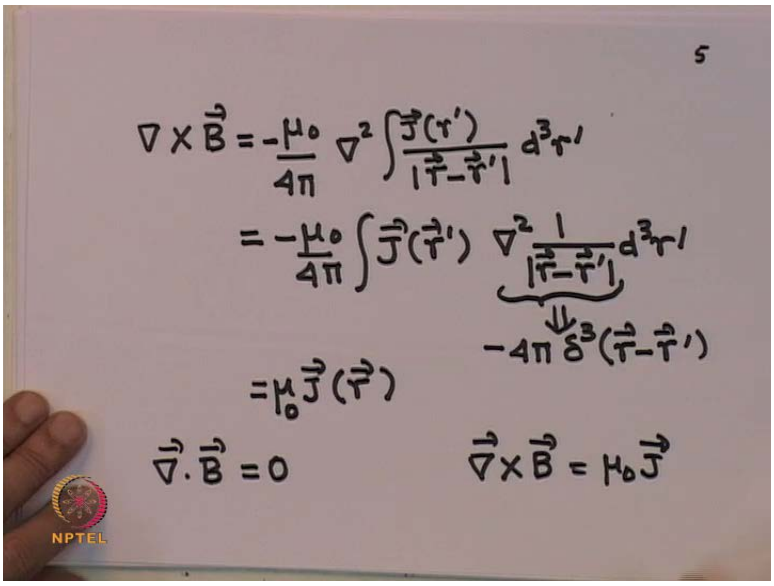

So, I am left with curl of B is equal to mu 0 by 4 pi, del square of j of r prime by r minus r prime, d cube r prime. So, this is, and there is a minus sign taken along with this. So, what we will do is this; as we have done several times, I will take the del square inside. But I know that del square acting on the variable r has no effect on this j of r prime, so this will be written as minus mu 0 by 4 pi integral j of r prime times del square of 1 over r minus r prime. Several times we have pointed out that del square of 1 over r is minus 4 pi times a delta function of r. So, this quantity is, this minus 4 pi times delta function, which is a 3 dimensional delta function. So, I write it delta cube of r minus r prime; these are all modules of vectors.

Now, because of this delta function, I can do this integration, there is a minus sign here, minus sign here, 4 pi cancels out, and I will be simply left with j r prime, delta r minus r prime, d cube r prime, which is nothing but j of r; and 4 pi r cancel out, I am left with a mu 0 j of r. So, therefore, combining the 2, I have got divergence of B equal to 0, absence of magnetic monopoles; and curl of B is equal to mu 0 j.

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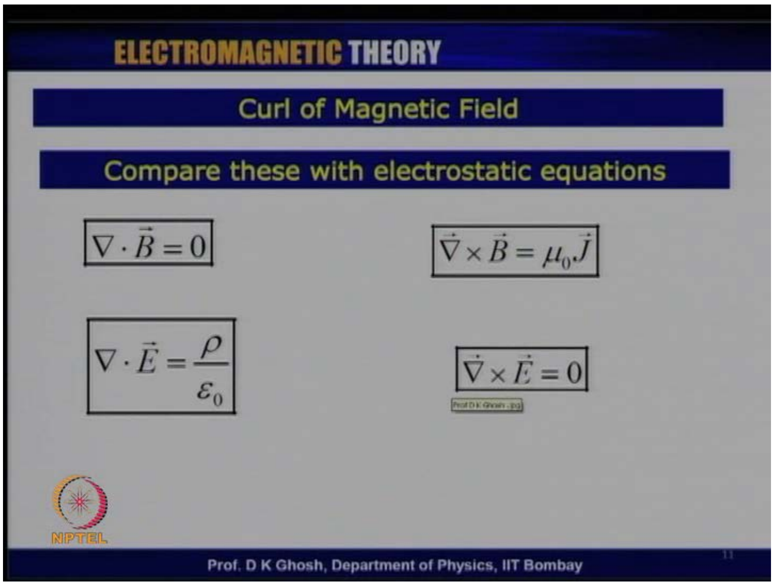

I would like you to compare these expressions with the corresponding expression for the electrostatic field. Remember, del dot of B equal to 0, I had del dot of E equal to rho over epsilon 0, that was my electrostatic Gauss's law. So, of course, we have already understood that why this is 0, and not something else. Del cross E, electric field was a conservative field, so therefore, the del cross of that was equal to 0, but you notice that del cross of B is given by mu 0 times j.

