Electromagnetic Theory
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Module - 2
Electrostatics
Lecture - 19
Special Techniques III

During the last few lectures, we have been talking about some special techniques, which solve electro static problems. We had seen that these special techniques are extremely useful, because of what is known as the uniqueness theorem while solving the Laplace's or the Poisson's equation which tells us that the solutions of these equations which satisfy a given boundary condition happened to be unique. These special techniques they allow us to guess a solution without having to rigorously find the solutions as we try to do in some cases.

Method of images which we have been talking about is as we have seen we have applied to the case of plane conductors. In the last lecture we had seen how it is adapted to the case of spherical conductors.

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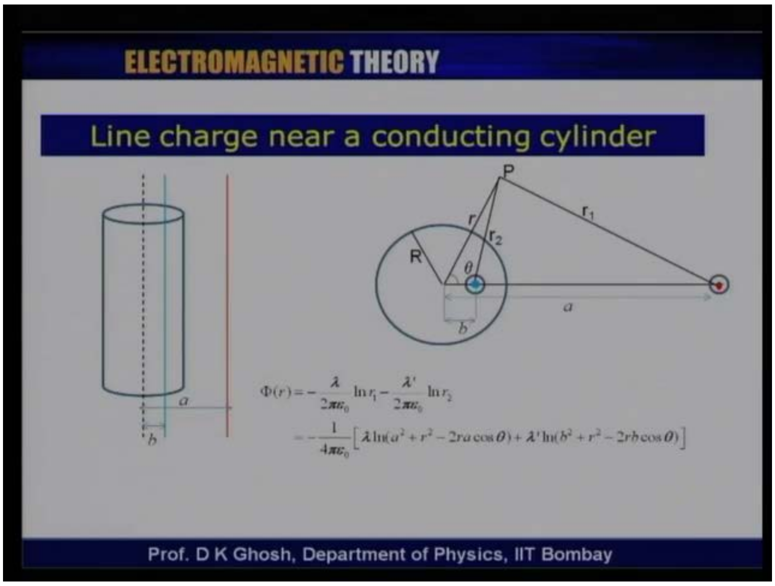

Now in today's lecture, we would like to use the same thing and take two different types of examples. One is connected with the cylindrical geometry. So, let us look at the situation where I have a line charge indicated by this red line here, and which is in front of a cylinder - metallic cylinder, which is grounded that is the cylinder is maintained at 0 potential. The distance of this line charge having a line charge density of lambda per unit length from the centre the axis of the cylinder is taken to be equal to a. By symmetry we know that the image charge has to be also a line charge, and let us take it at a distance b from the axis.

So, therefore I have a line charge of density lambda, and its image charge having a density let us say lambda prime. Now, this is a cross sectional view of what is happening? The situation is this that this is the red indicated the circle with a red at the centre is the original line charge, which is perpendicular to the plane of the diagram. The correspondingly the blue is perpendicular to the plane of the diagram as well having a line charge density lambda prime.

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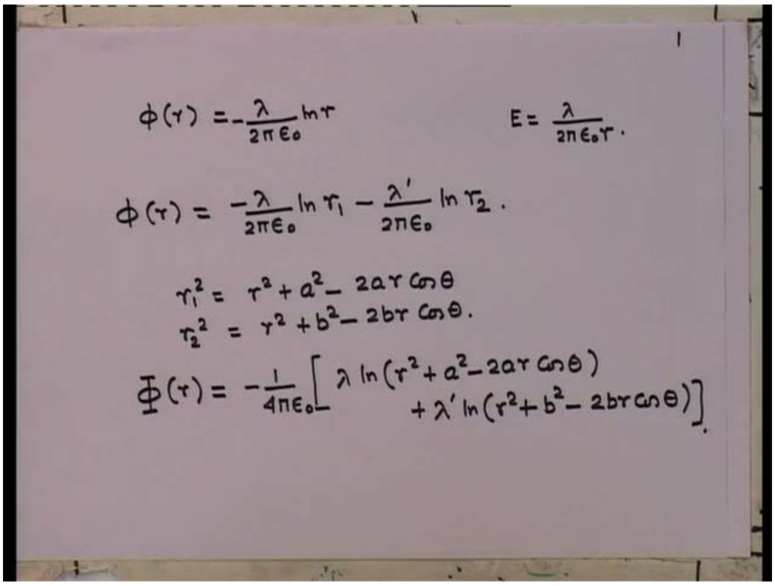

Now, we all know that the potential due to a line charge is given by minus of lambda divided by 2 pi epsilon 0 and therefore, the potential phi due to a line charge is lambda by 2 pi epsilon 0 minus logarithm of r. If you recall the electric field due to a line charge which we had found by using Gausses law was given by E equal to lambda over 2 pi epsilon 0 r. So, this is the corresponding potential of course, you could always add a constant but if I now, consider the line charge as well as the, its image charge.

So, this gives you a if potential or a field at an arbitrary point P. Let us suppose the distance of this point P from the original line charge is r 1 and from its image charge is r 2. Therefore, we write down the potential due to both as equal to minus 2 pi epsilon 0 logarithm of r 1 which is the distance of the image of the original charge line from the point P minus lambda prime by 2 pi epsilon 0 logarithm of r 2. Now, what we do is this we use the standard triangle inequality, we referring back to this picture again. So, referring to this picture again you notice that the distance of the point P from the line charge given line charge is r 1 and from the image charge is r 2.

I use the triangle law if the angle that is made by the vector r with the line joining the centre of the cylinder with the one line joining the axis. This point line charge is theta, then you can check that r 1 square is equal to r square plus a square minus 2 a r cos theta. Identically I have r 2 square is equal to r square plus b square minus 2 b r cos theta. So, what we do is this, that the potential then at an arbitrary point r is given by let me take out minus 1 over 4 pi epsilon 0 common. I have got lambda logarithm of r square plus a square minus 2 a r cos theta and plus lambda prime logarithm of r square plus b square minus 2 b r cos theta.

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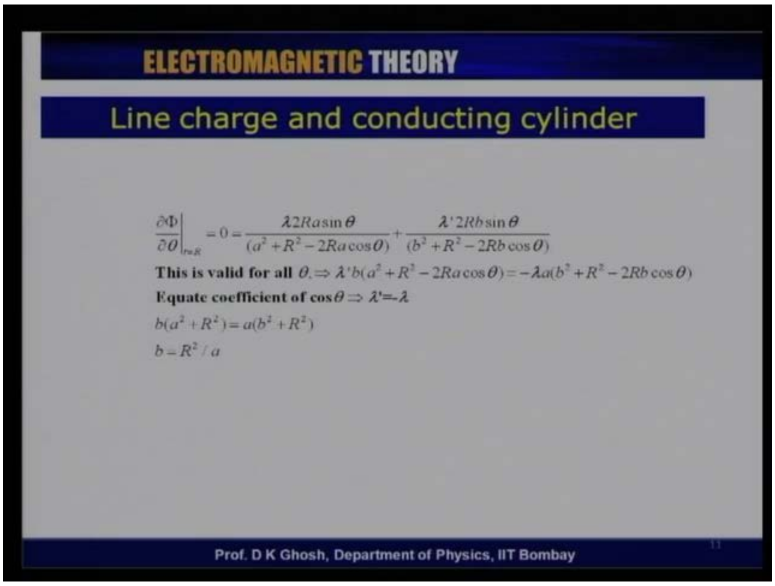

Now, notice that I require that the tangential electric field on the surface of the cylinder must be equal to 0.

