Electromagnetic Theory
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Module - 1
Elements of Vector Calculus
Lecture - 2
Line and Surface Integrals

In the last lecture we had introduced the subject of vector calculus, and we discussed fields in particular scalar and vector fields. We also brought out the meaning of gradient of a scalar field, and I emphasize that the gradient of a scalar field is a vector field. Today I would be spending a little more time in discussing special functions of vector fields or operators which act on the vector fields. But before that I would introduce what are known as the line and the surface integrals.

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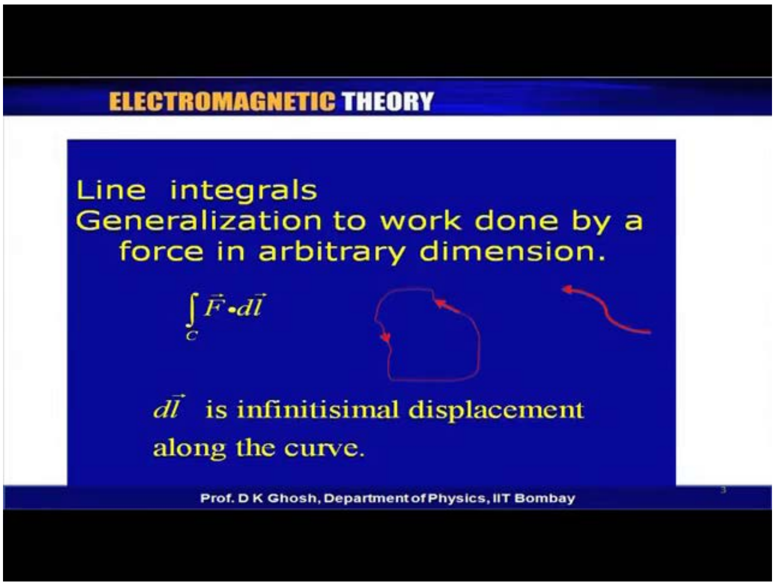

We are all familiar with normal integrals, as you are all aware that an integral is always regarded as a sound. Now, what we want to do is this, that lets first talk about a force which is acting on a particle and is in one dimension.

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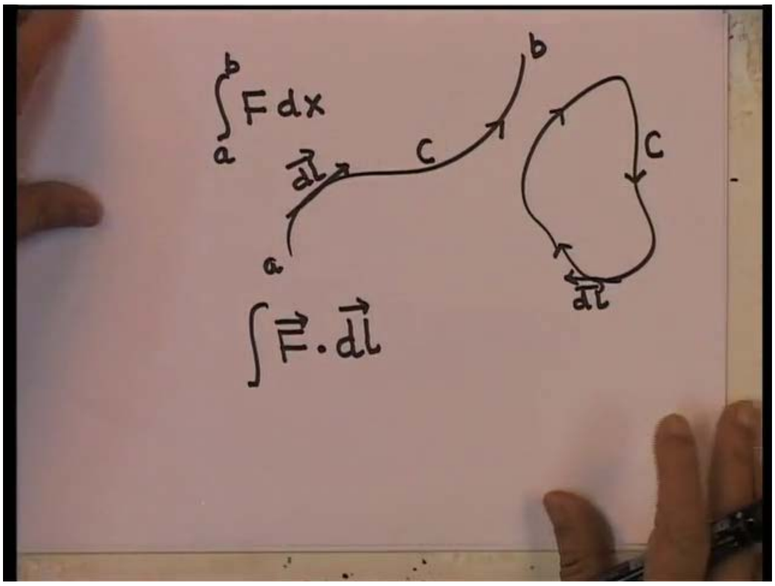

We are aware that integral F d x, if it is a one dimensional problem gives me the work done in let us say moving the particle from a point a to the point b. Now in computing this work done, we do a simple integrations. However, suppose my force acts on a particle and takes it along a curve. It could be two dimensional or it could be even a three dimensional path. It could be a open curve like what is being shown here or it could be a close curve for example, like this. Question is how does one compute the work done in taking the particle let us say from a to b in this case or in a circuit in the other case.

Now, we will represent the work done, let us call this curved size C. C is standing for curve, it could be a closed curve or an open curve. The work done as we are aware is given by dot product of F with d l where d l is along the path. Now, in particular the vector d l is directed, it is an infinitesimal element directed along the tangent to the path. For example, d l is directed like this here, d l is right directed like that there. Now, so what we want to do is to generalize the concept of computing work in one dimension to determining how much of work has been done, when a path, when a force acts on a particle and takes it from one point to another or in a circuit.

Now, the point is in general in general the this integral that is integral of F dot d l from a to b depends on not only on the end points as we are accustomed to while we compute integrals, but it also depends on details of the path taken. Of course, there are exceptions to this for instance if the force is a conservative force, exact details of these we will talk about later in the day today, but if it is the conservative force.

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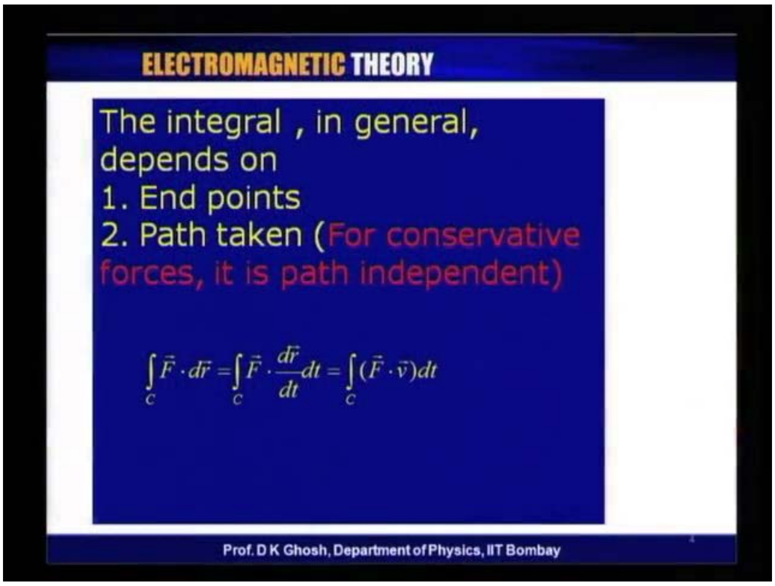

For instance a gravitational force or electrostatic force, then it can be shown that the work done is path independent. However, in such a case it would always depend upon the end points. Now, let us look at what, how does one write this.

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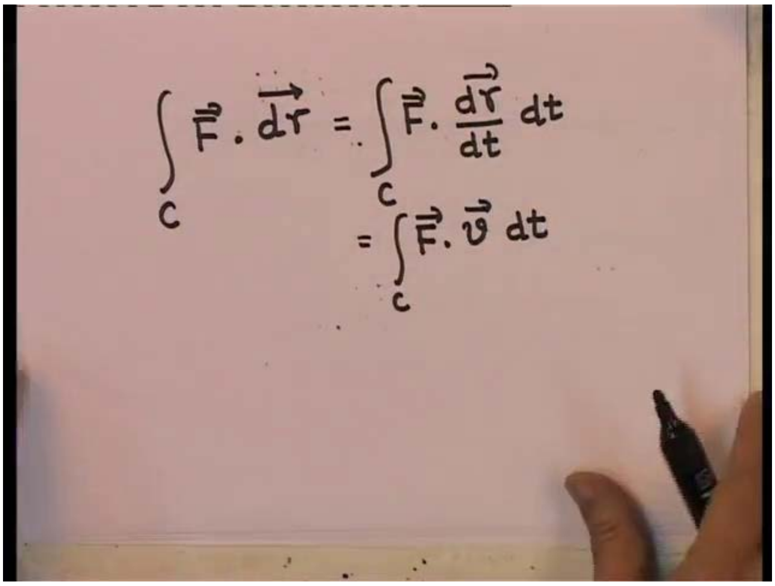

So what do we have is along the curve F dot d r let us say because d r is a an element along the curve and what we do is this that let me write this as F dot d r by dt dt where I have introduced time as a parameter time as a parameter and you notice that this is nothing but the dot product of force with the velocity with the velocity and I take a time integral. So, in other words what I have done is to paramatrize that with time. Now, along a curve along a curve I always had a single parameter and this is what we are going to exploit in computing the work or in general an integral like F dot d l along a given path.