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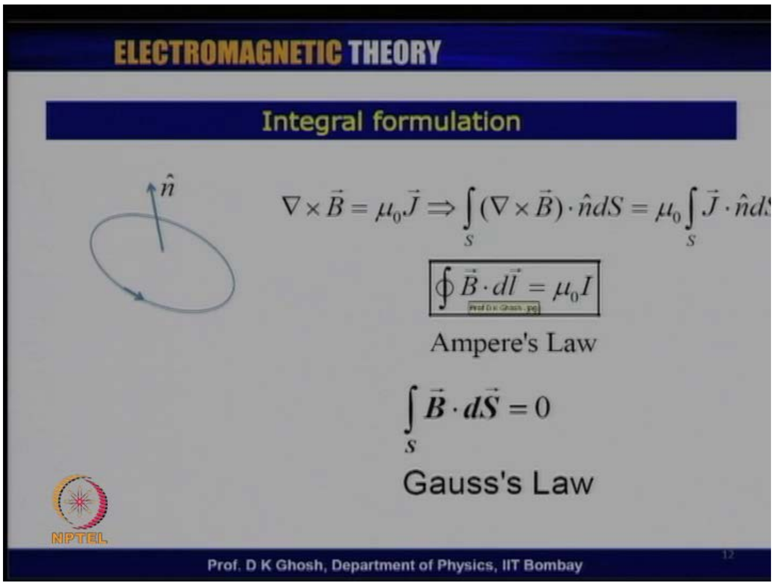

Let me do an integral formulations of these equation. So, del cross of B equal to mu 0 j. Now, suppose, I convert this by taking; so this is my circuit and I define as surface; and I say that, look, in that case, if I define a surface, then I can write down del cross B dot d S, is same as j dot dS. So, this will be a surface through which the current is flowing; that is surface perpendicular to the direction of the current. Now, since this is true for an arbitrary surface; so let me write it here.

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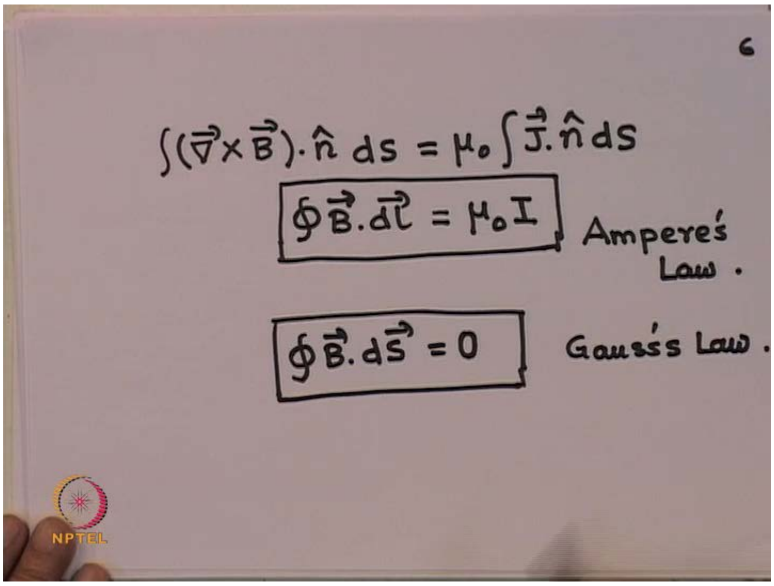

So, I get integral of del cross B, dot n d S is equal to mu 0 times j dot n d S; it is true of an arbitrary surface that I could take. And because of that I must have integral of B dot d l, remember that I have used Stoke's theorem here, which converts the surface integral of a curl to line integral of the vector field itself. And j dot n d S is my current. So, therefore, this is mu 0 times I. And this is usually called the Ampere's law of magnetism. In an identical way, del dot of B equal to 0 can be converted into integral B dot d S is equal to 0. In the same way as we had converted del dot of E equal to rho by epsilon 0 to integral E dot d S equal to q divided by epsilon 0. So, this is the Gauss's law of magnetism.

What I will do next is, to use the Biot Savart's and the Ampere's law, and illustrate a few problems. I would like to point out that the physical content of Ampere's law is no different from that of the Biot Savart's law. But however, frequently, if I particularly in cases I have symmetry, because it is expressed as an integral B dot d l; and if you want to use a definition like that to extract information about B, then I need this integral B to be expressed as a functional form; which would be possible only if I have substantial amount of symmetry. So, therefore, Ampere's law has limited validity, because it cannot be always used; but the, your Biot Savart's law is fairly general.