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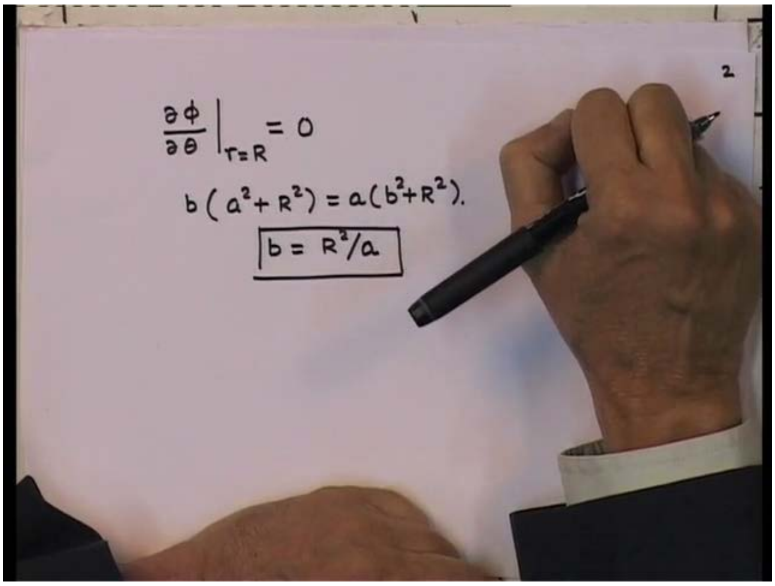

So, as a result minus d phi by d theta at so if I take d phi by d theta on the surface of the cylinder, which is evaluated at r is equal to R. This quantity must be equal to 0. Now, this is fairly straight forward because I have logarithm. So, I get 1 plus 1 over a square plus R square minus 2 a R cos theta and of course, the differentiation of this 2 a R cos theta with respect to theta which gives me these numbers. There now, this expression has to be valid for all theta this expression has to be valid for all theta.

You notice that since theta is arbitrary, I can cancel the sin theta from both sides of this equation and if I do say that these two denominators must lambda by this quantity must be equal to lambda prime by that quantity, I get this equation which says lambda prime b in to a square plus R square minus 2 a R cos theta is same as minus lambda a b square plus R square minus 2 R b cos theta. Now, if you make a statement that this is to be valid for all theta then obviously the theta dependent term here must go to 0. You notice that this is lambda prime b into a, and here also I have lambda into a into b. So, this tells me that lambda prime must be equal to minus lambda. that is the sign of the image charge is opposite to that of the real charge.

Now, with putting that in and cancelling out this, what I am left with it is also then lambda is equal to minus lambda prime. So, I get b into a square plus R square is equal to a into b square plus R square. You can solve this equation for a and b and you get b is equal to R square over a. Notice the interesting thing this is precisely the expression that we had obtained when we found out the location of the image charge for the sphere problem. That was the point of the inversion. So, you notice here that even in case of a cylinder the image charges location is exactly where it was for the case of a spherical conductor.

There is a however, in case of spherical conductor the charge that was induced that is the image charge magnitude was given by a different expression, but in this case it is very similar to the expression that you got for a plane conductor. That is the magnitude is the same and the sign is opposite of what it was.

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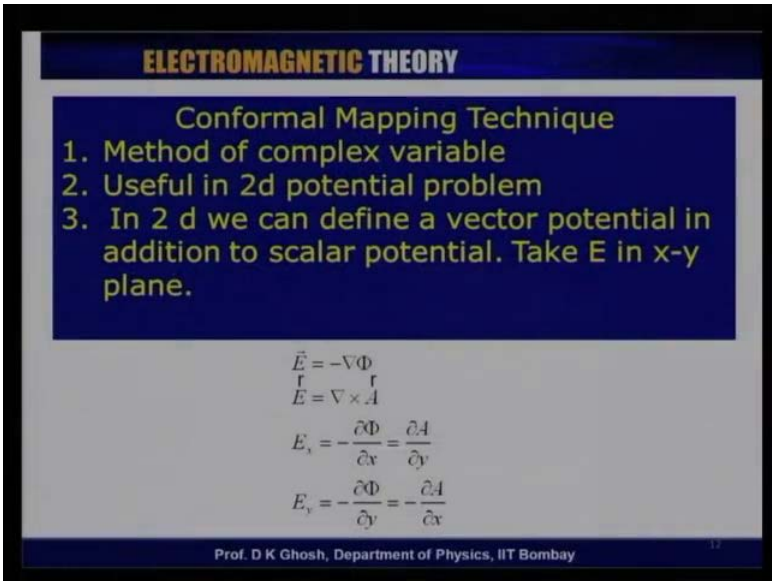

So, this is one of the ways in which we solve the cylindrical charge problem. Let me now go to a slightly different technique and this technique is useful for the case of only for that two dimensional case. That is very special technique which is applicable and we solve two dimensional potential problems. This is known as the method of conformal mapping mapping. Now, let me let me try to sort of give you a little back ground to this because it requires what is known as the method of complex variables and as I have said it is the only useful in two dimensional situation.

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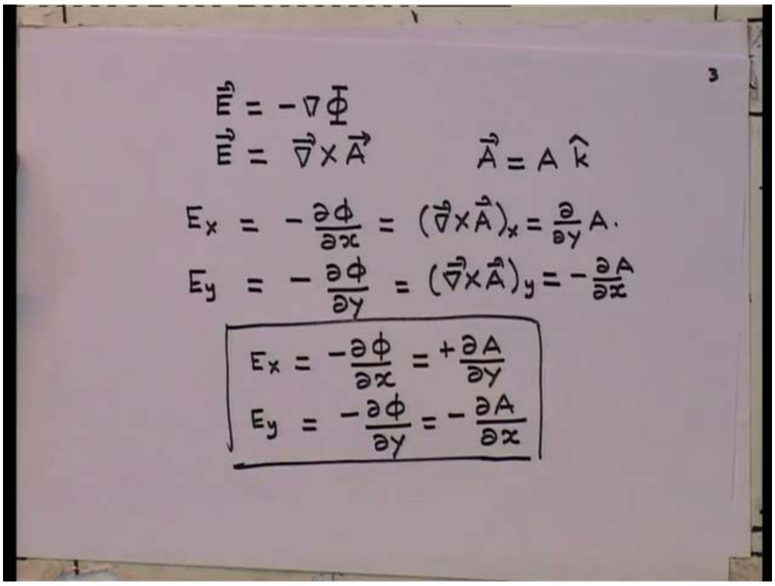

Now, notice one interesting thing that I know that the electric field is given by minus gradient of phi where phi is a scalar potential. Now, if I am in two dimension, I have another quantity which I can define and that is like a scalar potential. I can define a vector potential. Now, I would like to alert you that at a later stage when we talk about magnetic field, we will come across this same statement. Namely that there is a vector potential at this moment, I am talking about a vector potential corresponding to an electric field and that is rather special it is not generally applicable, but one can use it only if your dimensionality is two and what we do is this.