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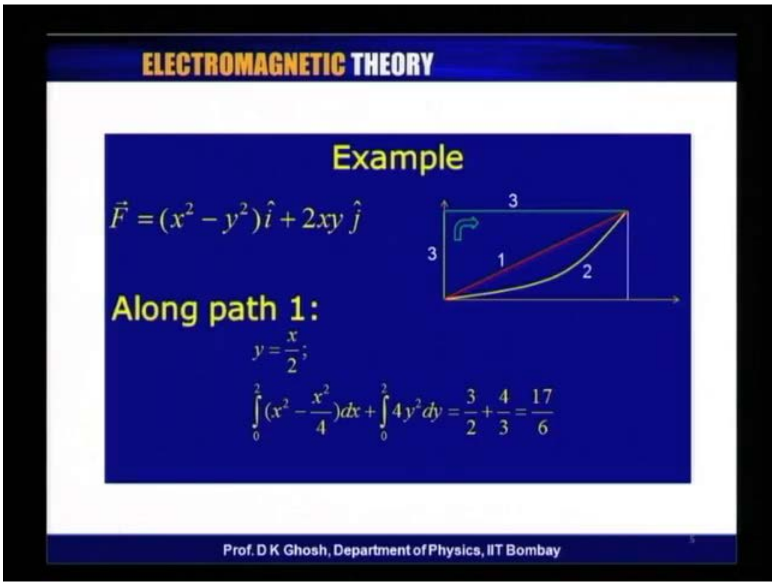

Now, let me start with an example.

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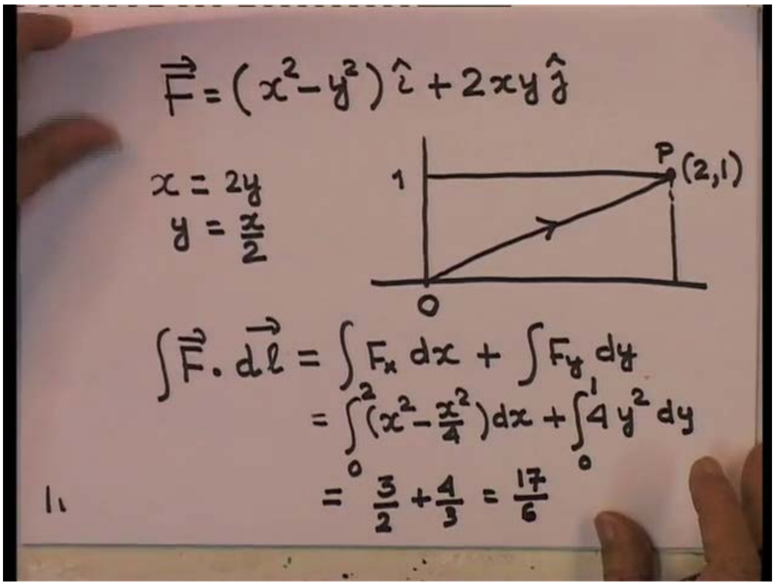

So, for instance I have given here F force which is written as F is equal to. Now, I have taken in two dimension. One could easily extend the curve to three dimension, but let me illustrate this by taking a force whose x component is x square minus y square and whose y component is two times x y. Now, what I am going to do is to compute the work along three different paths. In both the cases I am going from a point which is origin, let me draw it here again, origin to a point 2 1. So, supposing this is 1 along the y axis and this, the point 2 along the x axis, so from origin to the point P which is the point 2 1. First I will calculate the this is the force, the line integral of this force along a straight line, joining the point O to B O to P.

Now, notice since this straight line joins origin to 2 to 1, I can easily find out what is the parameter or what is the equation of this line. The equation of this line obviously is x equal to x equal to 2 y alternatively y is equal to x by 2. Now, let us look at how does one compute this? Remember, integral F dot d l if you like is F x d x plus integral F y d y. Now, F x is x square minus y square, but we have seen that y is equal to x by 2. Therefore, it is x s square minus x s square by 4 d x, the limit of integration over x is obviously from 0 to 2. I write the y part again. So, F y is given to be 2 x y, but x is 2 y. So, as the result I get 4 y into y is y square d y and limit on y is 0 to 1. You compute this. This is this trivial integration, you will get this equal to be by 3 by 2 plus 4 by 3 which is equal to 17 by 6. So, that is the work done along this line. Now, let us compute the work done along line along this curve 2. Now, I have taken the curve 2 to be at parabola.

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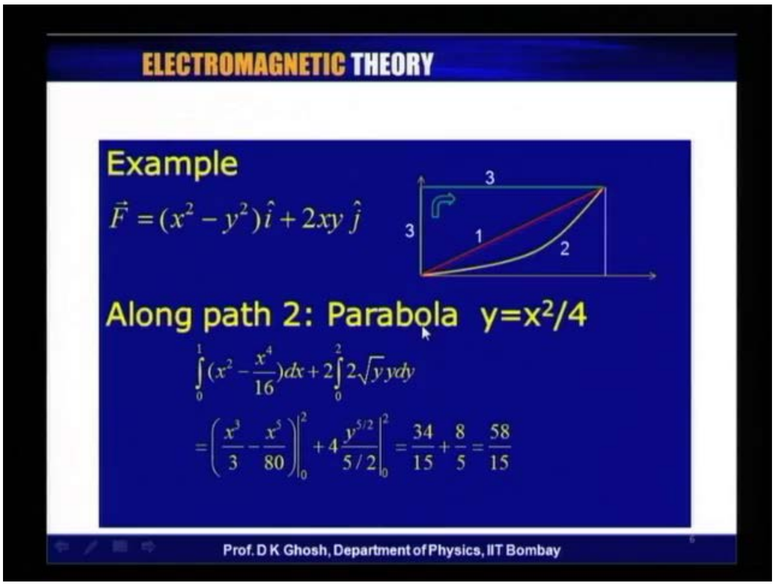

I have taken the curve 2 to be at parabola.

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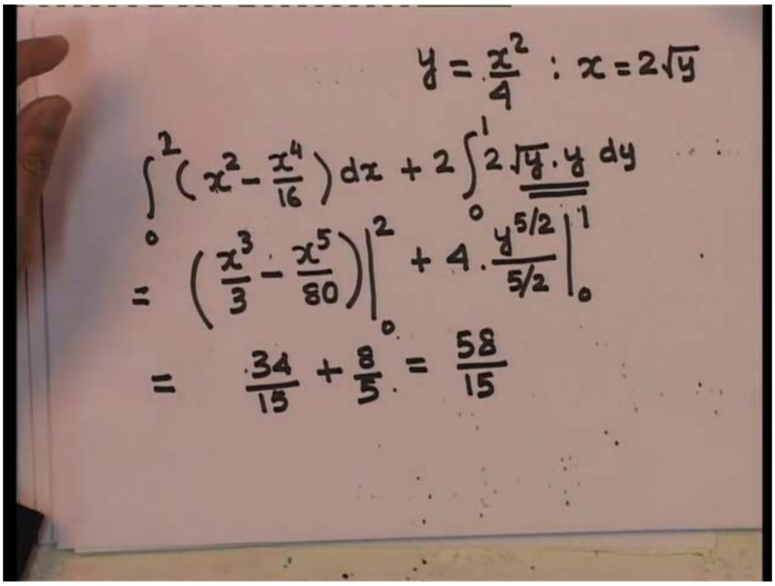

And my equation of the curve 2 is y is equal to x square by 4. I have given this, you can see that the it is of course, an equation to a parabola and when x is equal to 0, y is equal to 0 and when x is equal to 2 y is equal to 1, because it is 2 square by 4. Now, how does one do this for the same same force field or same of F, the vector field. So, once again I will compute this. So, this is now 0 to 1. I have got x square and I have got y square, but y is equal to x square by 4. So, y square is x to the power 4 by 16 d x plus two times integral x y.

Now, x is 2 root y. So, I have got 2 times 2 square root of 2 root y and I had already a y there d y and limits are once again the x limits are from 0 to 2 and the y limits are from 0 to 1. So, if you compute this now, this is very straight forward x square is x cube by 3, x to the power 4 is x to the power 5 by 5 which gives me x to the power 5 by 80. Let me draw from 0 to 2 and this is 4 integral of y to the power 3 by 2. So, I get y to the power 5 by 2 divided by 5 by 2, limits are from 0 to 1 and you can easily compute this. This happens to be 34 by 15 plus 8 by 5 and is equal to 58 by 15. Now, you already noticed something, that the value that we got for in the first case was 17 by 6 whereas, the value that we got in the second case is 58 by 15. Now, you can see that the value of the line integral indeed depends upon the path that we have chosen.