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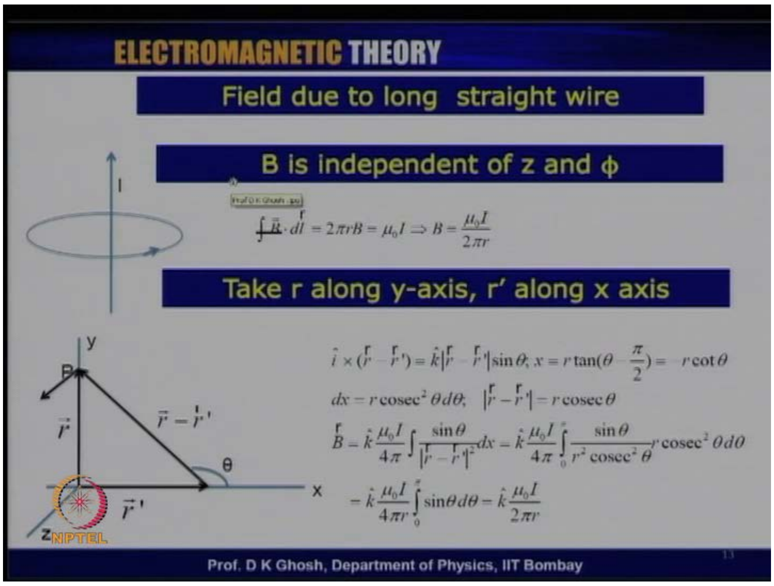

So, let me, let me start by taking a long straight wire; so long straight wire like this. Now, I am interested in calculating, what is the magnetic field at a distance, let us say r from the wire. Now, purely by symmetric consideration, I can imagine that the magnetic field will be the same, at the same distance from the wire; in other words, it can only depend upon the distance r from the wire.

So, if I take a circle of radius r, with the centre at the location of the wire, then the magnitude of the magnetic field everywhere must be the same. The direction of the magnetic field is given by the standard right hand rule; that is, if you hold the wire with your finger pointing in the direction of the current, the direction in which your fingers curl that gives you the direction of the magnetic field; in other words, the direction of the magnetic field is a azimuthal. So, it is depends upon phi.

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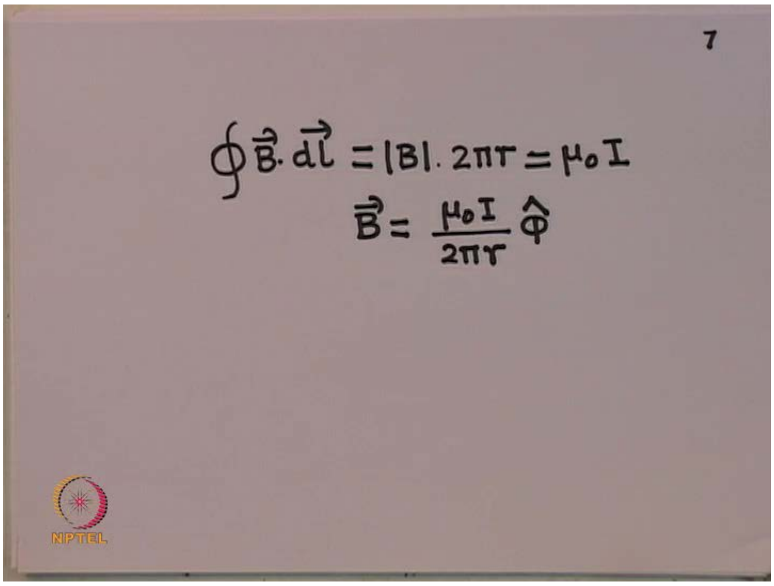

So, in that case, what I can do is, I can calculate what is the integral of B dot d l, by taking a circle of radius r. Since, B is the azimuthal, and the circle that I take is also in the same direction, B dot d l is same as magnitude of B times magnitude of d l; and then the integration, since magnitude of B is the same and the integration over d l will be matching dot 2 pi times r; and that must be equal to mu 0 times the current that is flowing through.

In other words, the field B at a distance r from the wire is given by mu 0 I by 2 pi r that is its magnitude, times of course the unit vector in the direction of phi; that was easy, because the problem has substantial symmetry. Now, what I am going to do is this, I am going to recalculate the same thing by taking a, by using Biot Savart’s law.