So, we say that let me define a vector potential A corresponding to an electric field by saying that let it be dell cross of a quantity called A. Now, I would like to recall for you that A cross product is actually a three dimensional concept. You do not only when you have three dimensions. I can talk about a cross product because when I talk about A cross B. Now, A and B are two vectors which can always be confined to a plane. But A cross B is a vector which is perpendicular to both A and B that is perpendicular to the plane of vector A and vector B.

So, therefore it is on a third dimension. Now, you can only use the concept of a cross product in three dimensions. That is precisely what we are talking about so what we are saying is this that suppose, I have an electric field which is a directed quantity. Let me say that this this field is in two dimension, it is in let us say x y plane. Now, then I can define a vector A which is because my electric field is in x y plane. The vector A has to be along the z direction or parallel to z direction. Therefore, A must be equal to A times the unit vector k. Now, this vector potential then let me define for what am I getting for the x and y component.

So, the x component of the electric field can of course, be written as d phi by d x, which is in terms of the scalar potential. But notice that this can also be written as del cross A s x component, which is a remember A only has z component. So, it is d by d y of A z which I will write as A. Similarly, E of y component of E which is minus d phi by d y which is equal to del cross as is y component which is d by d z of A x, which is equal to 0 minus d by d x of A z which is simply A. So, you notice we have written electric field as either as minus d phi by d x or as minus d a by d y sorry this is plus and electric field y in the y direction is minus d phi by d y, which is identical to minus d A by d x

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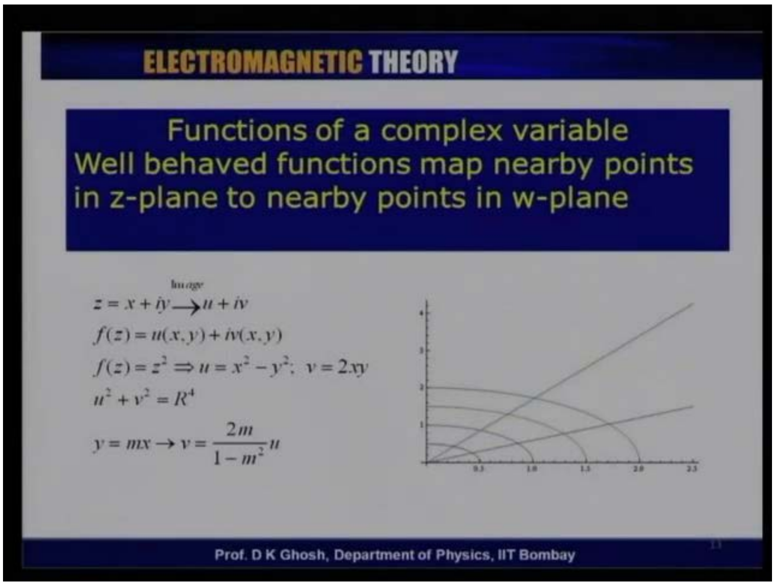

I will I will return to the significance of this equation as I go along, but let us talk about a little more about the complex variable, and then we will see what is the relationship between this.

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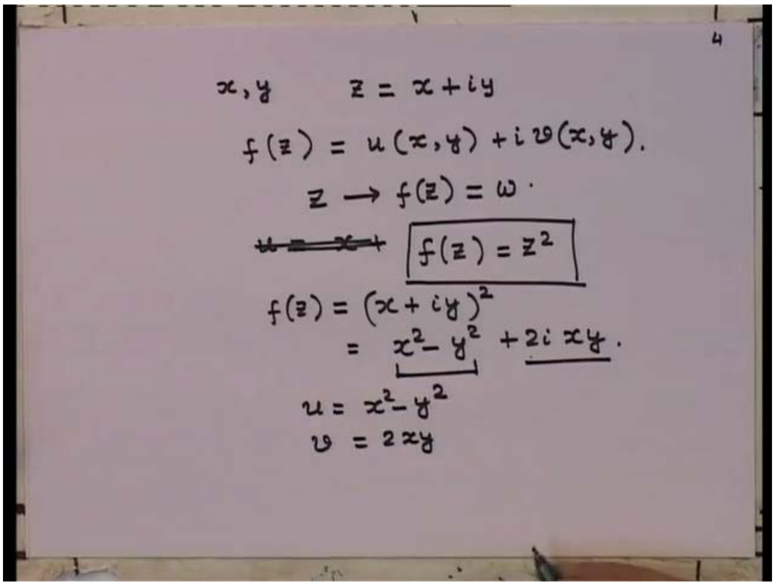

Firstly we instead of the x y I want to talk about a complex variable complex variable z. Not to be confused with the z direction, that we are talking about this is defined in terms of real and imaginary part x plus i y. So, the this is my z now, when I look at the functions of x y in. If I am talking about complex variable then becomes functions of z and this will be written as a real part which is u depending on x y plus i times v depending upon x y. So, this is actually a an image of z going to f z and this is what we will call as w which is a complex variable so the image of the z plane is the w plane.

Now, let us look at what it means in term of real and imaginary part so you notice this thing that the real part of f z which is real part of w which is equal to u so this this is written as x plus i y so x plus i y if you square it for example, let me let me illustrate this for the case of a special f z equal to z square is rather simple function. Now if f z is equal to z square then I can write f of z as equal to x plus i y square which is equal to x square minus y square plus 2 i times x y. Now, this is clearly the real part and this is the imaginary part. Therefore, I get u is equal to x square minus y square and v is equal to 2 x y. Let us look at what does that mapping actually mean. Now, I have I have sort of shown this picture here. So, what I am doing is this, let me consider just a a function like this, a region which is given by this. They are not particularly good, but let us say this circular region. So, my in my original z plane let me redraw it here.

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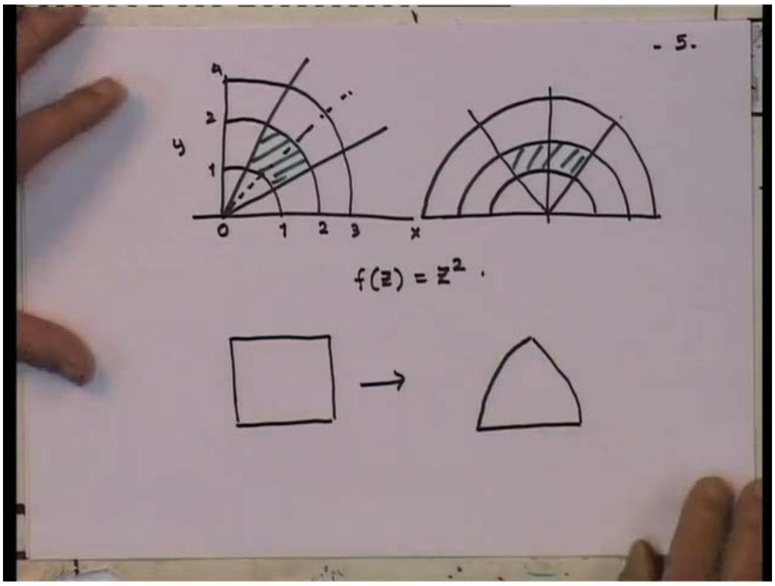

So, this is x this is y, I have families of circles. Now, this is what is drawn there. Now, so this is let me just call this as 1. So, this is 1 this is 2, this is 2 like that. Now, if you now do a mapping of f z equal to z square. Now, what you find is this, that if your f z is equal to z square, then this quadrant actually goes over 2 a half plane and the reason is not very difficult to see. Say for example, consider this circle. Now, if I want to use the polar coordinates for example, this circle is essentially given by you know the r theta relationship is a cos theta theta going from 0 to pi by 2 a cos theta or sin theta. But if I now do a plane like z square.