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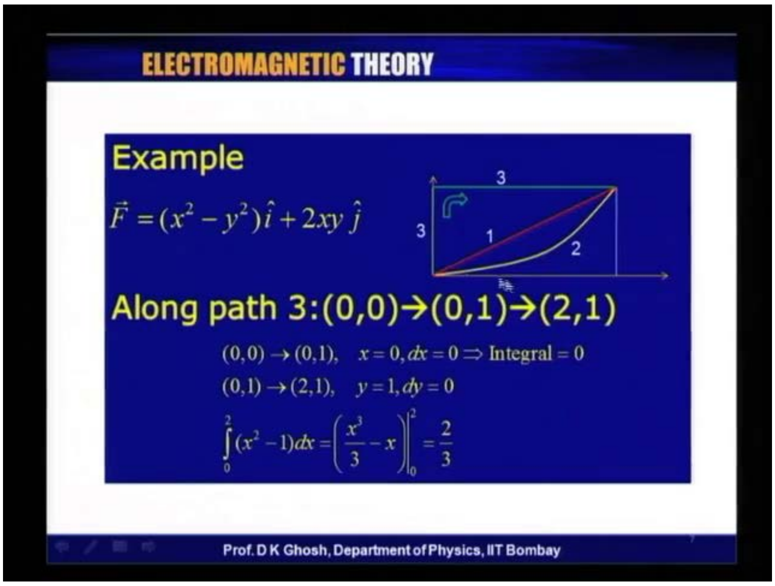

In the third case I am taking a path from origin to again to 1 by a little complicated path. I am first going from here 0 0 to 0 1.

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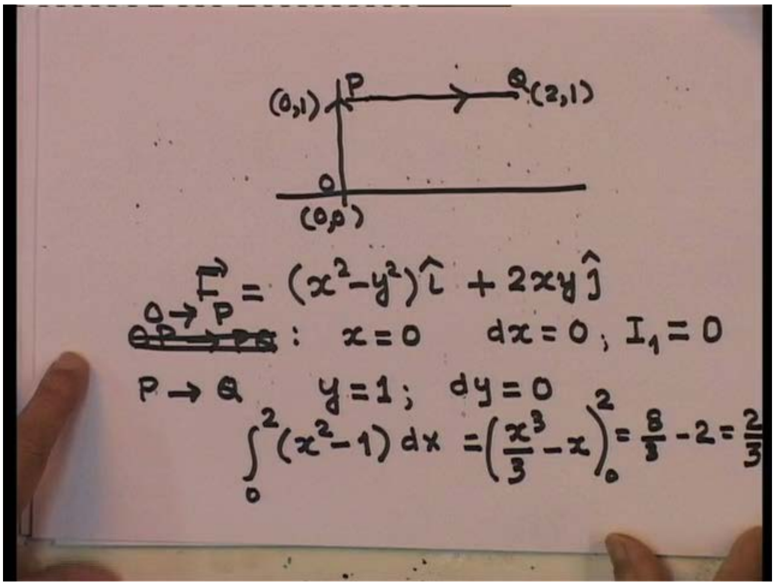

So, let us draw that path here. So, I want to go to 2 1 and I am going by a broken path like this in two steps. So, this is 0 1, this is 0 0. So, I go like that, first like this then like this. Recall my force is x square minus y square, x component and the y component is 2 x y j. Now, look at this when I go from origin to this point. Let us let us call this O P Q. So, as I go from O to Q my x remains 0, the value of x does not change. Now, if x remains 0 further along this path, along this path d x is equal to 0. So, notice that if I now compute this integral from O to P, integral of F x d x that is x square minus y square d x that gives me 0 because d x is 0. F y d y namely 2 x y d y gives me 0 because on this line along the entire path my x is 0. Therefore, contribution to the line integral from the path O P. So, let me write that as integral I 1 is equal to 0.

In the second path I go from P to Q. So, I will go from what I meant is, I will go from O to P and then I will go from P to Q. Now, when I go from P to Q when I go from P to Q. On the P to Q my value of y remains constant y is equal to 1. Now, once y is equal to 1 d y of course, becomes equal to 0 because constant, they have differentiation as equal to 0. So, since d y is equal to 0. I am only going to have to compute F x d x. So, this is simply 0 to 2 x square minus y square which is 1 d x and this is equal to x cube by 3 minus x evaluated from 0 to 2 which is 8 by 3 minus 2 which is equal 2 by 3. So, you notice one thing that I calculated this integral integral F d l, F dot d l in three different paths, 1, 2 and 3 and every time I got a different result.

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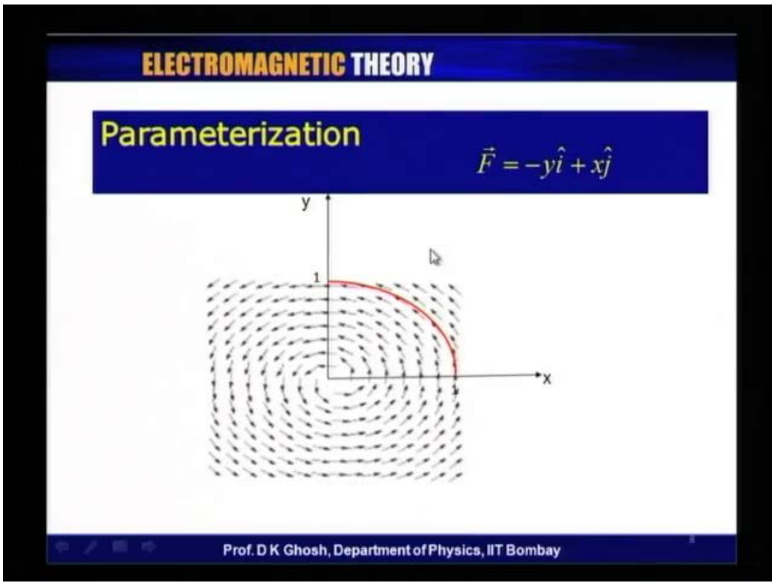

Now, I will talk about a slightly different technique now. I will return back to the same integral with a slight modification as I go along. Now, look at this field. Remember, we are talking about a vector field.

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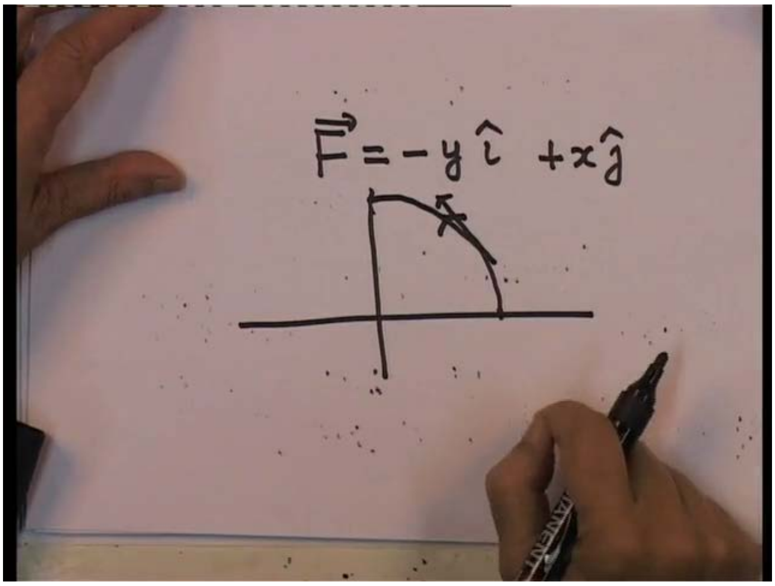

Now, this vector field is F is equal to minus y i plus x j. Remember, last time we gave a prescription for drawing the vector fields, sketching the vector fields. So, you could see this vector field, this is and I am going to compute the integral of this vector field go, in along a quadrant of a circle along a quadrant of a circle taking me from this point that is 1 0 2 0 1 in the anti clockwise fashion 1 0 2 0 1 in the anti clockwise fashion. Now, if you look at the sketch of the vector field that I have made the sketch of the vector field that I have made you will notice that in this quadrant in this quadrant the because x is positive because x is positive there is always a positive y component of the force field of the the vector field.

Now, as a result loosely speaking these vectors are all directed somewhat in the upward direction, very loosely speaking. Now, another thing that you notice is this that the vectors are directed in the upward direction and I am going to compute my line integral along this path and in this path the direction of d l like this. Therefore, the angle between the direction of the vectors, the, which we have shown in this sketch and the direction of the tangent to the curve always makes an acute angle. Now, since it makes an acute angle F dot d l which is the magnitude of F times magnitude of d l times cosine of the angle between the two remains positive. Now, it therefore, what I expect it is a clearly going by geometry I expect this line integral to be of positive value.

