Electromagnetic Theory
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Module - 2
Electrostatics
Lecture - 21
Dielectrics

In the last lecture we had brought in the concept of a dielectric. We said that in while in conductors the charges are free. In the sense that the carriers of electricity they do not remain tightly bound to an atom or a molecule. There situations in insulators where what you find is that the electrical charges retain their identity to the molecule or to the atom to which they belong the.

We had talked about two different types of materials. One where even in the absence of an electric field there is a separation of the positive and the negative charges. Thereby the molecule getting a what is we can say as a permanent dipole movement. On the other hand there are molecules or systems where in the absence of an electric field the charge centers. That is the positive and the negative charge centers they coincide. But however, if you apply an electric field the these charge centers are pulled apart and as a result the systems or the atoms in the molecules they develop what is what can be called as an induced type of moments.

So, basically these are the class of materials which go by a common name of dielectric as oppose to the conductors where the charge carriers are said to be free.

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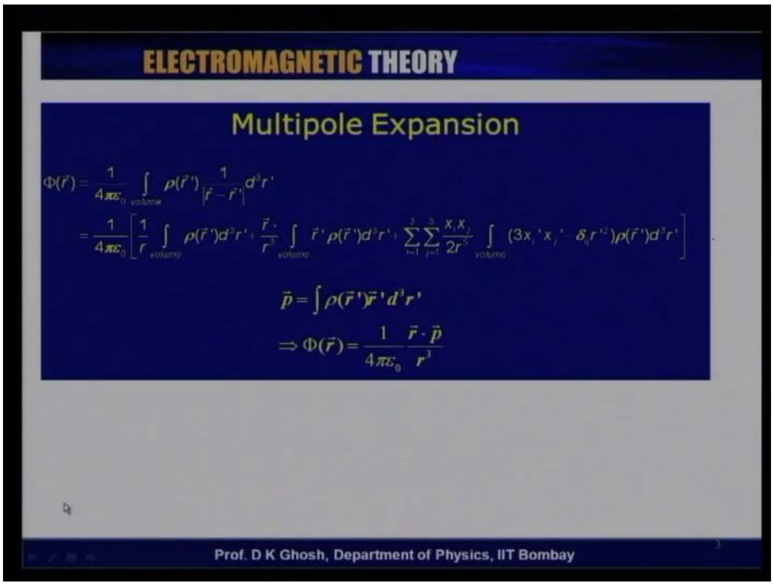

So, what we did last time is to look at the concept of what happens in a situation like this? And what we said is that the system because in the presence of an electric field for instance the there is a separation of the charges.

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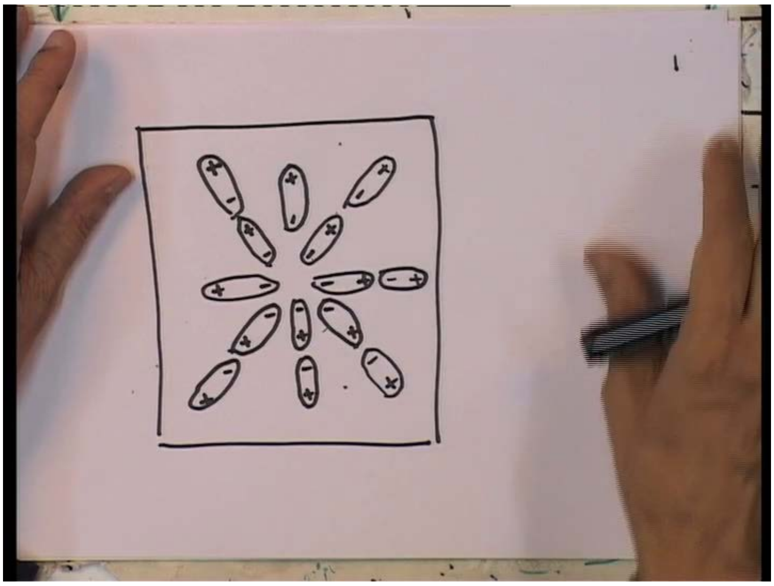

So, the molecules they align in the way it is shown here. For instance you notice the negative? Their end to end negative positive negative positive etc. Now, what you notice is in this area, in the central region? There is a negative charge. For instance which sort of acts as a source for the electric field.

So, notice that here I have a source which has come because of the way the molecules of the material have become aligned. And this is what we have been calling as the bound volume charge the other thing that happens is that.

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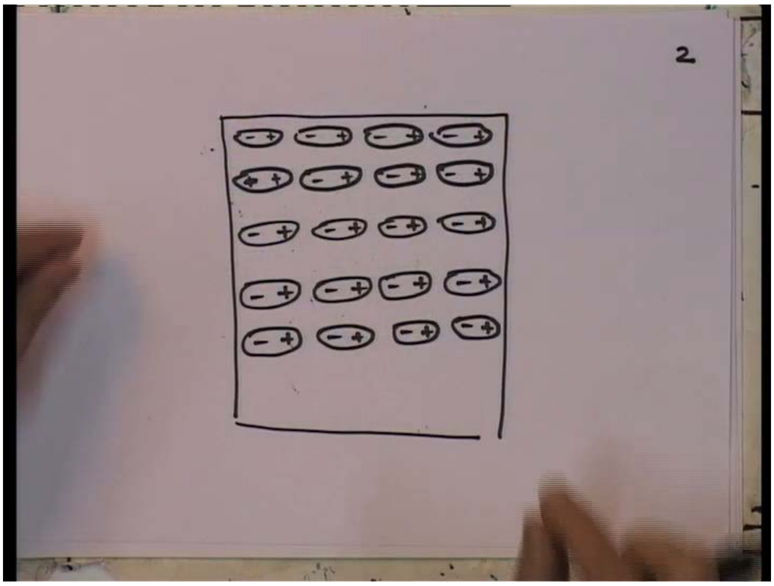

If you look at the boundary of the thing; then you notice again that there is a separation of charge that is the edges seems to get charged. For instance the here I have shown on a particular type of alignment in which this edge becomes negatively charged and this edge becomes positively charged.

Now, the second thing that we did last time is to look up a what we call as a multi pole expansion. Basically we are interested in finding out the potential due to a an arbitrary charge distribution and the. So, as you know that the arbitrary charge distribution is I have 1 over 4 pi epsilon 0 supposing rho of r is my density. So, the potential at the point r is given by the usual expression 1 over 4 pi epsilon 0 integration over the whole volume rho r prime 1 over r minus r prime d cube r prime. r prime is my integration variable.

Now, what we do here is this that since r is a fixed position at which we are interested in finding out the potential? This quantity we expand there is a 1 over r minus r prime which is being expanded and I sort of in principle this in finite series. But, what I have been generally found is that for most physical situation; if you retain up to the first the first time as you realize is just I have I am expanding 1 over r minus r prime by r and 1 over r I have taken out. So, in the first term I simply get rho r prime d cube r prime.