Now, it is clear that this region will get transferred to or mapped on to a region like this and there will be I will similarly, get sequences of circles in the half plane. Now, you can see for example, supposing I consider two lines. Now, you can let me take for convenience that one of the lines is less than 45 degrees the other one is greater than 45 degrees that with x axis. Then you will find that one of these lines will be like this, while the other line will be on the other one. So, let me draw one more area. Now, supposing you concentrate on a region in this picture. Now, this gets mapped on to a region like this. So, this is this is what is. For example, if you had a square in the original one, if you do a mapping it is fairly easy to see that this sort of becomes something like a distorted triangle.

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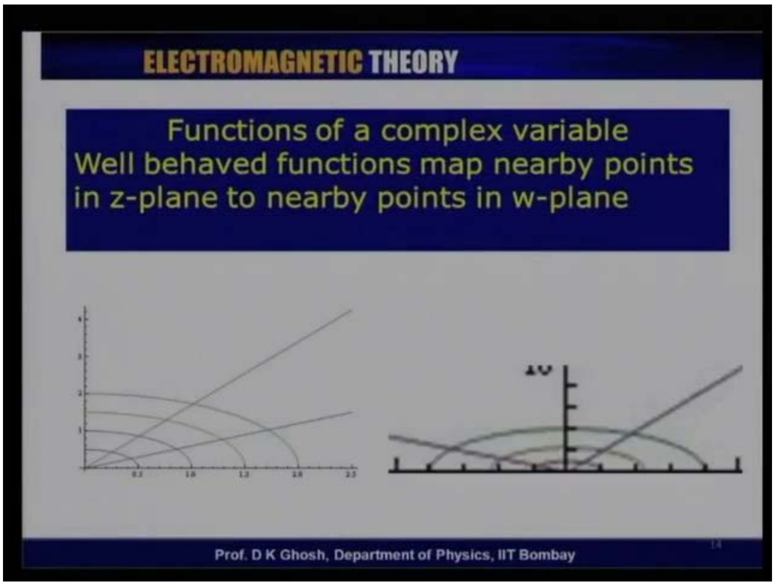

Now, this is what we have been trying to talk about. Now, in discussing our techniques as applied to the electrostatic problem, we use what are known as well-behaved function. Now, an well behaved functions is one where when you are mapping neighbouring points in the z plane by the same transformation. Let us say f z equal to z square. They get mapped on to the neighbouring points in the w plane. So, the points which are close in the z plane remain close in the w plane. That is what is general this is a loose definition of what we are calling as a well behaved function.

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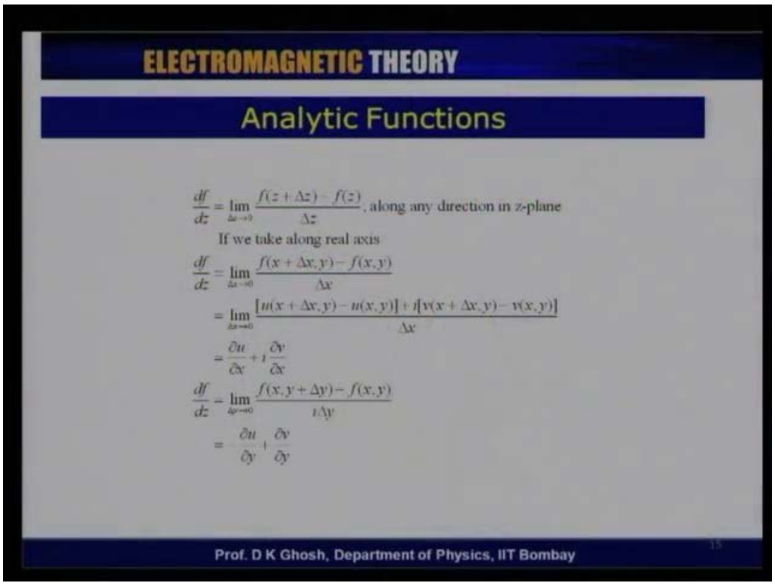

Now, before proceeding further, let us recall the definition of a derivative, the way we have learned it with respect to real variable. Now, remember that in a normal function of a real variable x.

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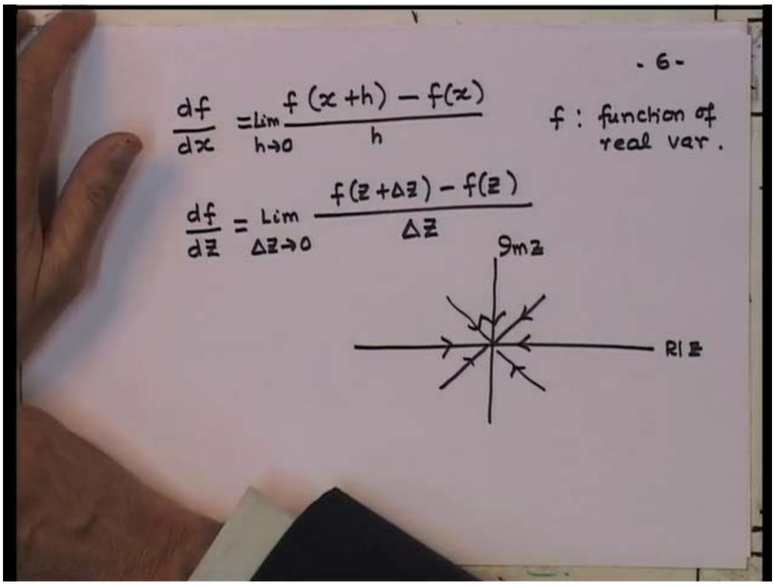

Let us say, when I define d f by d x, I say this is given by f of x plus h minus f of x divided by h limit h tending to 0. This is f is a function of real variable. Now, if you instead if you consider a function of complex variable, then I want to write down what is d f by d z. Now, I borrow essentially the same definition and instead of h. Let me write it as a delta z going to 0, I have f of z plus delta z minus f of z divided by delta z. Now, notice important difference between these two cases. Now, here we did not talk about how the thing goes to 0. So, let us look at supposing, I have a z plane this is the real z, which is the usual x axis and this is the imaginary z.