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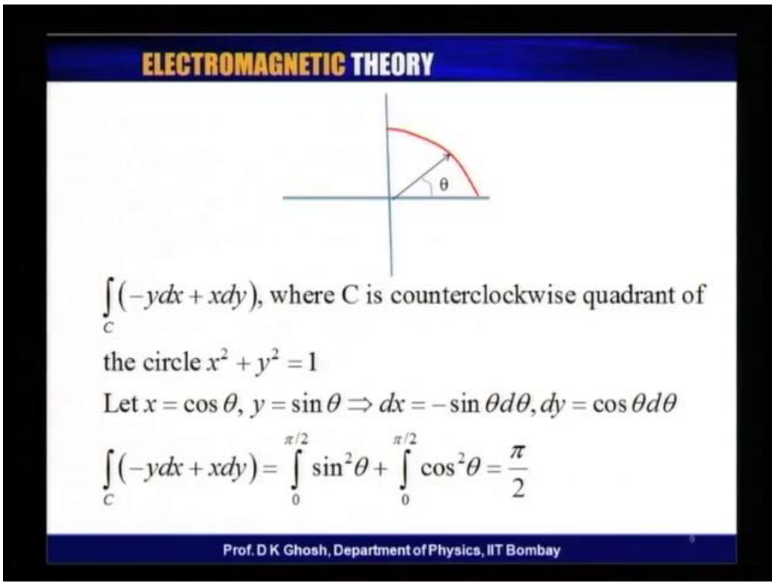

Here I will use a slightly different technique. The technique is that along the curve I I need to have a parameter. I have already said that a along a curve I have a single parameter. Now, notice that if I have this curve which is quadrant of a circle.

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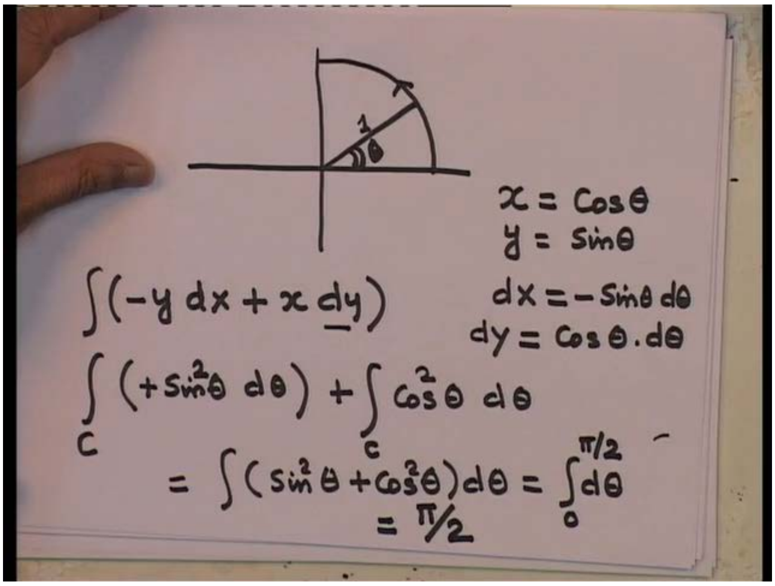

Now I can use a parameter like this using this angle theta as a parameter. Now, if theta is a parameter notice that in term this is a quadrant of a circle of radius 1, because I have said the equation is x square plus y square equal to 1 and I need to compute minus y d x plus x d y. Now, in terms of this theta since the radius of this circle is 1, I can write x is equal to cos theta and y is equal to sin theta. So, d x is minus sin theta d theta and d y is cos theta d theta.

So, you notice that this integral now which is over the curve C minus y is minus sin theta, d x is another minus sin theta d theta. So, minus minus plus sin square theta d theta plus integral again over the same (( )) x which is cos theta, d y is cos theta d theta so I get cos square theta d theta. Now, I can simply add them and get it as sin square theta plus cos square theta which is equal to 1 of course, d theta so which is nothing but integral of d theta and the limits on theta we know is from 0 to pi by 2. So, as a result, the result of this integration is simply pi by 2. So, this is the parameterized.

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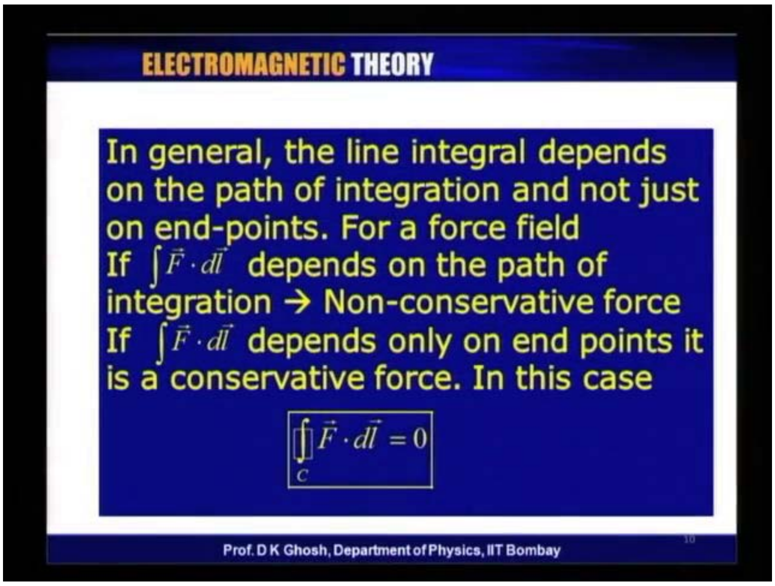

Now, in general as we said in the beginning the line integral depends on both the path of integration and the end points. Now, if the line integral F dot d l depends on the path of integration and if F is a force field then we say that this force is a non conservative force. That is a non conservative force is one where the line integral depends upon the path of integration. As we had seen in the first example that we gave you, that there were three paths along which we calculated the line integral and found the values to be different.

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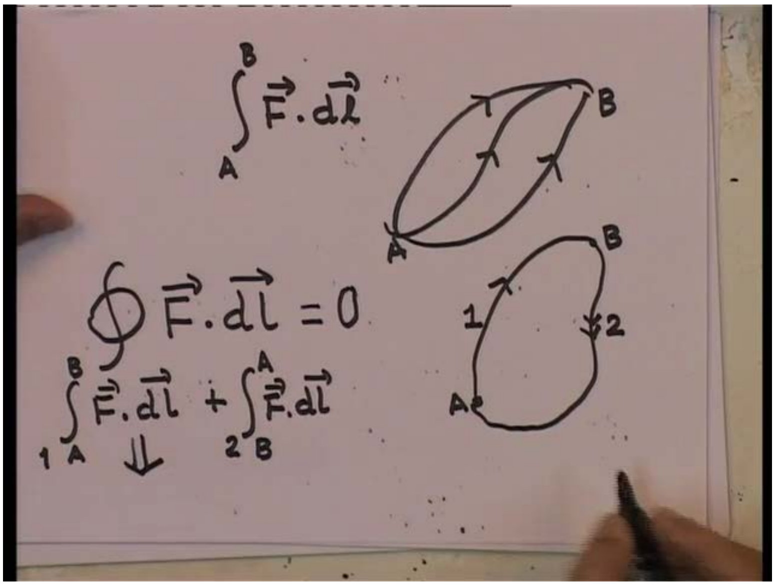

However, if F dot d l only depends upon the end points supposing I am going along some path from A to B and it does not depend upon exactly which path do I take, do I take this path or for example, that path or this path, I do not care, result is always the same. In other words this line integral in such situation depends only on the two ends point A and B. Such a force is called a conservative force such a force is called a conservative force and you can see that if I have a closed path. Supposing, I am calculating the line integral like this and closed path integrations are written like this with a circle over the integral sign it is also called contour integral, closed contour. For a conservative force for a conservative force you can show that this is equal to 0.

Now, the proof is really straight forward. Supposing, this is the path you are going starting from the point A you are returning back to the point B. Let me take any path arbitrary point B on the way. So, you notice my integral is from A to B by path 1 and then once I reach B I am following a segment of the path which is 2. So, we will say B to A by path 2. Now, what we have said is this that this integral from A to B depends only on the end points depends only on the end points. So, in other words this integral F dot d l.

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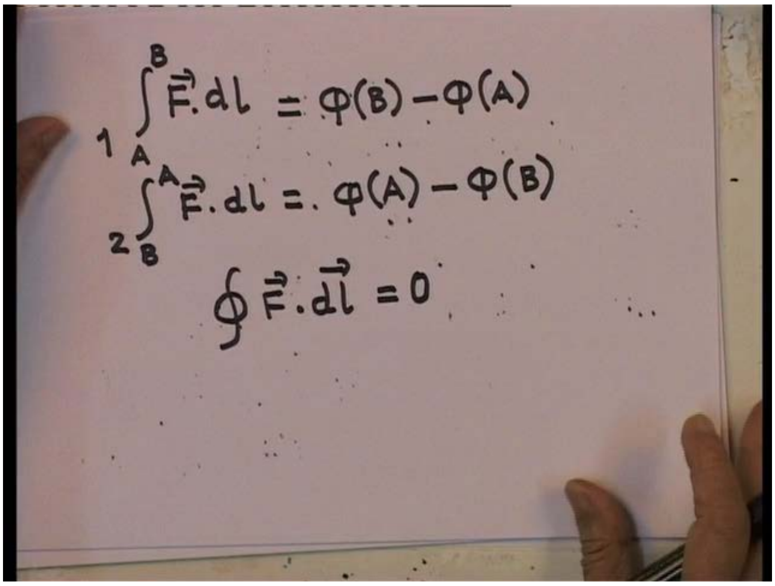

So, A to B F dot d l along path 1 can be written as something like phi B minus phi A. Just I evaluate this and then subtract from the value of that integral, the standard definite integral result at B minus result at A. Now, when I am going on path 2 I am going from B to A F dot d l. Now, once again we have said that it depends upon only the end points so as a result this is going to be phi at the upper limit minus phi at the lower limit which is B. Now, add them up. Obviously the integral over F dot d l over the contour is equal to 0. So, for a conservative force the integral, the line integral from A to B depends on only the end points and if you take the force field through a closed contour the line integral is identically equal to 0. Now, let us spend a little more time on the conservative force field.