Now, as you can see this is just the total charge q. So, therefore, this term is the total charge q divided by r which is exactly the expression that we have. When a point charge located at the origin; its potential is calculated at the point r. Now, notice the reason why it is true. I have made this expansion for r very large. That is further away from the charge distribution. Now, if r is very large if I am looking at a potential at a large distance form a charge distribution as a very crude first approximation. It turns out that that charge distribution seems like a a point charge. Now, since it looks like a point charge to the leading order in the expansion; I just get the potential due to a point charge which is q by r which is exactly what is shown here. Now when you go to the next term.

So, I have got one minus r prime by r to the power minus one. So, therefore, my leading term now becomes this. The other than one over r square which comes out. So, I get r r by r cube dotted with this integral integral r prime rho r prime d cube r prime.

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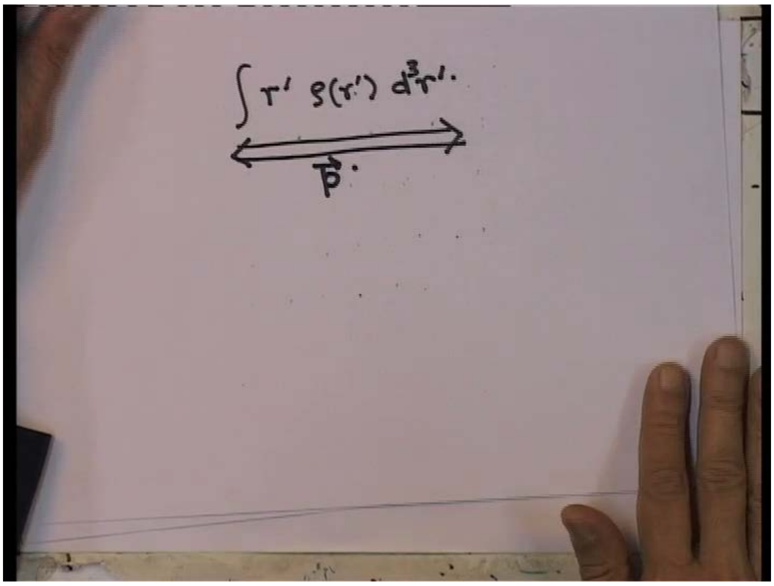

Now, let us look at this quantity here. So, we said integral r prime rho r prime d cube r prime. Now, this is the quantity which we defined to be the dipole moment the. As you can see that this is the straight forward generalization form the way we defined the point dipole where we had simply two charges which were separated by a distance d and we said that the dipole moment has a magnitude q times d. And its direction is form the negative charge to the positive charge. Now, what I have done here is to essentially extend the same concept to a charge distribution. And here again I get a distance times the charge density and the direction will still be from negative charge to the positive charge. Now, this is what we call at the dipole and we are going to be looking at this term in lot more detail as we go along.

The next term in the expansion is what we call as a quadruple term and I had shown you last time that the quadruple term is given by. I am basically looking at this expression here which is three x i prime x j prime minus r prime square delta i j rho r prime d cube r prime. And you notice that this x i and x j. These are the components of the position r prime. So, for example, one two is x y etc and the. So, this quantity this integral is what is known as a quadruple moment tensor. You can see there are nine quantities there because i and j can each run form 1 2 3 that is x y and z. So, and for most practical situations; this essentially is the distance to which we have to do the expansion and further expansion which would be octal pole and things like that they are normally not of much used.

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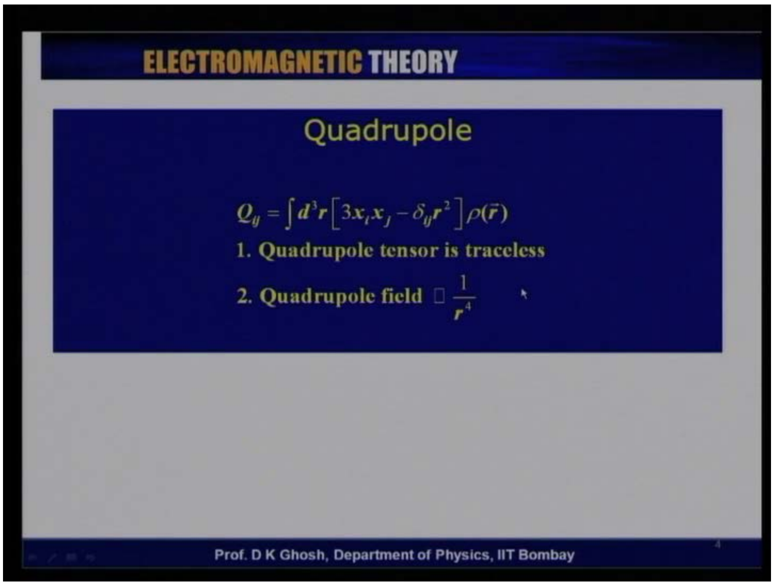

Just now a comment on the Quadrupole the notice that Q i j is d cube r 3 x i x j minus delta i j r square rho r. Now, this quantity has certain properties which you can very easily show. First is that quadruple to quadruple moment tensor is trace less. Now, you can see that you remember that trace is simply the sum of the diagonal components of a matrix. So, if you just look at how much is the diagonal component you can trivially prove that this quantity is 0.

The second thing that you can prove; which I will talk about it in my next lecture that typically the quadruple field goes as 1 over r to the power forth. Now, like point dipole one can think of quadruples.

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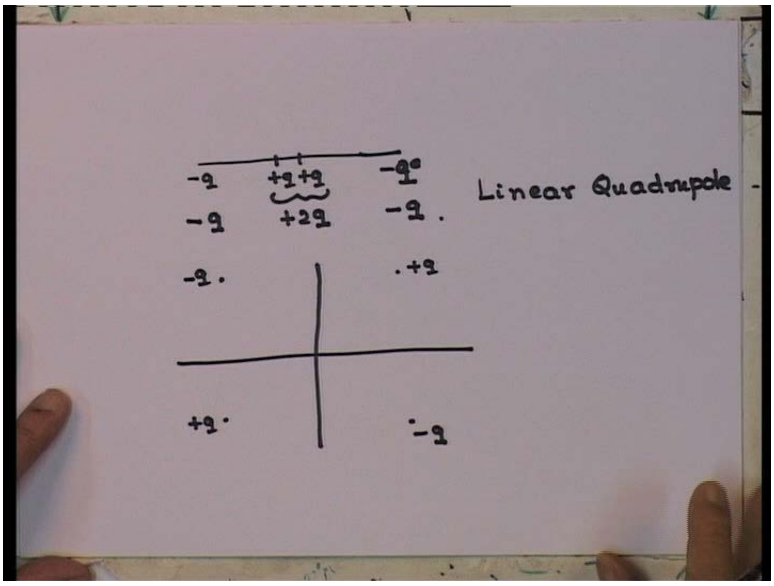

Now, the quadruple there can be various types. For instance we could think in terms of a end to end quadruple meaning. Thereby, I have got let us say minus q then plus q. Then plus q and minus q that is a that is a charge plus 2 q. At 1 end at the centre and a minus q minus q. There this is what I will call as a linear quadruple. You could think of four charge distribution in a slightly different fashion. For example, they could be distributed at 4 corners of a square. So, I get q here minus q there plus q there and minus q there. So, this also a is an orientation or a configuration for a quadruple. We will be looking at quadruple as we go along with the lectures later.