Now, when z delta z is going to 0, supposing I am talking about going to 0 in this picture, you notice there are many ways in which I can go to 0. It can go like this, it can go like this, of course, it can go like that, it can go like this, this etcetera. Now, in dealing with the derivative of a complex variable it is very important to realise that this definition make sense, if the value returned is unique irrespective of the direction in which delta z goes to 0.

This very important now, this as also have very interesting consequences as we go along and look at this statement here. So, I have got d f by d z. Now, suppose I take this delta z going to 0 along the x axis that is delta z is the same as delta x. So, I can write down f of x plus delta x y minus f x y divided by delta x. Since, f is u plus i v so I will write this as u of x plus delta x y plus i times v of delta x y and like this. Now, if you now split it up in to two parts you get, d f by d z is d u by d x plus i times d v by d x.

Now, supposing instead you consider the way delta z is going to 0 is along the imaginary axis. So, then what I have is d f by d z is given by this expression here is i delta y there, which is the minus because I am dividing by i minus d u by d y plus i times d v by d y i is missing here. Now, notice that these two definition must be the same.

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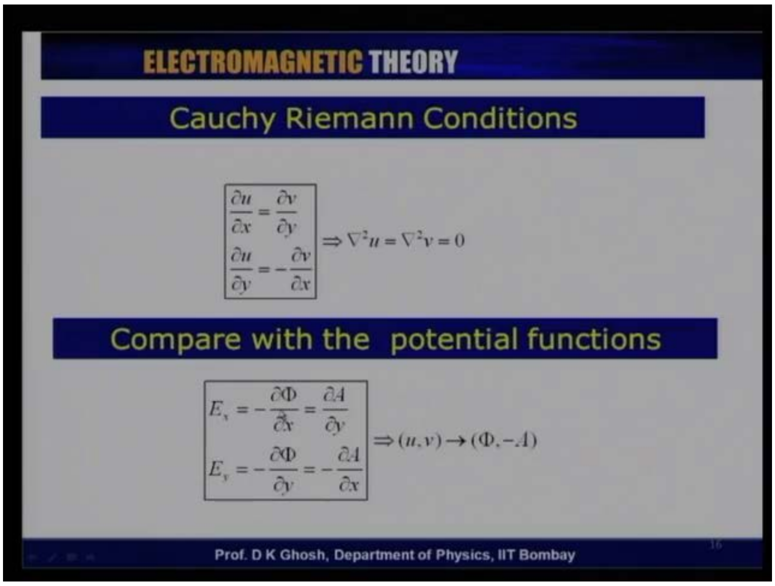

So, if you want to now equate these two definitions because the derivative is unique.

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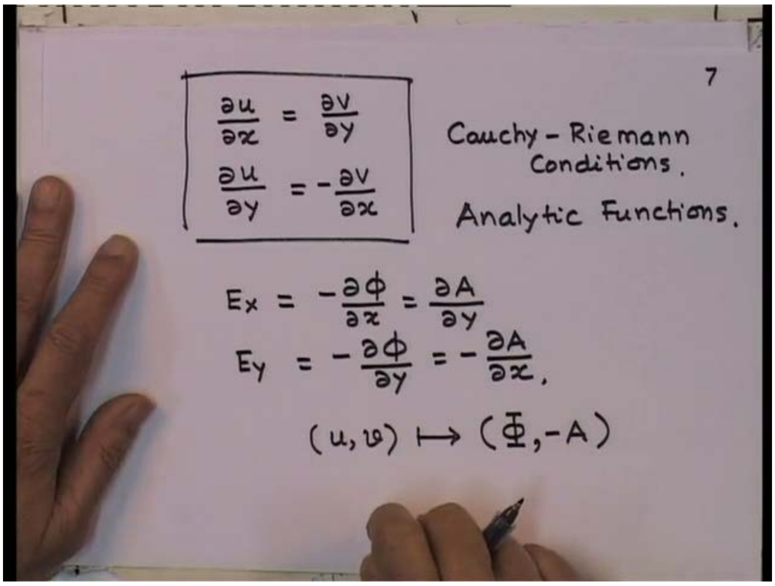

What I get if I equate the real part? I get du by d x is equal to d v by d y and d u by d y is equal to minus d v by d x. Now, these relationship, they are known as a Cauchy-Riemann condition and a function which satisfy Cauchy-Riemann conditions. These are known as analytic functions. Now, I would like you to compare these expressions with what we had obtained little while back. E x is equal to minus d phi by d x which is equal to d a by d y and E y was given by minus d phi by d y which is equal to minus d A by d y d A by d x.

Now, if you compare these two expressions, you notice that the role of u and v are taken respectively by the potential phi and negative of the vector potential A that is the pair u and v essentially are the same as the pair potential scalar potential phi and the minus of the vector potential A. I do not put in a vector sign there because it is essentially a one dimensional quantity because it is only in z direction. Therefore, we notice that I can now think in terms of what I would call as a complex potential, where the real part is given by what we have been so far calling as the potential. That is the scalar potential and the imaginary part of that complex potential is given by minus that is the negative of the vector potential. That we have defined just now, that is del cross of A is equal to E. So, even this.

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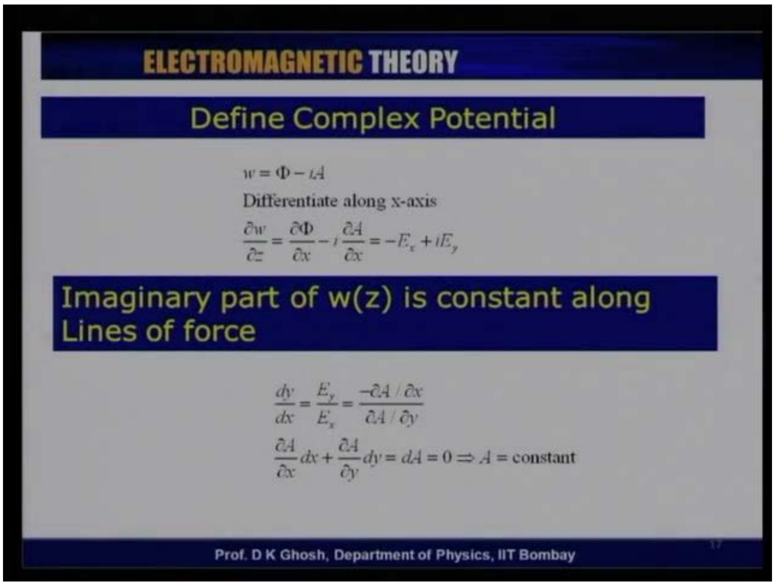

We define what is known as a complex potential.