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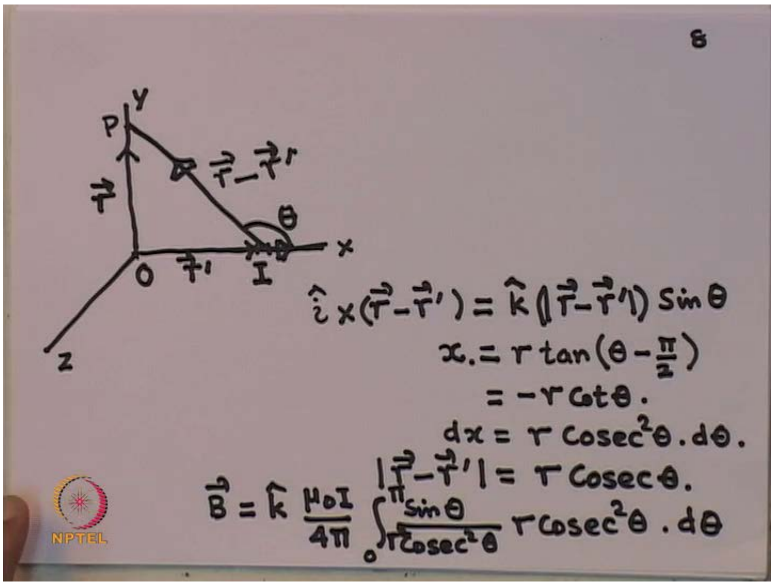

So, if I use Biot Sarvat's law, let us look at, how does it work? So, let me, let me define the x, y, z axis in the same way; this is my x axis, this is my y axis, and the z axis is out of the plane of the paper. Now, my, I will take the wire in the direction of x axis, and the point at which I am interested in calculating the magnetic field to be at, on the y axis; remember that I can always do it, because I have an infinite wire.

So, whichever point I am interested in calculating the magnetic field, I simply that and I, I use it for defining my axis. So, what I do is, this is my point p; and this I will take as my, this is my origin. So, this is my vector r. So, if I take a current element at a vector position r prime, then this is my vector r minus r prime. So, let us look at, how does it go? If you recall your expression for the Biot Savart's law, let me go back a little bit. So, remember that, what I had was d l prime cross r minus r prime. In this case, my d l prime is the same as in the x direction. So, it is nothing but I times d x.

So, let me return back to that. So, what I need is, i times, i cross r minus r prime. Now notice, i cross r prime is of course 0, because r prime is also along the x axis. So, left is, i cross r, and r is taken along the y axis, so therefore, i cross r is along the z axis. So, this is unit vector k; unit vector k times r minus r prime magnitude, times sin of this angle, times sin of this angle. So, what I do there now is this. That, I can write this as equal to, I want to write it in terms of this fixed distance r. So, notice that I got r minus r prime.

Since, this angle is theta, I can write this distance in terms of r times tangent of theta minus pi by 2, which is equal to minus r cot theta. I use this that, my d x then. So, this is, this is my, this is my x; x is equal to, this is x is equal to r tan of this angle. So, therefore, it is minus r cot theta. So, d x is equal to r cosec square theta d theta. And I can write down r minus r prime magnitude itself, as equal to r cosec theta. So, therefore, my magnetic field which is along the direction k mu 0 I by 4 pi; if you collect all these things, you have got magnitude of r minus r prime, times sin theta, coming from i cross r minus r prime.

You had a 1 over r minus r prime cube. So, this one will go left with r minus r prime square, which is r square cosec square theta; and I have a sin theta there. So, I get it as sin theta divided by cosec square theta, r square cosec square theta; and of course, r cosec square theta d theta; that is, that is my d x there.

Now, notice this thing that, things cancel out there; I am left with simply an integration of sin theta from 0 to pi. So, I am left with simply integration of sin theta from 0 to pi, which gives me minus cos theta; and when I put the limit I get a factor of 2. So, therefore, I am left with again k mu 0 i divided by 2 pi r, which is, of course, was obtained in a much simpler way in, from the Ampere's law.