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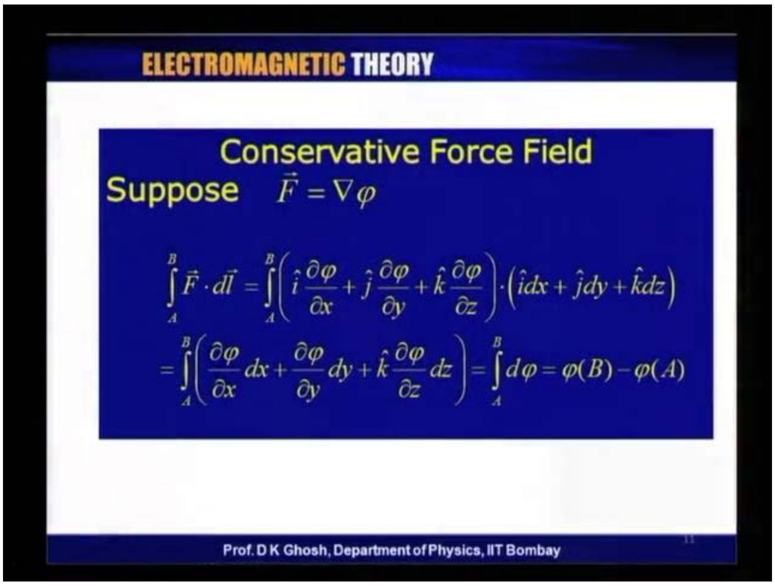

The reason is that the field which is of importance to us in this course namely the electrostatic field is a conservative field, so is gravitational field.

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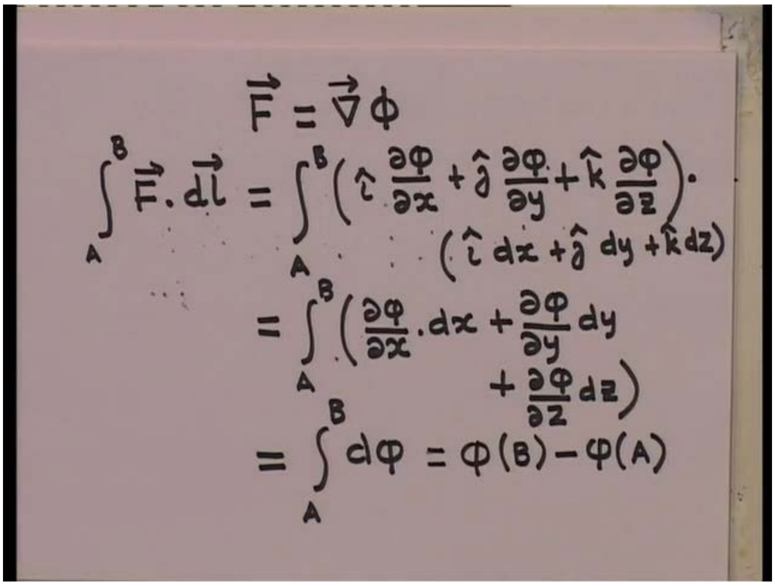

Now, suppose I have a force field vector field which can be written as a gradient of a scalar field. Force field can be written as a gradient of a scalar field. Now, remember we had said the gradient of a scalar is a vector. Now, I am not saying that all force fields can be expressed as a gradient of a scalar field, but supposing it can be done. Now, let us see what it means. What it means is this, supposing I am going from A to B again I have to calculate F dot d l. Now, since F is a gradient you remember the expression for the gradient of a scalar function in Cartesian and we had seen that this is equal to i partial phi with respect to x plus j partial phi with respect to y plus k partial phi with respect to z dotted with dotted with d l which is obviously i d x plus j d y plus k d z.

Now, take the dot product. Remember i dot i is 1 j dot j and k dot k are also one, but i dot j i dot k etcetera are 0. So, what I am left with is A to B d phi by d x partial phi partial x d x plus partial phi with partial y d y plus partial phi with partial z d z, but this quantity d phi by d x d x d phi by d y d y plus d phi by d z d z is the differential of the function phi itself. So, in another words this is from A to B just d phi because this expression includes changes due to variation in x y and z. Now, this is the an exact differential so as a result the integration is very trivial, it is phi B minus phi A. Notice, what we have done, we have said or we have proved that if a force field is expressible as gradient of a scalar function then integral F dot d l only depends upon the two end points. In other words such a field is conservative field, such a field is conservative field.

Now, we make a statement at this stage that if I have a conservative field of force, for instance an electric field or a gravitational field I can always express the force as a gradient of some function.

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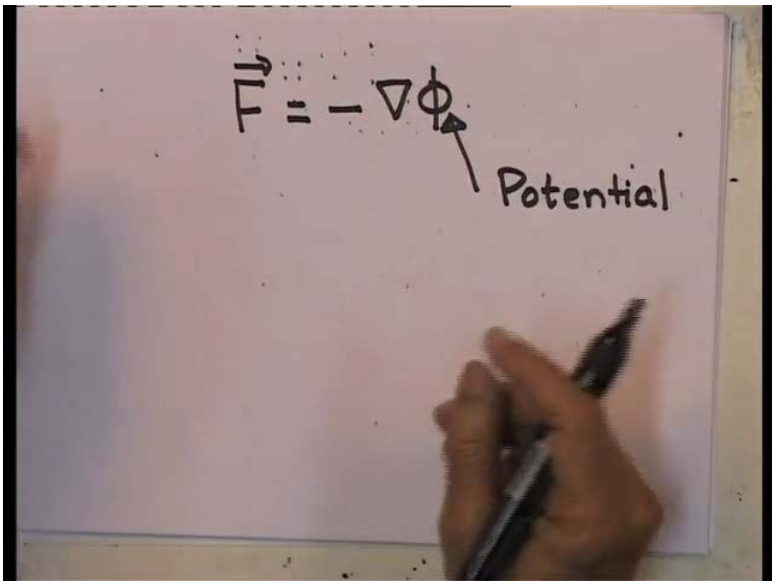

Generally, because of various reasons instead of writing F as equal to gradient of phi, we define F as equal to minus the gradient of phi and this phi is then given a name called potential, this is conventional, but in principle a mathematician would call phi as a potential whether you have a minus sign in front of it or not. But we will of course, in electrostatics will be using this convention, namely force is negative gradient of a potential function and the potential is a scalar function the potential is a scalar function.

So, this is about line integrals. So, we have seen line integrals in general depend upon the path that we take. In some cases when they do not we call the vector fields as non conservative field in which case the line integrals only depend on the end points, line integrals only depends upon the end points. So, line integrals was essentially one parameter (( )) though I am not saying it is in one dimension, but I go along a line, I go along a curve. So, there is a single parameter in such a situation. Now, let me now extend this now let me now extend this to define what is known as a surface integral.

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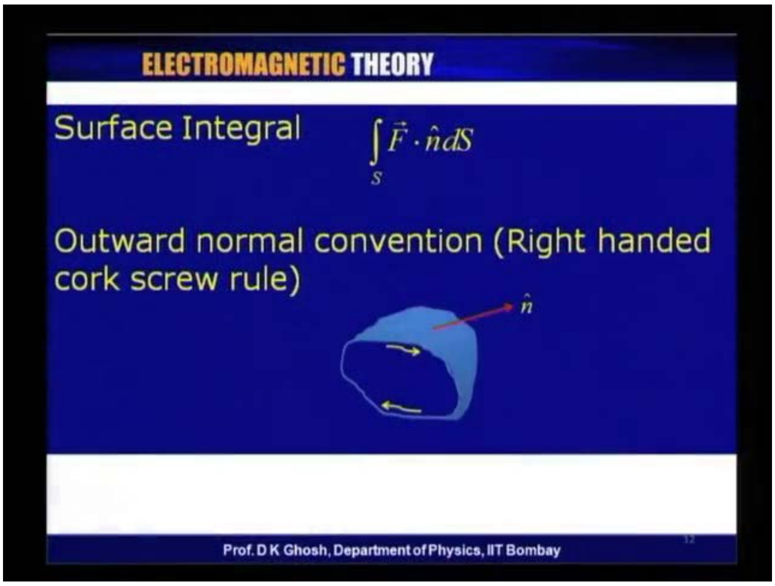

Then surface integral we will write this surface integral as integral of F dot n.