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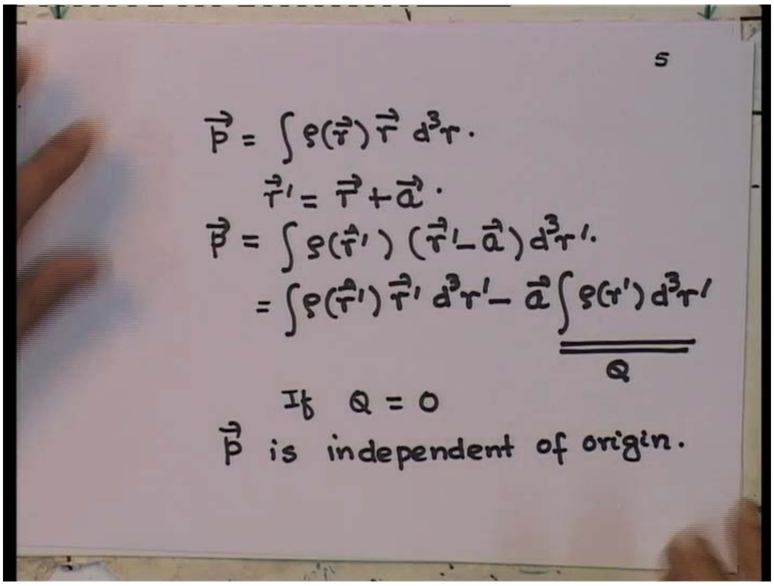

The point about dipole which is going to be the primary subject of discussion in today's lecture is that if you look at the dipole expression dipole moment expression vector p equal to integral rho r vector r d cube r; you notice this could in general depend upon a choice of origin. For instance, if you defined a vector r prime equal to r plus a; then the my dipole moment becomes rho r prime. This is just a variable change and r is r prime minus a and d cube r of course, is d cube r prime.

So, look at the first term you notice that this is nothing. But, the dipole moment with respect to the new variable that I have chosen minus since a is a constant I have got rho prime d cube r prime this quantity is nothing, but, the total charge. So, the expression for the dipole moment will not depend upon the origin if the total changes is equal to 0. If q equal to 0 p is independent of origin. So, this is about some general comments about dipole moments and lets proceed with our what we were talking about.

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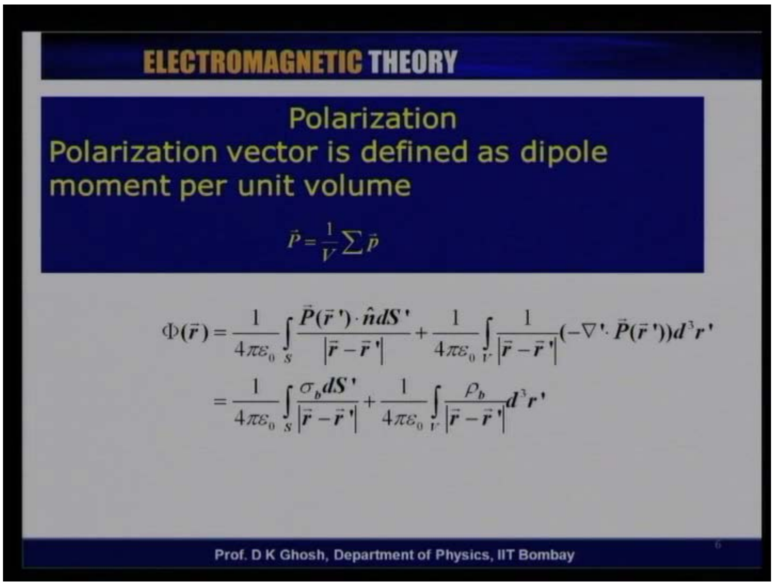

So, I defined the polarization vector as the dipole moment per unit volume. So, I take a unit volume. I will take a volume v make a vector sum of all the dipoles there in and divided by the volume that have being considered. Now, we derive the expression for the potential of this collection of dipoles potential. Due to this collection of diploes and what we found last time was that the general talk general expression for the potential due to a collection of dipole is given like this. 1 over 4 pi epsilon 0. Of course, and I have got a. This is actually consist of two terms; the first terms is a surface term and the surface term is p dot n dS prime. P is the polarization vector that we talk about over r minus r prime. And the second term which is a volume integral term which is 1 over 4 pi epsilon 0 integral 1 over r minus r prime minus del prime dot p r prime d cube prime.

Now, if you look at these expressions you notice that this is the type of expression we would get for the potential due to a charge distribution. For example, in this case it is a surface charge distribution. So, if I had a surface charge density sigma in place of this p r prime dot n which is nothing, but, the normal component of the polarization vector. If I replace this with sigma b then this going to be the potential expression due to such a surface charge distribution. And similarly, if minus del prime dot p prime is replaced by a volume charge density rho b. Then this should be identical to the potential that I get due to a volume charge distribution rho b within this volume.

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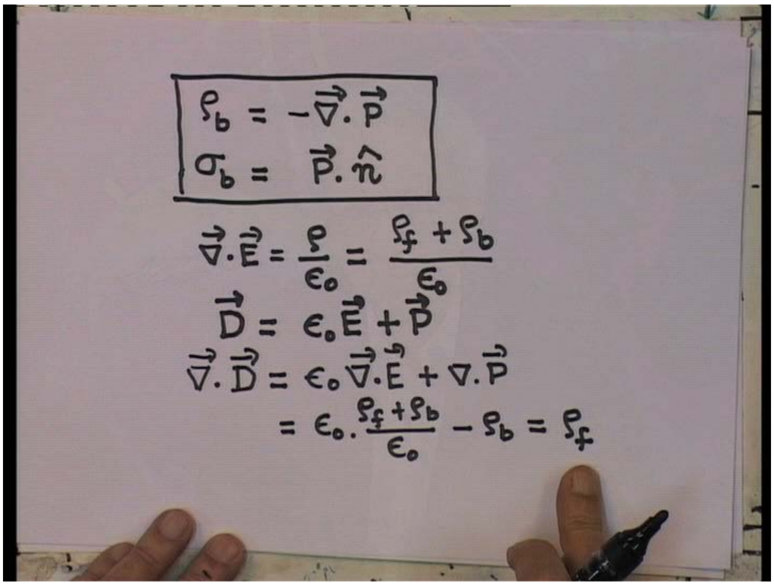

So, this tells me that the polarization vector is related to the bound charges. By this expression that rho bound let me give the symbol rho b this is nothing, but, minus del dot p where p is the polarization vector. And sigma b which is the surface charge distribution is given by the normal component of the polarization vector.