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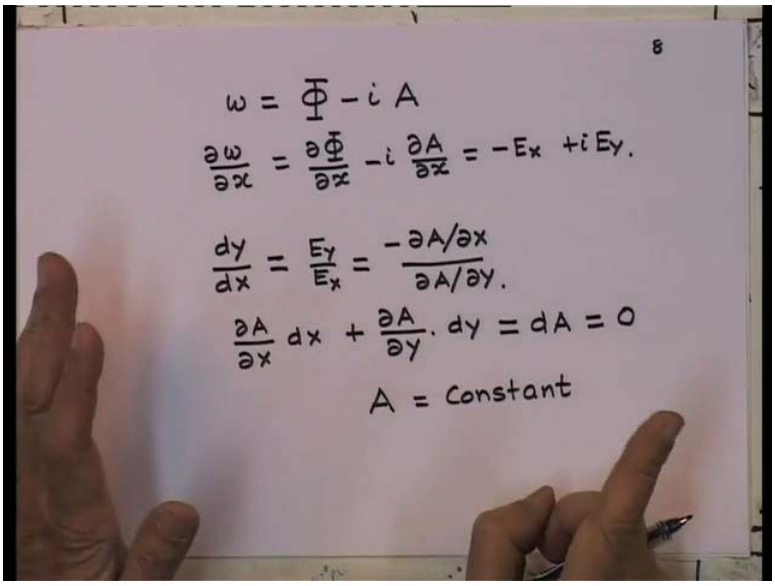

So, let me define complex potential with phi, I mean w which is equal to phi minus i A. Now, suppose I have to differentiate this along the x axis. So, I get d w by d x which is equal to d phi by d x minus i times d A by d x, but this is nothing but minus E x plus i E y. Now, if you look at for example, the complex potential function w and we say that supposing I am looking at a lines of force. Now, what is what is meant by lines of force?

Now, what will show just now is that along the lines of force the imaginary part of the complex potential remains constant.

So, let us look at how does it work? I know by definition lines of force equations are this this is d y by d x because I know that the direction of lines of force, is along the tangent to the curve. So, d y by d x is E y by E x. Now, if you write it in terms of the complex potentials then this is minus d A by d x divided by d A by d y. Now, this tells me that d A by d x d x plus d A by d y d y, which is identical to d A that must be equal to 0, which implies that A is constant.

What does it mean? It means that the if you look at a complex potential having a constant imaginary part, means that you are on the lines of force. Now, I leave to you to show that if the real part of the complex potential is constant. This refers to the equipotential, well I do not have a surface because I am in two dimension, but it refers to the equipotential lines. The the technique that we use is what is known goes by the name of conformal mapping. Now, let let me define what is meant by conformal mapping. So, we have talked about a mapping from the z plane to the w plane.

Now, if the mapping is such that is preserves the angle between curves. So, on z plane you have some curves you are mapping it to the w plane. You look at the corresponding curves if the angle between them happens to be the same in both the magnitude. The sense, then the mapping is called a conformal mapping this is here.

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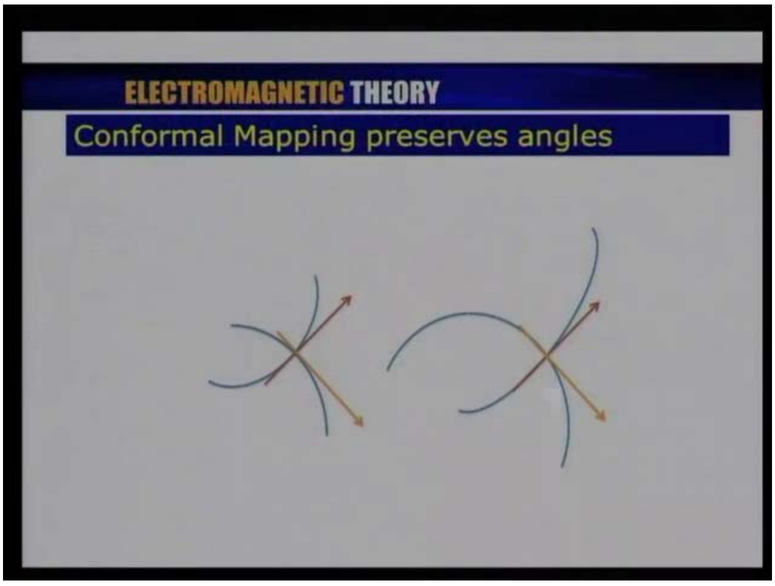

This picture will illustrate what I am talking about. So, notice this are two curves and by some you know transformation this curve has become like this that curve has become like that. I draw the tangents in the same sense and you notice that this angle is the same as that angle this is what I would call as a conformal mapping now one of the things that you must realise, which I am going to not going to prove.

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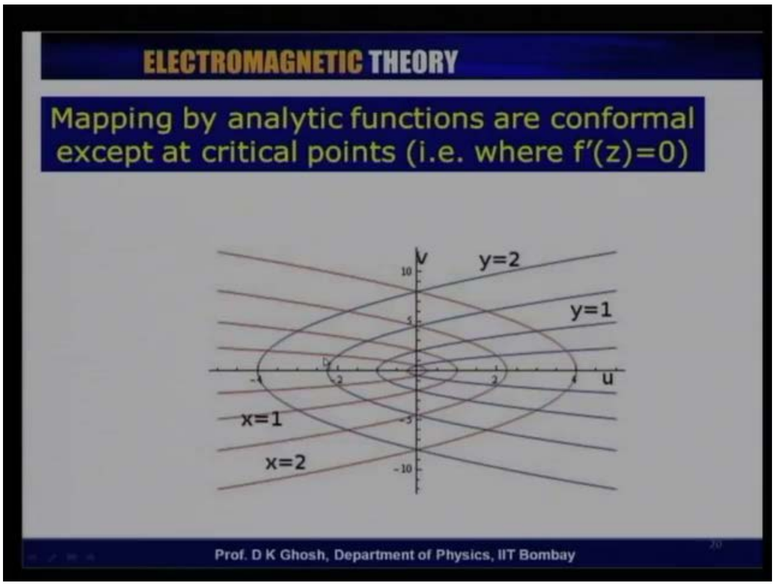

That is when you map things by analytic functions the example that we gave you z square that is an analytic function, it is a differentiable function. Therefore, I will take it. Now, when you do that, the mapping is conformal excepting in one case. Excepting if you are at critical point, that is a point where the derivative of the function becomes equal to 0. So, for example, if you take the function z square now, I know that the derivative of this function which is 2 z that becomes equal to 0 at z equal to 0. So, other than at z is equal to 0, the derivative of f z equal to z square does not vary anywhere in the complex plane, which means the mapping from z to z square is conformal, everywhere other than at the origin, because origin happens to be a critical point.

Now, this picture tells you that what what happens to these. See what we are done is this that I when I am the this z plane I am I am talking about x y. Now, suppose if I keep the imaginary pattern namely the y constant. Now, I am asking how does y constant translate when I am do the mapping? Now, this is fairly straight forward this has drawn using mathematical. So, I know that u becomes x square minus y square and v becomes 2 x y. So, this is this is what I am actually plotting.

So, these are these blue coloured ones they actually have been drawn keeping y equal to constant, the imaginary part equal to constant which would correspond to the lines of force. The red ones are drawn assuming x equal to constant which will correspond to equipotential. So, this is this is the way the mapping would look like.