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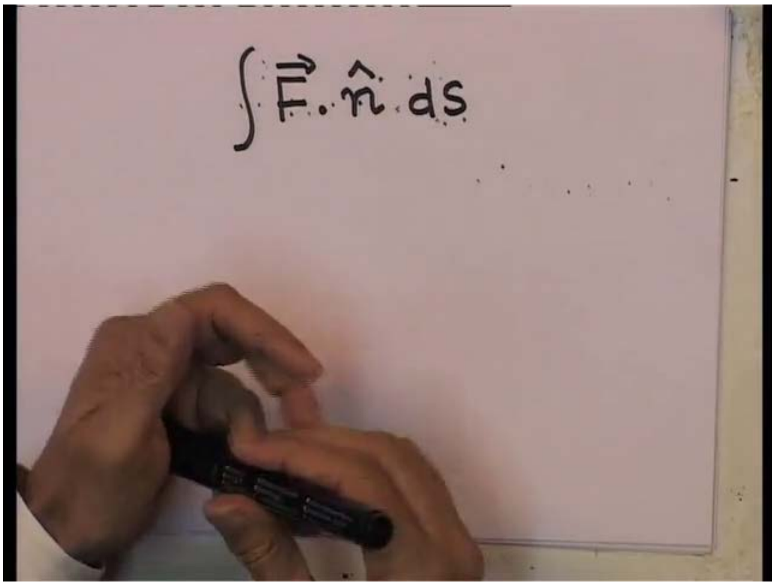

I will explain what it means d S. Now, in order to understand the surface integral look at this situation, let me take an open surface for example, a hemisphere which is an open surface, a better situation is to have a net.

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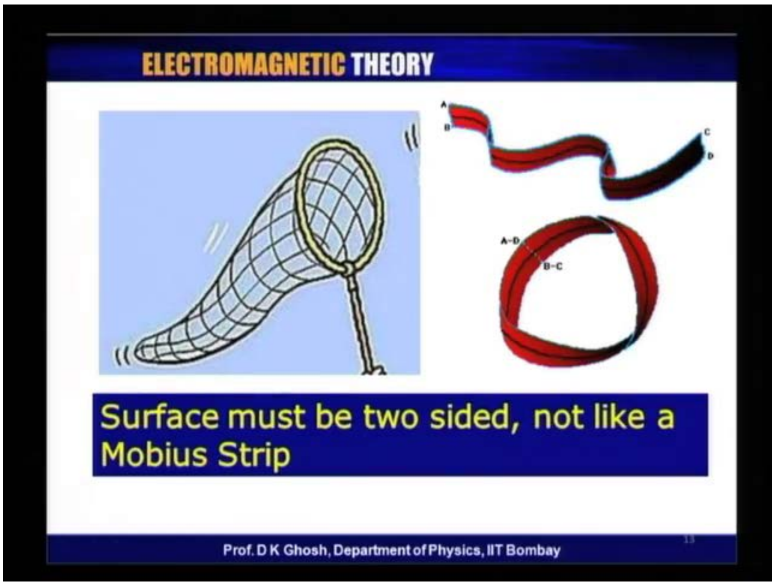

For example, a butterfly net as it is called, you notice that there is a rim here there is a rim here and there is a net which is bounded by this rim and this is an open surface in the sense that this surface is there only to one side of the boundary. Now, this is exactly the picture that you gave in the other one. So, I have an open surface and so what we do is that whatever is the bounding curve of the open surface. So, imagine I have a cup imagine I have a cup is a hemispherical cup or even an ordinary cup it does not matter. Now, this is my bounding surface this is my bounding surface and a curve which binds it, again going back to my butterfly net this is what it is, this is the rim this is the rim and this net is the surface. Now, when that happens we say that direction of this end, it is a unit vector which is perpendicular to the surface. Now, it is not a constant vector it changes from point to point it changes from point to point.

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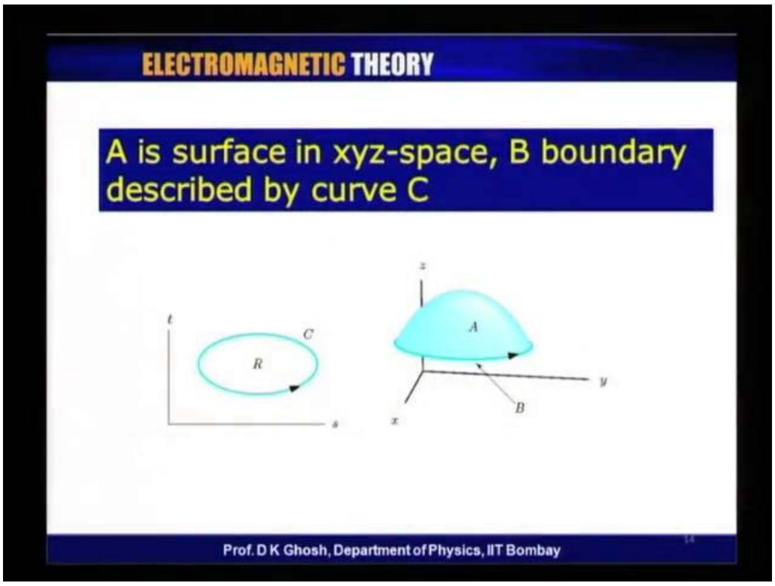

So, for example, you are at this point. So, draw a perpendicular at that point and draw it outward. Now, what do is meant by outward because after all the outward needs to be defined. The, so what we do is we adopt a convention which is called a Right handed cork screw rule which is with which you are all familiar, that is if I am rotating or if I am moving suppose this curve is being turned along the direction of a cork screw then the direction in which head proceeds is the direction of my normal to the surface, the outward normal and so that is why I said a butterfly net is very clear. Now, there is some some issues connected with it here.

Number one that I need to have what are known as two sided surface, in order that my definition of a surface integral makes sense I need a two sided surface. What is a two sided surface? For example, an open hemisphere it has an inside surface and an outside surface, you take a cup, you have an outside surface and an inside surface. Now, on the other hand there are certain surfaces which are one sided surface and for example, one of the examples is what we call as a Mobius strip, it is diagrammatically shown here, but let me try to explain what is a Mobius strip.

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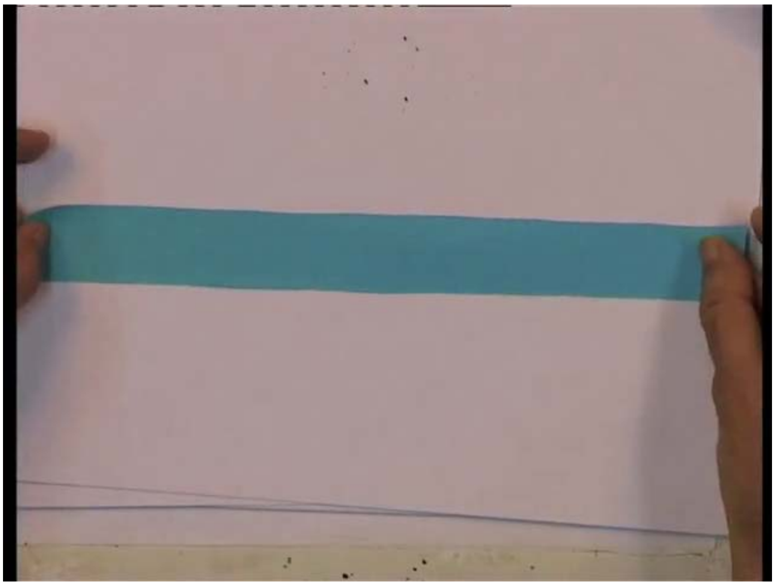

Now, I have this strip of, blue strip of paper. I have taken a small strip with it. Now, notice that if I just fold it like this making at a ring then I have two sides in other words supposing I am on the upper surface and I want to go to the lower inside surface then I have to always cross an edge I have to always cross an edge and this is known as a double sided surface.

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This is true of for example if I take this glass, at this moment there is water in it, but this is the outside surface and there is an inside surface of the glass. If I want to go from the outside to the inside, I will need to cross this rim, I will need to cross the edge, but on the other hand please look at it this way. Supposing, I take this again, this is diagrammatically shown here. Supposing, I take this strip again and instead of folding it like this I give it half a turn, I give it half a turn and fold it in this fashion, fold it in this fashion that is the same side does not get stuck. I will get, so I have actually already stuck it with a glue and this is the type of picture that I get.