So, therefore, the effect of the dielectric is to essentially given by a sum of a volume charge density rho b and a surface charge density sigma b. And how does it modify the Maxwell's equation? That we are familiar with remember del dot of E was equal to rho by epsilon 0. Now, this is the actual electric field. But, now my rho can have two components first is of course, could be the free charges that are there in the system.

So, let me call it rho free and the bound charges that have been developed because of the polarization. So, it is rho f plus rho b divided by epsilon 0. Now, notice this makes it more difficult to compute the electric field which is the actual eclectic field that a test charge at the location where the electric field is calculated will feel.

Now, in order to make some sense out of it; what one does is to define an auxiliary vector which we called the displacement vector or is simply called the vector D. Now, this is defined to be epsilon 0 times the electric field plus the polarization vector p. Now, note that the vector D and vector E do not have the same dimensions because there is a permittivity constant coming in there.

Now, using this I can write down the Maxwell's equations like this what is del dot D. So, notice del dot D is epsilon 0 del dot E plus del dot of p and we have just now seen that del dot of p is rho free plus rho bound divided by epsilon 0 and minus del dot of p p is just rho bound.

So, therefore, this is minus rho bound. So, after canceling the epsilon 0s what I am left with is simply rho. free First thing there are two interesting points to note the displacement vector satisfies the Maxwell's equation that its divergence is given in terms of the free charge density. Now, remember I it is much easier than to calculate d because I would know how I would have a better information about free charge density than I have about the bound charge density. Because bound charge density is an internal mechanism mind you that the bound charges are as much of a real charge as the free charges are there is nothing they are not fictitious charges. They are actual charges excepting that they still remain attached to their parent molecules.

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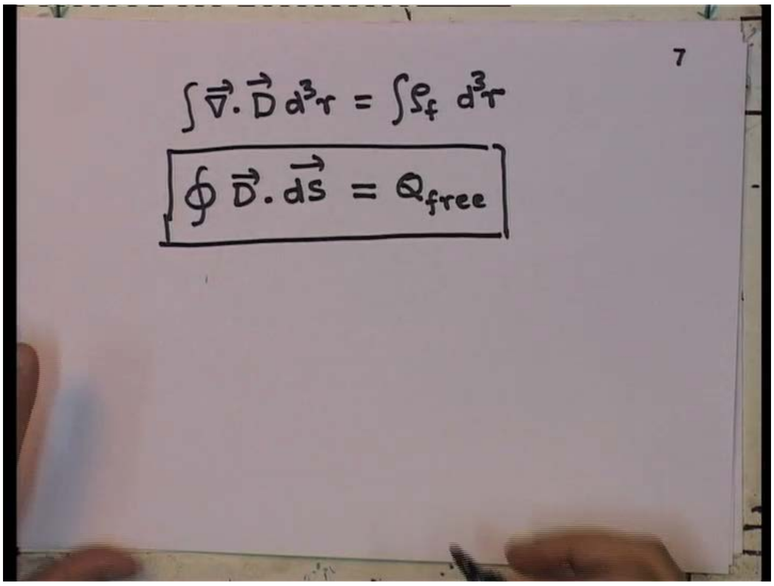

So, del dot of D is free. Now, one can then write down the Maxwell's equation in integral form. Recall that integral of del dot of D d cube r is then equal to integral rho f d cube r and I used the divergence theorem to convert this divergence of d over a volume to a surface integral of d over a closed surface containing the volume.

So, it will be D dot d s and that is equal to integral rho f d cube r which is nothing, but, the free charge total free charge. So, this is the integral form of the Maxwell's equation and once again note that the vector D is a related to the free charge density where as the vector E is related to both fee charge density. And the bound charge density and the realistic field is actually the vector E though we might occasionally find it easier to work with vector D which is determined in terms of the free charge density about which we may have little more information.

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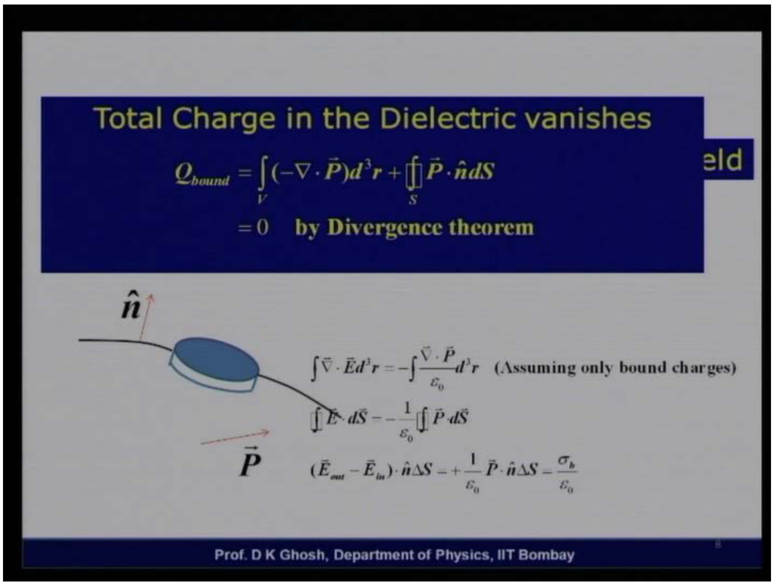

Having done this, let us look at the boundary conditions that we satisfy. Now, notice this is very similar to what we did in case of a conducting boundary excepting. That you have to remember that when we took a Gaussian pillbox half outside a conductor and half inside a conductor the electric field inside was 0 that is no longer to here.

So, let me first just take assume there are no free charges makes life simpler assume that only bound charges. Now, if now have a Gaussian pillbox with a of a very small width height half into the polarized medium namely the dielectric and half outside namely the vacuum then I can write del dot of Ed cube r because remember that I have no free charges. So, E we have already seen that this. So, good generally then mean this is minus integral del dot P divided by epsilon 0 d cube r. We only assume bound charges that is integral E dot dS is 1 over epsilon 0. If I convert this to a surface integral this becomes 1 over epsilon 0 P dot dS.

Now, what we do is his that since these are now being converted to surface integrals; I look at the two surfaces remember that the thickness is. So, small that from the curved surfaces there is no contribution to this integral. So, what we do is this and we also realized that the direction of the normal for the outside surface is opposite to the direction of the normal for the inside surface. Now, as a result what we get is this that e out that is the field outside minus E in dot n dS the minus sign comes because the direction of n is opposite and that is equal to P dot n dS and the reason is that there is no polarization outside. So, therefore, I have only contribution from inside and we already know that this P dot n dS is noting P dot n is nothing, but, my bound charge density the surface bound charge density there is a delta s missing here.