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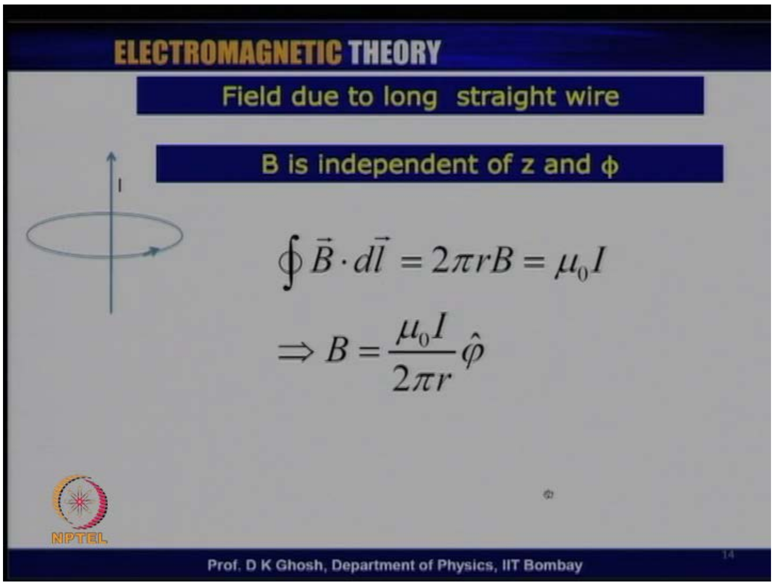

I will use another application now; to calculate the magnetic field on the axis of a circular loop, now say an expression which is used several times for calculating other things, in particular the field inside a solenoid.

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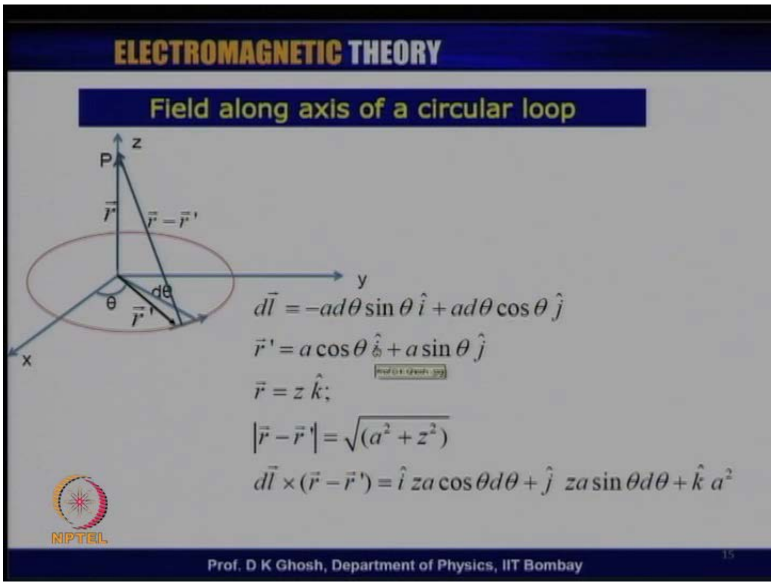

So, let us refer to this picture here. So, I have taken my circle which is in red, in the x y plane. So, this is x axis, this is y axis, and z axis is along the axis of the circle that is perpendicular to the plane of the circle passing through its centre. So, remember that, what I am going to do, I need my, to apply Biot Sarvat's law. So, I first need an expression for d l, remember I d l cross r minus r prime. So, let us look at the vector relationship first.

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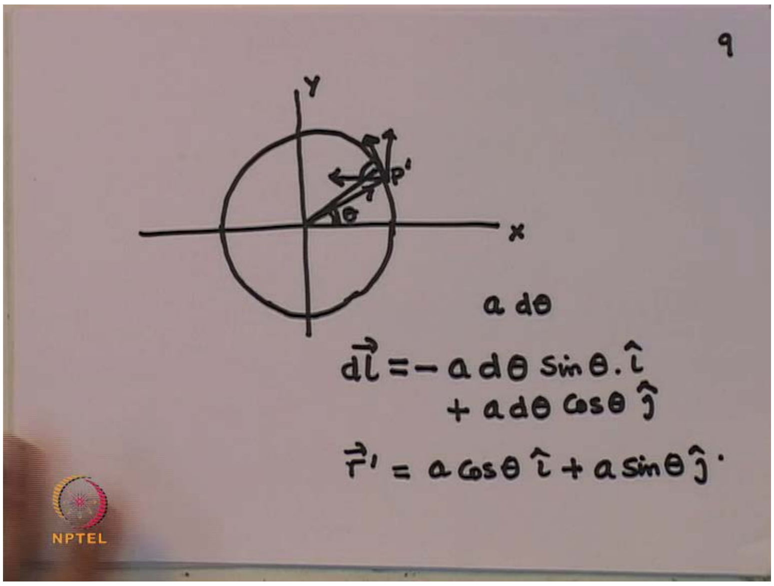