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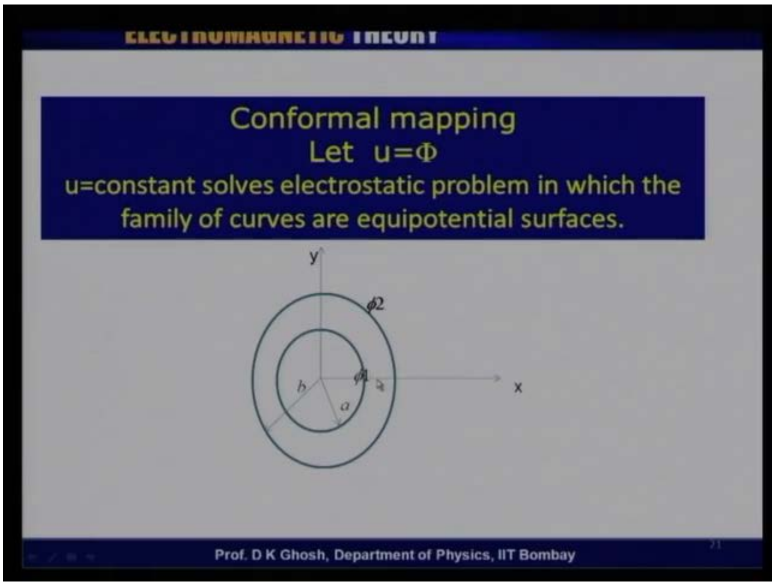

Now, so what what we are trying to say, we have seen that the electric field can be written as a derivative of either scalar potential or vector potential. By equating the relationship or how the x component of electric field or y component of electric field can be written in terms of this scalar and the vector potentials? We have found those equations are identical to the Cauchy-Riemann's conditions for a complex variable. Since, they are identical to Cauchy-Riemann condition, I am using the method of complex variables.

Now, I will illustrate the use of this, in the potential theory. We have not quite talked about that, how does it work? That we have talking about how does the Laplace equation coming to the picture?

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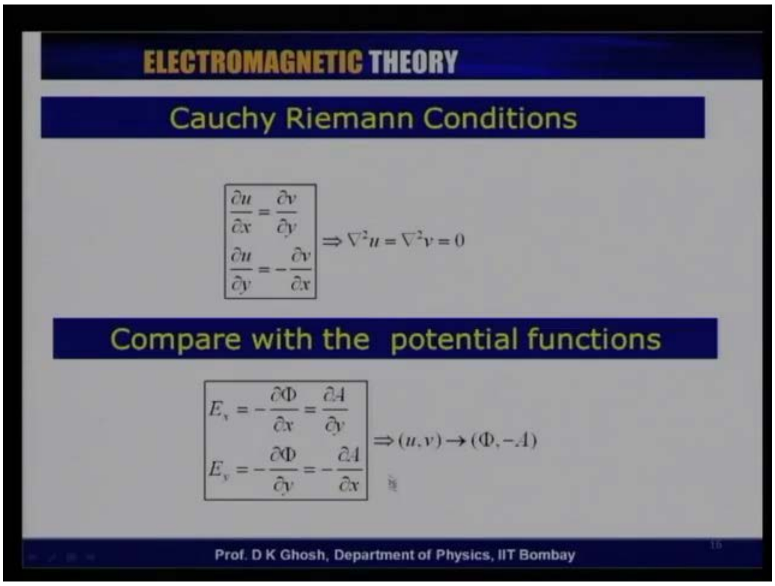

So, notice this thing that when we note down the Cauchy-Riemann condition d u by d x equal to d v by d y or d u by d y equal to minus d v by d x. Now, since the order in which differentiations are taken is immaterial then if i for example, if I differentiate this with respect to y and differentiate this with respect to x and add them up, I find then I get del square u equal to del square v equal to 0. In other words the set of Cauchy-Riemann conditions are equivalent to Laplace's equation the u and v are known as harmonic function. So, essentially solving a complex variable problem in two dimensions is equivalent to solving a Laplace's equations in two-dimension.

This is this is the relationship between the conformal mapping problem and the original electrostatic problem. Now, let let me give an illustration. Suppose, I am interested in finding out the electric field in the region between two conducting cylinders one of radius a, and another of radius b.

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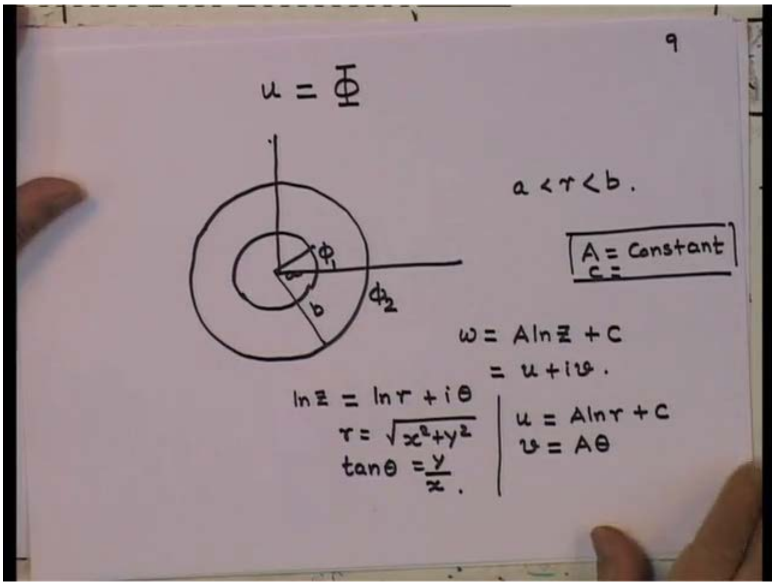

Now, we know that. I now, let me choose the complex potential u the real part to be equal to the scalar potential. Therefore, if I can solve the problem of u equal to constant this is same as solving the problem of phi equal to constant, that is the finding the corresponding equipotential surfaces. So, let let me illustrate this with this example I have got a an inner cylinder of radius a and an outer cylinder of radius b. Suppose the inner cylinder is at a potential phi 1, and the outer cylinder is at a potential phi 2. Now, I am interested in solving what happens to the region between a, and b. Now, firstly let me consider a complex potential of the type a log z.

Let me, I will tell you in a movement. How do I infer this? Let me consider a complex potential, which is written like this remember. z is a complex variable and write this as equal to u plus i v. Now, I know that log z remember that in polar representation z is r e to the power i theta. Therefore, log z is nothing but logarithm of r plus i theta, where r is of course, equal to square root of x square plus y square and tan theta is equal to y by x. Therefore, I can write u is equal to A log r plus constant and v of course, is equal to simply A times theta. Incidentally let me say that the in this A is taken as a constant and not the vector potential that we talked about little while back. So, C, A and C are both constant. So, what we are interested now is to find out what happens to the potential in the region between. Now, look at this.