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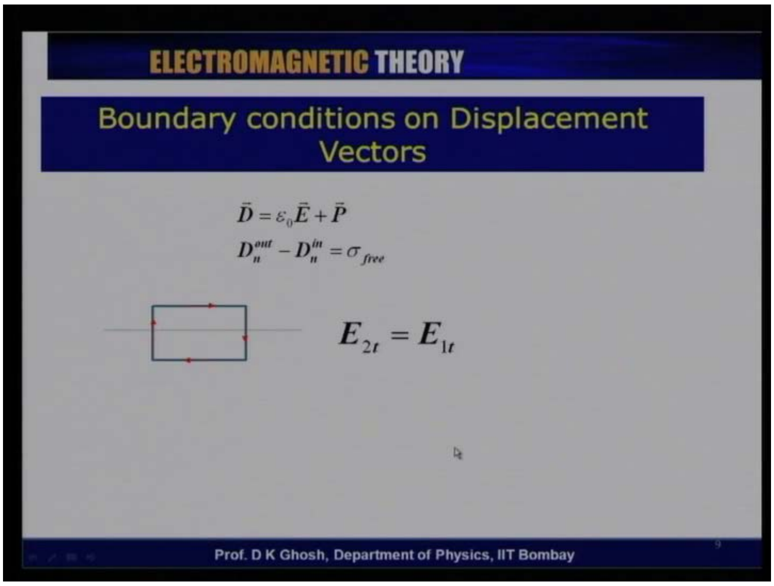

So, if you look at that what I get is that the discontinuity in the electric field E out minus E in as was before given by the surface charge density divided by epsilon 0 and this is precisely what we had earlier as well excepting that in this particular case the surface charge density is purely due to the bound surface charge density. This is one of the boundary conditions that we have.

The second boundary condition is actually on the tangential component and this is done exactly the same way as before that is take a circuit. And since the electric field is conservative field just go round the circuit. And you get if this is the medium two; this is the medium one I get E to t is equal to E 1 t. the Look at this we had said that D is equal to epsilon 0 E plus P and so, therefore, you can now convert this boundary condition that we did for E to a boundary condition on D and that gives me D out normal component out minus D normal component in is free charge density surface charge density of the free carriers.

So, at all stages the vector D is related to free charge density whether there all the surface as it happens in case of a conductor or inside the material. So, D n normal component of D is discontinuous only if there is a charge surface across which there is a discontinuity. This exactly the same thing as we had talk about for the normal component of the electric field when we were talking about conductors, the differences that because of the difference in their dimensionality the D and out minus D in simply could be sigma b. No epsilon 0 etc and the tangential component of the full electric field D is continuous. So, these are the things that we need to learn.

Now, there is a class of dielectric known as linear dielectric. Now, in a linear dielectric the polarization vector that is the dipole moment per unit volume is proportional to the electric field E itself now.

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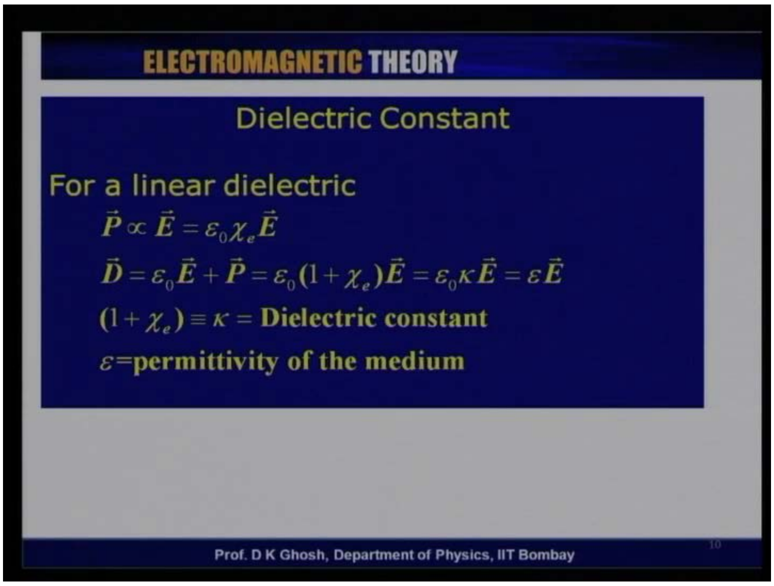

So, therefore, what one does is to write the polarization vector as some constant which will we write as epsilon 0 times some number chi. And chi is normally defined to be a susceptibility and so, epsilon 0 chi times E now notice that D is epsilon 0 E plus P by definition.

So, therefore, if you replace P by epsilon 0 chi. This gives me epsilon 0 into 1 plus chi times E. This quantity 1 plus chi is called the dielectric constant that is a number kappa. So, epsilon 0 times kappa. This kappa is the dielectric constant and the epsilon which is the product of epsilon 0 times K kappa is usually called the permittivity of the medium. That is the polarized medium just as epsilon 0 was a called as the permittivity of the free space. So, epsilon and epsilon 0 are related by a dielectric constant.

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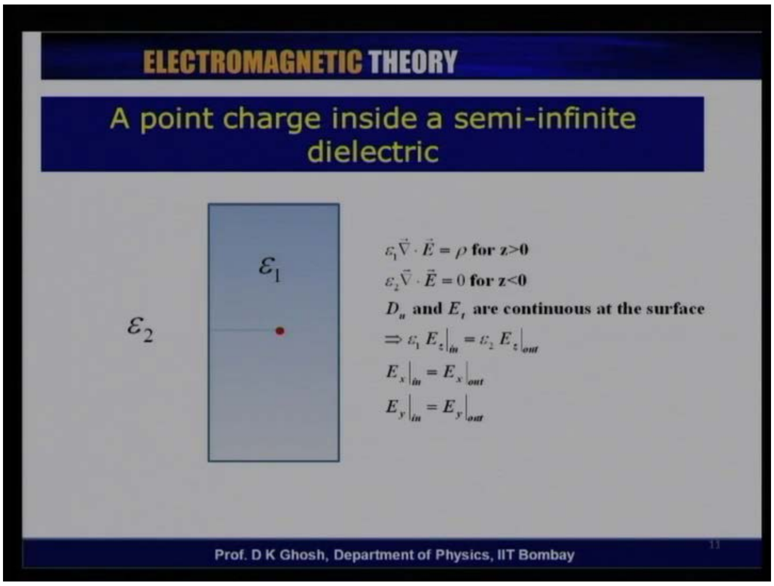

Now, let me now go over and the try to illustrate some specific methods of doing problems would directed this problem is where we consider two semi infinite dielectrics. Well, one we will take to be epsilon 2 which could be epsilon 0 for example. And epsilon 1 and I assume that the epsilon 1 feels the entire half space z greater than 0 and the interface is z equal to 0 which is perpendicular to the plain of the figure and epsilon 2 you could take it as vacuum. But, once again it is in general another dielectric.

Now so, what we want is this in the dielectric? When I use the dialectic; I mean epsilon 1 here and I will occasionally call epsilon 2 as epsilon 0 which is my vacuum. So, epsilon two exists in the space z less than 0 and therefore, there is since there are no charges in the region z less than 0. I know that epsilon 2 times del dot E must be equal to 0 the in the region z greater than 0 epsilon 1 del dot E I know is nothing, but, the charge density rho. There is a charge at least here because we have put a charge at the position d. Now how does one solve this problem?