So, I will find it easy to show it just for the circle, let me show it along x axis and y axis. Supposing I have a circle here, this is essentially recalling some vector algebra that we have done; this is the x y plane. Now, what I am going to do is this, I am looking at an element, at an angle theta here, making an angle theta with the x axis; and the direction of that vector is along the tangent at the point p; the magnitude of the vector, because I am taking a small element there is, a times d theta; if I take the element to be at the position theta making an angle d theta. So, I have a vector along the tangent to the circle of magnitude a d theta. Now, how do I resolve it along the x axis and the y axis. So, notice that I can resolve it like this, and that.

Now, so therefore, since this is the vector which is of magnitude a d theta; the x component is minus a sin theta because this angle is theta. So, therefore, this angle is theta, and hence, this is 90 minus theta. So, therefore, my d l will be minus a d theta; minus because the direction of that component is along the negative x direction; sin theta times I; and of course, a d theta cos theta, because cos theta direction, the direction of, along the y axis is positive j; the point p, this is of course, really not the point p, but the point let me call it p prime, that is the position at which I have got the current element. Now, the vector r prime itself, which is this vector, I can resolve it; and that simply gives me a cos theta times i, right, because this is, this is in this direction, it is outward direction. So, r prime vector itself is a cos theta times i, plus a sin theta times j; and the vector r, returning to the slide, the vector r is along the z axis, I have rewritten the two relationship that we have proved. So, vector r is z times k.

So, therefore, and further the magnitude of r minus r prime; by Pythagoras theorem is r square plus a square, square root. Now, you put all of them in. What is d l, cross r minus r prime. Now, so that is very easily calculated, because r minus r prime, r prime I have written down; r is along the z direction. So, therefore, you just write this d l cross r minus r prime, this is along the k direction. You have got the d l expression there; you can easily show that this as i, j and the k component. Important one is the k component because d l has a i and a j; and r minus r prime will give you this relationship there.

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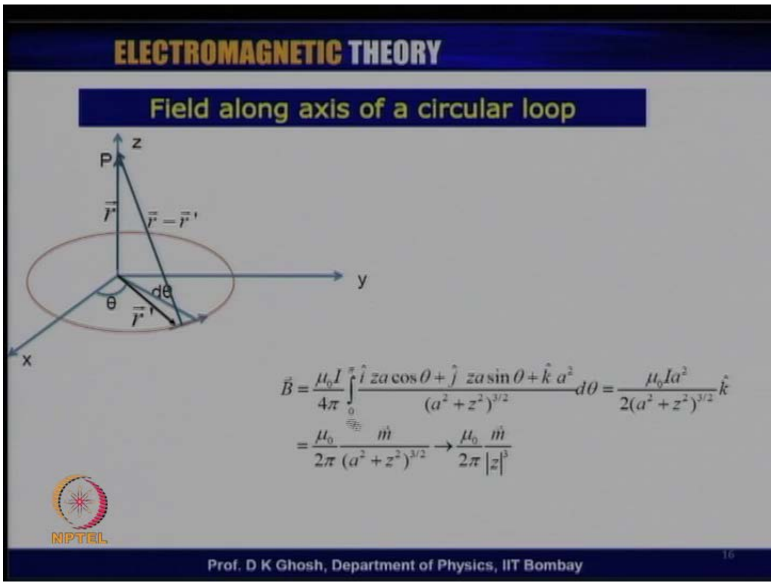

So, if I divided this by distance which is a square plus z square, if there is a cube there. So, a square plus z square to the power 3 by 2 d theta. All these quantities are constant, other than theta. So, therefore, the integral of cos theta or integral of sin theta, they give me 0; and I am simply left with a term which is dependent upon k. So, this gives me mu 0 I a square by 2 a square plus z square to the power 3 by 2 k. I would like to point out that the current times pi I a square, which is the area of the loop is equal to, is defined to be the magnetic moment of the current loop; and so therefore this is also written as mu 0 by 2 pi, m by a square plus z square to the power 3 by 2. And if you take z to be very large, then you notice that this gives me a 1 over r cube dependence of the magnetic field. We will see that this is nothing but the way the dipoles fields behave.