Now, notice what is so special about this. The special about this is I can come from any point on this surface and traverse both sides, both inside and outside by ever having to cross this rim. So, in other words at a given point the outward normal is not uniquely defined. The outward normal can be both outward like this or inward on the inner surface. This is a one sided surface so for such surfaces this exam this a called a Mobius strip.

My definition of a surface integral is not valid so let me talk about normal surfaces which are two surface, two sided open surfaces and in going from one side to the other side I need to pass through a rim, need to pass through an edge. Well here the same thing the this is the hemisphere, surface is A and the rim of the hemisphere is the curve C and I have to sort of define that if I am going in a particular direction then the direction which the head proceeds will be the direction of the normal, outer normal. So, that is about two dimensional surface integrals.

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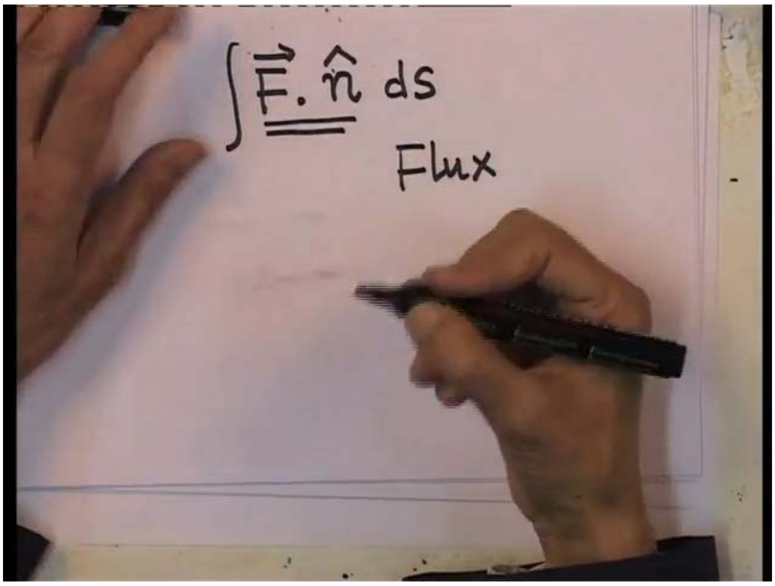

Now, one can now go and define. Now, remember the surface integral is dot product of a vector field with the outward normal and integrated over the surface, this is usually called a flux of the force field F through this surface flux of the force field f through this surface. Now, we can now extend this concept to a three dimensional integral, but this is much simpler because the integral is simply a volume integral over a scalar function, a scalar function is phi and so this is nothing but normal triple integration which you are familiar with.

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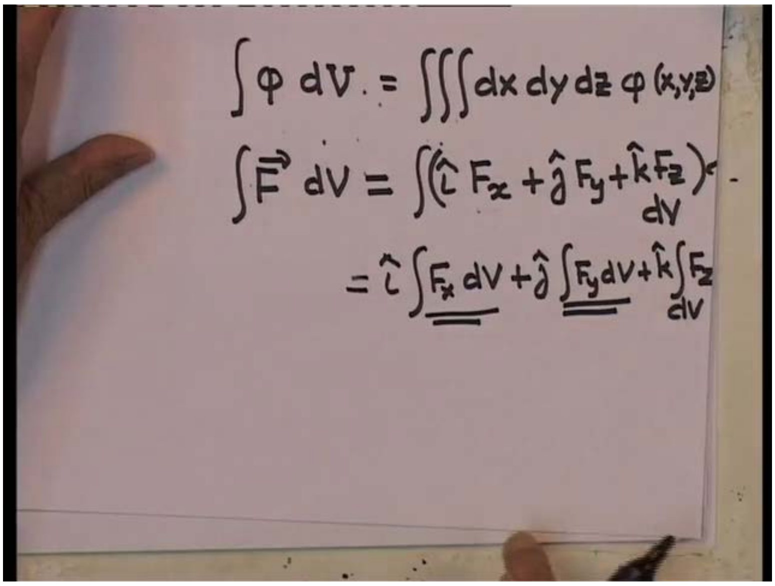

So, volume integral phi d V, V here is the volume is nothing but triple integral over x y z of phi of which is a function of x y and z. Now, I can define another type of triple integral or volume integral which is F d V that is an integral of a vector field over a volume. Now, this is to be interpreted as actually three integrals because I can write my F as i F x plus j F y plus k F z d V. So, this is nothing but i integral F x d V. Now, remember F x is a scalar function plus j F y d V plus k F z d V and each one of them we have just learnt what it means because this is just like any other volume integral of a scalar function which is nothing but a triple integral. Therefore, this is i times 1 triple integral plus j times the second triple integral plus k times the third triple integral.

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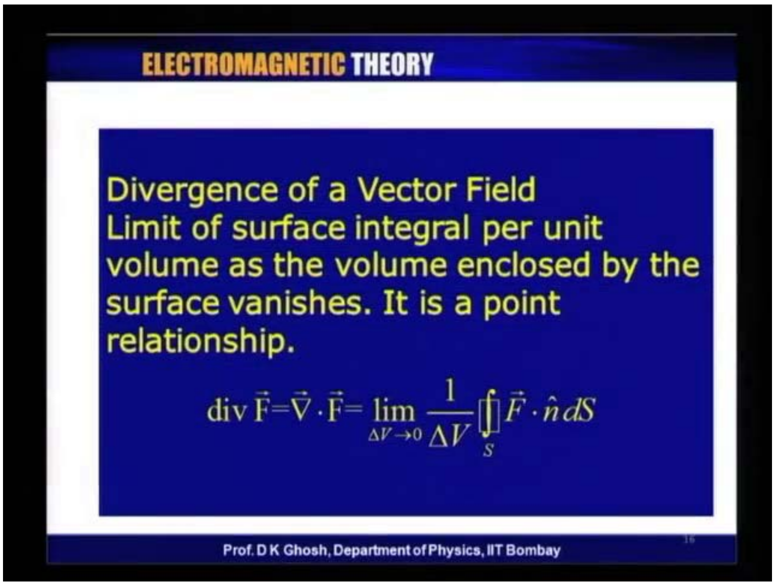

So, this is all about the line surface and the volume integrals. Now, let me concentrate a little more or spend a little more time on vector field once again. Now, I define what is known as divergence of a vector field. The reason for the name will become clear in a little while. Now, I define divergence of a vector field as a limit of the surface integral per unit volume, I will explain this with figures as I go along, as the volume enclosed by the surface becomes smaller and smaller becomes infinitesimally smaller.

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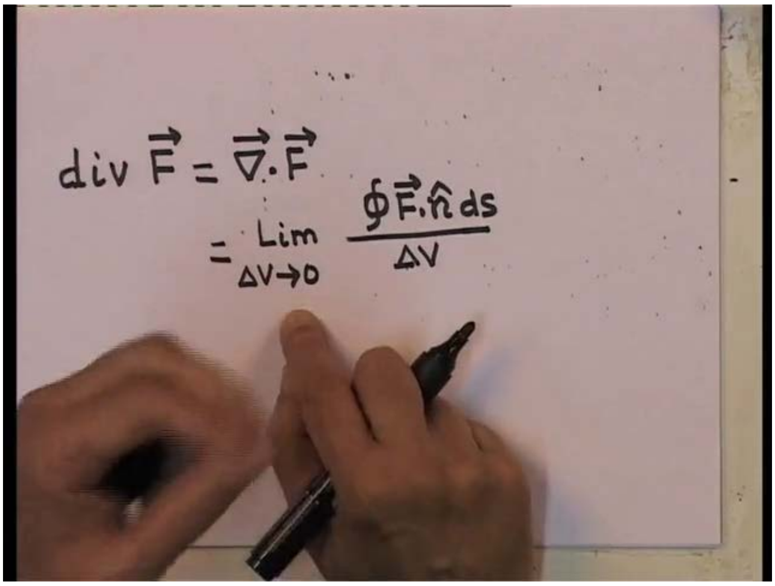

Now, let us look at this relationship a little more carefully and now divergence of the vector field is written as a div, div F, one also writes it for reason to become clear as del dot F both of them are identical ways of expressing divergence. So, what I have said is this, that now this is a point relationship, this is a point relationship. In other words you define divergence of a vector field at a particular point.

Now, supposing around that point around that point I think of a small volume, infinitesimally small volume described by a surface d S. Now, so what we do is this, take this volume to be infinitesimally small. Let us call it delta V. So, I define di del dot F as limit of delta V going to 0 of the surface integral F dot n d S. So, I have taken a closed volume, very small closed volume at the point where I am computing the divergence of F. So, that is that is why this circle. It, circle essentially means it is closed. So, over a closed surface I have taken and divided by the volume enclosed by that surface, take this as the volume becomes smaller and smaller I reduce that volume essentially to that point and that is why I say, it is a point relationship.