The basic principle of solving this problem is to observe that the normal component of the vector D and the tangential component of the vector E must be continuous act to the surface which is at z equal to 0 and since my surface is z is equal to 0 my normal direction is the z direction.

So, since the normal component let me assume an linear. So, if I have a linear dielectric the relationship between the electric field and the field D is given by D is equal to epsilon that is the permittivity of the medium times the electric field. So, therefore, the normal component of the electric field in this medium is epsilon 1 times E z and the normal component of the electric field in the other medium is epsilon 2 times E z well I have said in out.

So, these are essentially the z components. I know tangential component of the electric field E is continuous and since z is equal to 0 is nothing. But, the x y plane you could take any two directions. There let us call it x and the y direction. So, that I have x component of the eclectic field E is the same as the x component inside and the same is true for the y component. So, basically the tangential components are continuous.

Now, you recall that when we did the problem of a charge in front of a conducting grounded conducting medium or conducting plane. We had seen that the problem can be solved by what we called as a method of images and what we did there is to put any equal and opposite charge at the image position now this was done. So, that the boundary condition at the surface of the conductor namely the electric field becoming equal to 0 can be satisfied.

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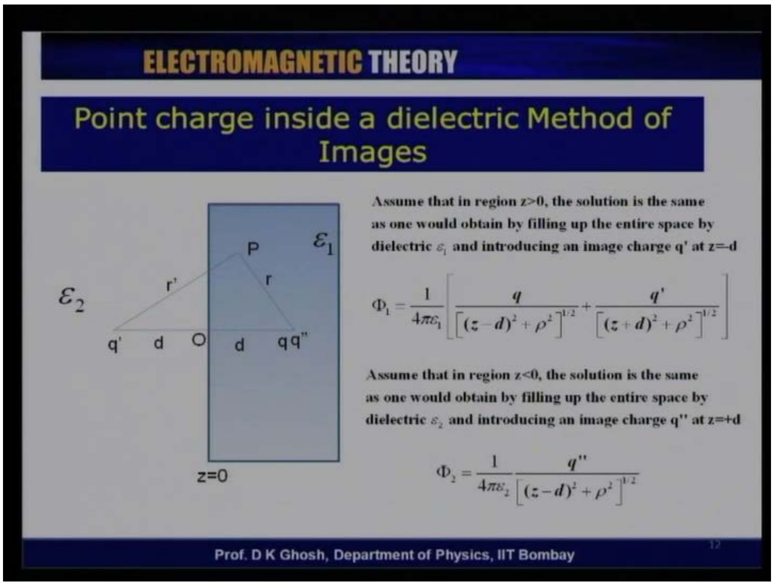

So, let us look at what happens in this case. Now, what we do is this that look at this picture once again. Before I describe the method I recall that whenever we are doing solutions of problems like this we depend on the famous uniqueness theorem namely that the solution of the Laplace's of the Poisson's equation that we are getting they are unique. And so, therefore, it is immaterial how you get it once you have got it and it satisfies this equation you are perfectly.

So, let us look at that situation. Now, in the z greater than 0 that is in the medium which I have been calling as the dielectric; now what we could do is that the solution in this medium is the same as what one would obtained it one were to fill up the entire space with a dielectric medium having a dialectic permittivity epsilon one and replace it and putting a charge q prime at a distance d from the origin in the left hand side.

So, notice that what I have done id to this. Q is my original charge and this is at a distance d into the medium. Now, I am saying that this imagine that the entire medium is uniform medium and you put a charge q prime at a distance d r. So, that I have got a charge q and its image q prime. Now, since these are just two point charges in a dialectic medium epsilon 1; I can write down at any arbitrary point p the potential due to such a charge distribution and I notice that I can write phi 1 is equal to 1 over 4 pi epsilon 1 q by z minus d whole square plus rho square rho is nothing, but, x square plus y square and q prime by z plus d whole square plus rho square to the power half.

Now, notice that if you took the Laplacian of this equation, this gives you the singularity at the point where the charge q is situated. So, therefore, this indeed is the solution of the Laplace‘s equation that we set out or Poisson’s equation that we set out to solve.

Now, what happens to the field in the left hand side that is z less than 0? Once again I assumed that the entire space can be considered to be filled up by a charge by a dielectric epsilon 2 alone and in this case I substitute or I put an image charge which I have called as q double prime.

So, in other word q double prime gives me the solutions of the Laplace’s equation in this problem. And this is again written down 1 over 4 pi epsilon 2 because I have filled up the entire medium with a dielectric or permittivity epsilon 2 q double prime by z minus d whole square plus rho square t to the power half. Once again if you did the Laplacian of this you will find that this the Laplace equation is satisfied; there are no similarities in this problem.

The so, what I do now. So, what we do is this I have got two expressions. I have got an expression for phi 1 which is the field in the right hand side and I have got an expression for phi 2 which is an expression for the potential in the region to the left. And I need to satisfy the boundary conditions. And these boundary conditions are things which we have discussed a little while back namely the normal component of d is continuous.

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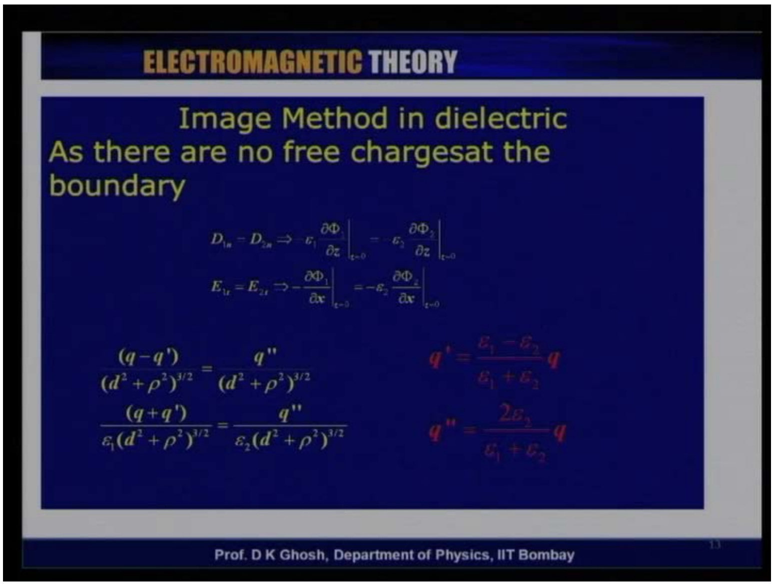

So, let us look at what is normal component of D. So, the normal component of D, now just do calculation for one of them.