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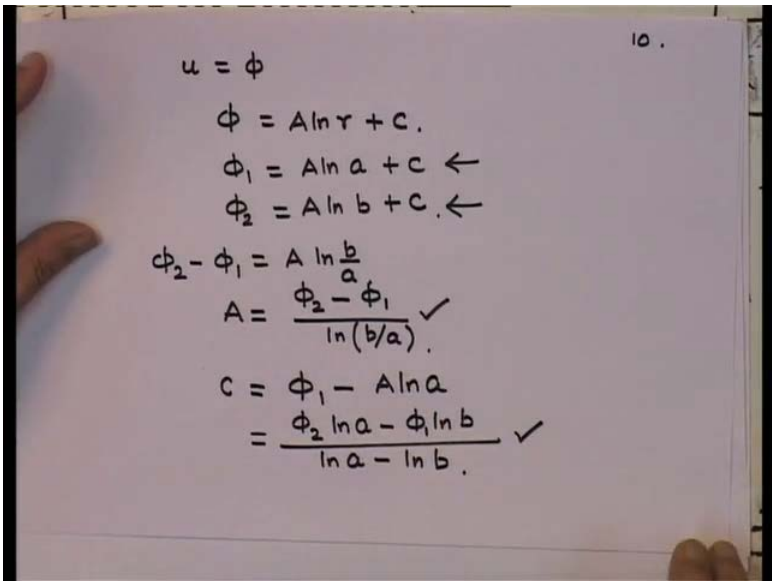

So, let me so let me take u is equal to phi. Therefore, phi is given by A log r plus C. Now, since phi is constant on the two planes r is equal to A and r is equal to b. I can write phi 1 is equal to a log a plus C and phi 2 is equal to A log b plus C. Now, the difference in these two potentials phi 2 minus phi 1, that will cancel out this. Unknown quantity C, that is equal to a logarithm of b by A, A log b minus b log a. Therefore, the constant A is determined in terms of the given potentials phi 2 minus phi A phi 1 divided by the logarithm of b by a.

Now, I can use any one of these expressions to determine what is C. For example, C is phi 1 minus A log a. So, put this back there and you can show that this becomes phi 2 log A minus phi 1 log b divided by log A minus log b. Now, I have determined A and I have determined C. So, as a result my general expression for the potential phi is given by a, which is phi 2 minus phi 1 by log of b by a times log r plus the constant.

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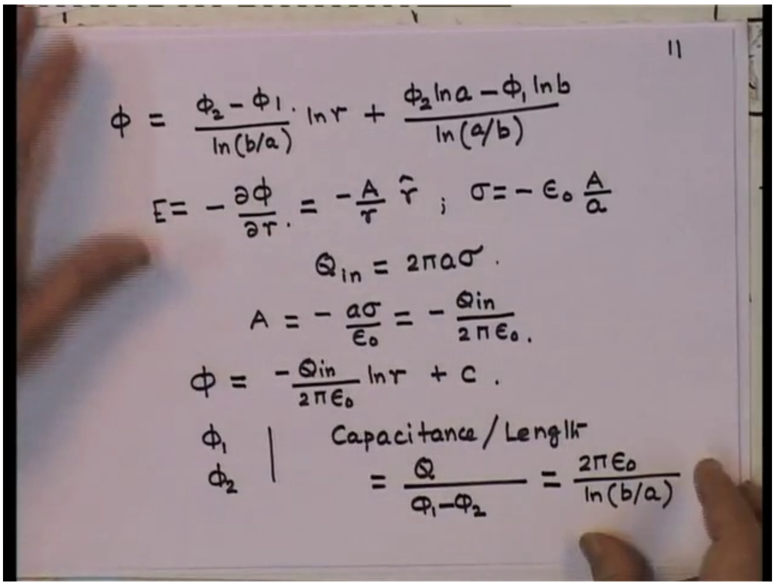

Which we have seen is given by plus phi 2 log a minus phi 1 log b divided by log of a by b, recall that log of a by b is negative of log of b by a. Therefore, you can sort of write it in a little more compact way. Now, let us look at the electric field which is minus d fi by d r, which is minus A by r unit vector r. Well actually this was known to us from the Gauss's law and we concluded on the potential format from there. Now, once I know the electric field, I know that I can find out the normal component of the electric field, which is of course, the radial component and write it as sigma naught times the normal component, so sigma to the epsilon naught times normal component.

So, sigma the charge density becomes epsilon naught times A divided by a. I am writing down this sigma on the inner conductor. I know that the charge on the inner conductor Q in is given by 2 pi a times sigma. So, as a result my A which is minus a sigma by epsilon 0, can be written you just put sigma is equal to Q in divided by 2 pi a, so that becomes minus Q in by 2 pi epsilon 0. So, this tells me that the potential phi at an arbitrary point can be written in terms of the charge in the inner conductor divided by 2 pi epsilon 0 times logarithm of r plus constant.

I can write down now, phi 1 and phi 2. So, I write down what is phi 1, I write down what is phi 2 and can define the capacitance per unit length which is equal to Q divided by phi 1 minus phi 2. Rather straight forward algebra and you will find this is given by 2 pi epsilon 0 divided by logarithm of b by a this this technique of using a complex variable method for solving two dimensional potential problem is very useful, sometimes very intuitive and we could of course, talk about more examples connected with this, But let us briefly summarise what we have done in the two or three lectures, what we have done is to say.

That we are interested in solving the Laplace's equation in charge free source free region, and Poisson's equations in regions with their out sources. We have talked about the fact that solutions which satisfy a given boundary conditions are unique and we have been using in the last few lectures. Our intuitive ability to guess these solutions and we have made a statement that if we have a solution that solution must be necessarily unique using. This we have talked about the image problems, we have seen that in case of a plain conductor the image of a source charge a behaves like an image in a mirror, that is located at a distance d, which is exactly the same distance as the object is from the conductor.

The same technique though in a slightly modified way was used for talking about spherical conductors. What we found in case of spherical conductors is that a very similar technique can be used by talking about an image, but however, the image does not quite behave the way it behaves in case of a plain conductor, but more interestingly in case of a spherical conductors, we found that there is a point of inversion and about which there is a beautiful symmetry. That is if I know the potential at a particular point in space due to a charge, which is located at a particular distance from the centre. We can relate it to a potential at a different point, that a point of inversion due to a charge which is also calculated from the same principle.

So, that is the way the potential problems were solved in the spherical geometry. We found that in simple cases, we can use the image problem in case of cylindrical geometry as well that does not give either. The first case or the second case, but we found for example, in case of a cylindrical conductor, a line charge has a, an image charge density which is equal and opposite to that of the object charge density. But its location is like the case of a sphere. Having done that we have talked about conformal mapping and a conformal mapping is one which we have said preserves the angles of mapping.

It is a techniques of complex variables, where what we have done is to find out the analogy of the complex potential which consists of a real part which is our usual scalar potential and an imaginary part which is the vector potential whose cross product gives me the electric field. We have found that the scalar and the vector potentials satisfy a pair of equation, which is identical to the Cauchy-Riemann condition for a complex variable. This allowed us to use the techniques of complex variable for the case of two dimensional problems, electrostatic problems. We found that if you put the imaginary part namely the vector potential to be constant, then what we are talking about is lines of force and if you put the scalar potential to be constant, you get the equi potential.

These two are perpendicular to each other with this, we have come to the end of our discussion of electrostatics of the conductors. What do you want to do next is to talk about the electro statics. What happens if there are dipole moments in the medium? The dipole moments in the medium will polarise and they will have effect on how strong the electric field is. So, next lecture we will define what is meant by the polarisation and continue with electrostatics a little more.