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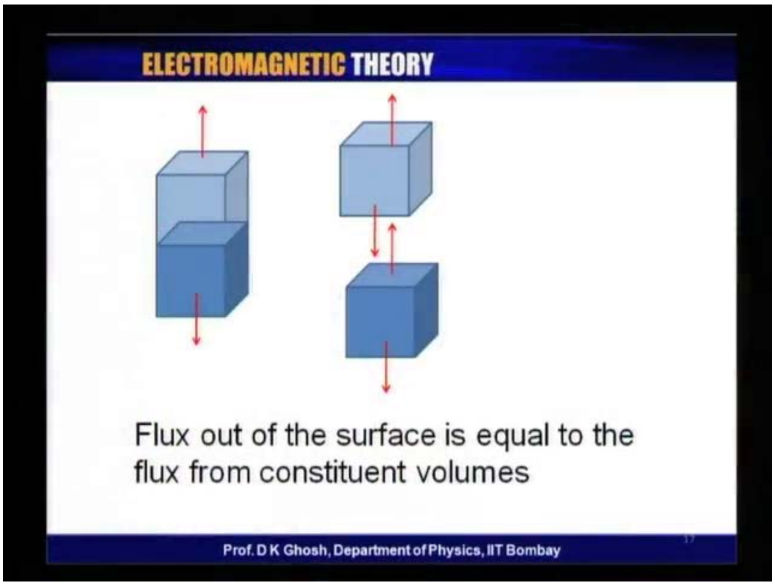

Now, this picture is what explains what happens. Now, supposing I have a volume bigger volume. Now, what do I do? Let us consider a volume. Now, I can think of splitting that volume into a large number of infinitesimally small volumes. Now, let us look at what I am trying to do in the process. Now, look at this supposing I have a volume of this type, this is a rectangular parallel bite and I have a volume and notice and at this moment concentrating only on two surfaces, though it has six surfaces I have a upper surface whose direction is outward normal is this direction and a lower surface the this surface the bottom surface whose outward normal is directed like this. Now, supposing this surface I am to split into two smaller surfaces.

So, in other words this big surface, this big volume gets cut into two, I take a section and just split it. Now, when I split it remember nothing has happened to the direction of the normal to this surface. Nothing has again happened to the normal to the bottom surface, but once you cut it, in this section, in this section the normal is directed like this and in this section the normal is directed like that, I do a mental exercise. So, if I just cut it since the vector field is the same over this surface which I am going to cut then the contribution due to the surface integral from this surface and this surface because they are directed oppositely will become 0. So, what it means is that if you have a volume, if you have a volume and you split that volume into.

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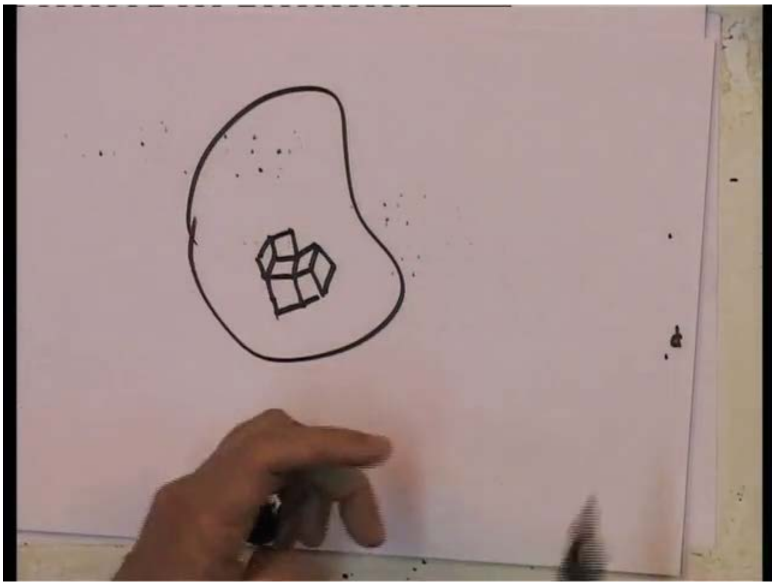

So, supposing I have a volume like this, and I split it up into smaller volumes. Now, I can reintroduce, talk about the same argument once again and say that the contribution from things which are inside the pairs will always cancel and what will I be then left with? I will be left with only contribution from the outside surface only contribution from outside surface.

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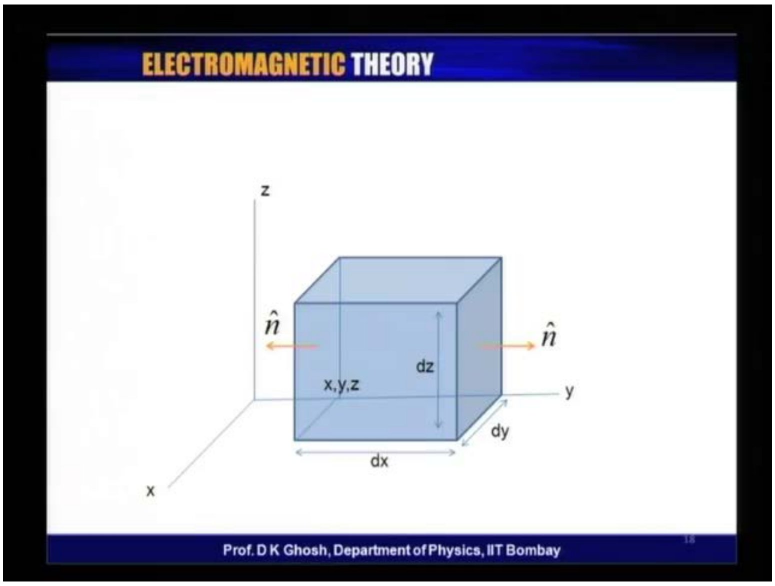

Well, let us now try to compute, we have said that divergence of a function is the ratio of the limit of surface integral over the vector field at a point, I have said it is a point because I have an infinitesimal volume there, bounded by an infinitesimal surface. I divided by the volume and I let the volume and the surface go to 0 and this ratio is finite and that quantity I call as the divergence of the vector field at that point. So, what is this divergence of a vector field?

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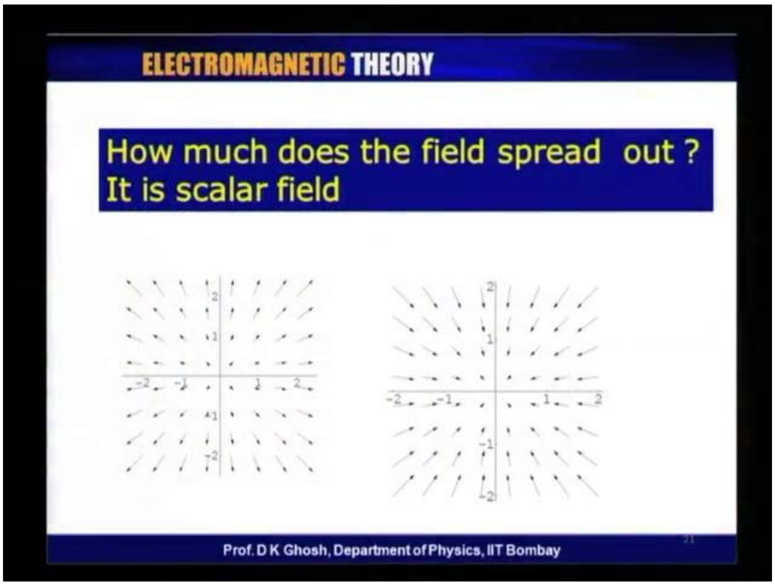

This picture here sort of gives you an idea of if here, we have drawn the pictures of the vector field. You notice as the name suggests, the field gives an idea of how much the vector field diverges from a point. The situation becomes clear when we look at for example, a velocity field in a vector. We know that if there are, if I look at a closed region then if the fluid is flowing in to a point, then of course the density inside would increase. And similarly, the fluid is flowing out the density would decrease.

So, here is an example where the field is spreading out from the origin. So, here the divergence is positive. In this case however the divergence is negative. This is the physical meaning of a divergence of a vector. So, to summarize what we have done today is we defined a line integral and emphasized the fact that in general the line integral of a vector field could be path dependent. In case it does not depend on the path it is called a conservative field and we have seen for a conservative field I can write the force as a gradient of a potential.

We defined a surface integral and also defined the divergence of a vector field through a ratio of a surface integral of a closed small surface at a point with the infinitesimal volume that it encloses. Next time we will be obtaining an explicit expression for the divergence in Cartesian coordinate system.