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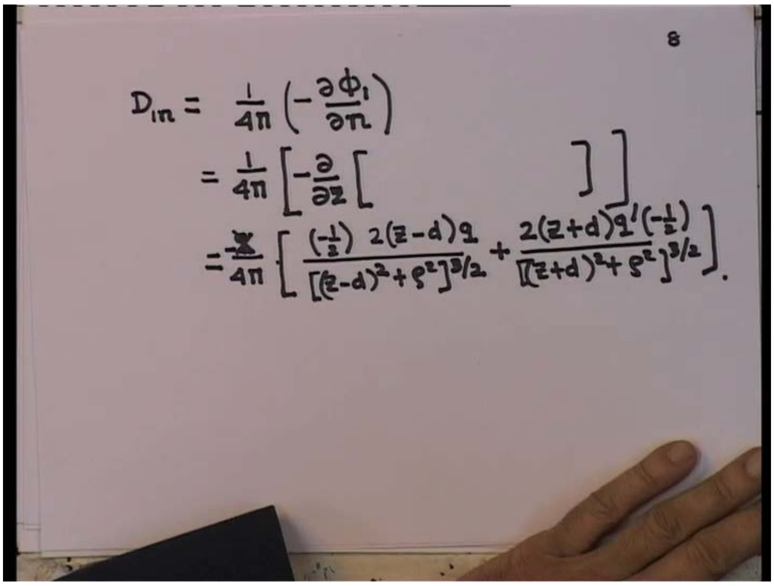

So, what is D 1 n I know that D 1 n remember that it is a linear dialectic. So, all that I need is epsilon 1 times E 1 m. So, that will cancel out this epsilon 1 that is here and I will be left with 1 over 4 pi. This is I know that negative gradient of the potential is what I need to calculate. So, it is minus d phi 1 by d r because this is a normal derivative.

So, since well d actually this is d n and since the normal direction is just the z direction. So, D 1 n is same as 1 over 4 pi times minus d by d z of this big expression that we wrote down. Now, this is a very easy differentiation to do because I have got 1 over z minus d whole square. So, you notice this is 1 over 4 pi minus sign I have got q. Of course, let me take the q outside I have got z minus d whole square plus rho square to the power 3 by 2 and a minus half and d by d z of this things.

So, which is simply two times z minus d and a very similar calculation for this one. So, let me put the q inside here and a q prime there. So, this will be simply replacing with 2 z plus d q prime and by z plus d whole square plus rho square to the power 3 by 2. And of course, minus a half again you can simplify this and notice that you have to take z equal to 0. So, that both these term will have the same denominator and as a result these two terms can be added up very simply. So, this will give you one equation. I do a similar job for the second term not working it out. And you notice that this will give me an equation of this type q minus q prime by d square plus rho square to the power 3 by 2 equal to q double prime divided by d square plus rho square to the power 3 by 2.

The second condition that I must satisfy is that the tangential component of E must be continuous and what is tangential component tangential component is differentiation with respect to either x or y remember rho square in my denominator is nothing, but, x square plus y square.

So, once again the it is a fairly straight forward derivation and with this you got these two equations they are fairly easy to solve. And what you find is that the this gives you the amount of two image charges that you have turns out q prime is epsilon 1 minus epsilon 2 by epsilon 1 plus epsilon 2 times q and q double prime is 2 times epsilon 2 by epsilon 1 plus epsilon 2 by q. So, this is a method of solving the problem at with image charges.

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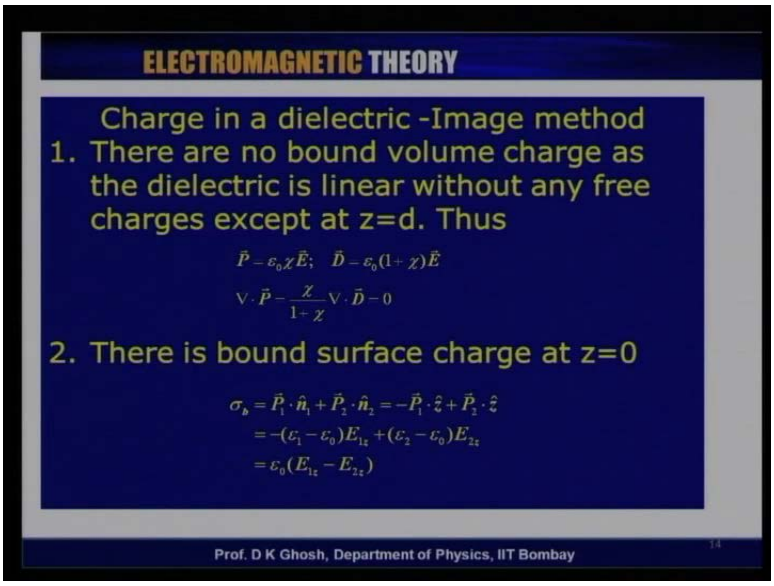

So, notice that what we have done is this. We have assumed that the dialectic is linear and there are no free charges excepting probably at z is equal to D and you notice this implies since its a linear dielectric. I have my polarization vector is proportional to E. So, that the dielectric vector D is proportional to E.

So, that del dot of P becomes equal to 0 and since del dot of P is equal to 0; the there are no volume charge densities. However, there is a bound surface charge and the bound surface charge you notice can be easily calculated because we had seen that the surface charge density is the normal component of the polarization vector.

So, that it is p 1 dot n 1 plus p 2 dot n 2 and I know that the direction of the normal is the z direction which is towards the positive z for the medium to the right hand side and towards the negative z for the medium on the left hand side.

So, therefore, one can write this as minus p 1 dot z plus p 2 dot z. We have just now written down that p is proportional to E. And so, I have just rewritten it here and you notice that since in the normal component of the electric field the D field is continuous that will cancel two of the terms there and I will be left with simply sigma b equal to epsilon 0 u 1 z minus u 2 z.

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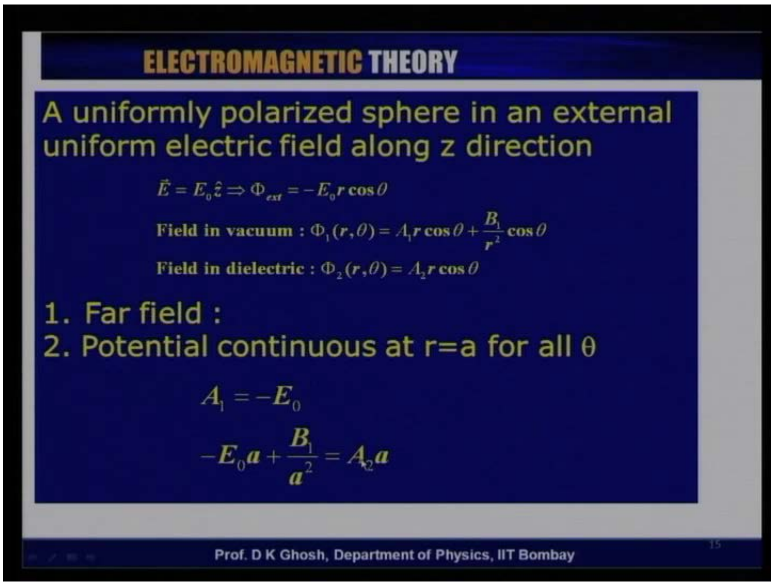

So, this is the surface charge density. So, this is the way one does use uses image charges for these problems. Let me look at a problem which once again. We did it for a case of a conducting sphere. We had seen how the field lines are modified near a conducting sphere if a it is put in a uniform electric field along the z direction. So, let me look at what happens to a dielectric sphere? Let me say that there is an electric field E along the z direction.

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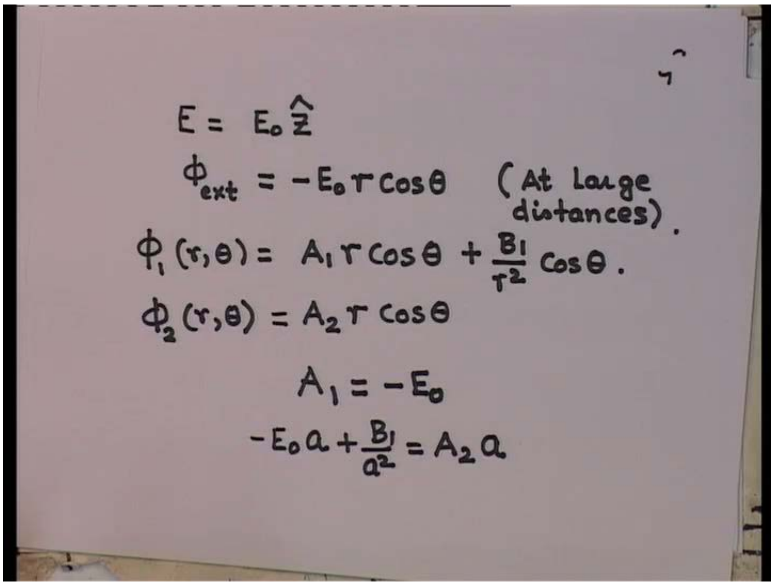

So, let me write it down. My electric field E is E 0 z the corresponding potential which should be applicable at far distances external potential at large distances should be minus E 0 z and since I am going to be using spherical symmetry let me write it as minus E 0 z cos theta.

So, this should be the form of the potential at large distances from the sphere. Now, we had seen that one can have a an expression for the potential. In general using associated legendary functions which is nothing, but, an expansion in various cosine theta cos theta cos square theta etc etc. Now, since I know that whatever is the form of the potential it must reduce to minus E 0 cos theta at large distances. So, what we will do is this that the field in vacuum has to be given by. So, let me call this phi 1 of r theta that is equal to A 1 r cos theta plus B 1 by r square cos theta.

So, this is the field outside the dielectric notice that all that can have is a cos theta term. So, that at large distances I recover this back and there is no 1 over r term because there is no point source of charge. So, B 1 by r square cos theta is what I have which is of course, a term which would vanish at infinite distance leaving me only with A 1 r cos theta. The field in the dielectric I should write in a very similar way excepting that the constant. Let it be different let it be A 2 r cos theta. I do not write down B 2 by r square cos theta because the point r is equal to 0 is part my dielectric.

So, as a result the I cannot have the potential diverging at that point. Now, first thing is to notice that at large distances this must boil down to minus E 0 r cos theta. That tells me that the constant A 1 is nothing, but, minus E 0. The other thing to notice is that the potential is itself is a continuous function because if it were not then the field at the surface of the dielectric would become infinite.

So, since the potential is continuous for all theta. So, we put r is equal to a in these two expressions and equate them. So, that tells me that minus E 0 a plus B 1 by a square is equal to A 2 a cosine theta has been cancelled because this relationship must be valid for all angles. Theta the third condition is there are no free charges on the surface.

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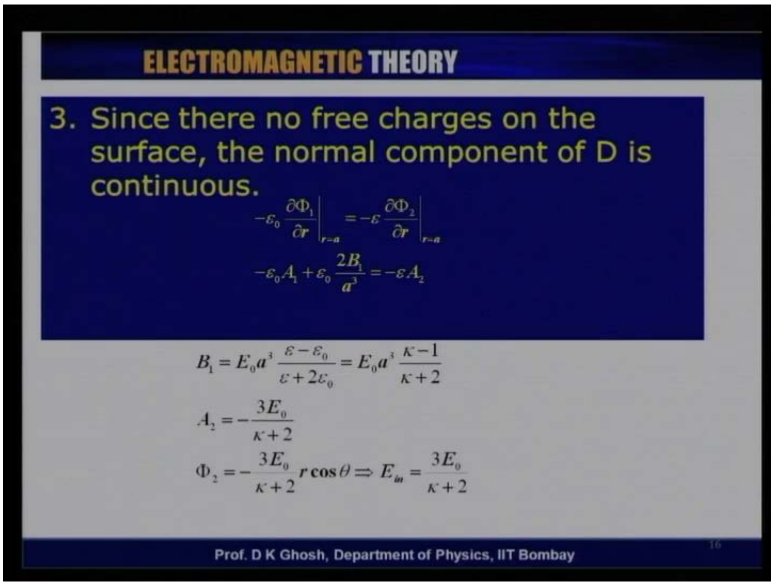

So, as a result the normal component of D must be continuous. That gives me a another relationship that I am looking for. the If you solve these expressions you can write down what are a A 1 A 2 and B 1 and get an expression for the potential phi 2 for instant and get what is electric filed.

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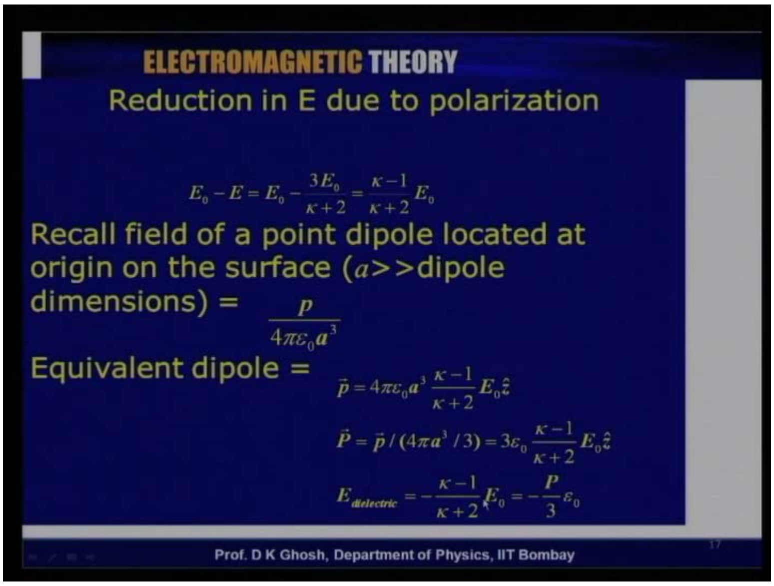

What we will do is that take this result next time and find out what is the effect of the polarization of the medium on the electric field lines that are there due to the external uniform electric field.