Electromagnetic Theory
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Module - 2
Electrostatics
Lecture - 18
Special Techniques

We have been talking about special techniques in solving problems of electrostatics using the advantage that is given to us by the uniqueness theorem in solving Laplace’s and Poisson’s equation. What we did last time is to talk about the problem of a point charge in front of a grounded conducting sphere. What I mean by that is that the sphere is to be maintained at 0 potential. What we look at is the following.

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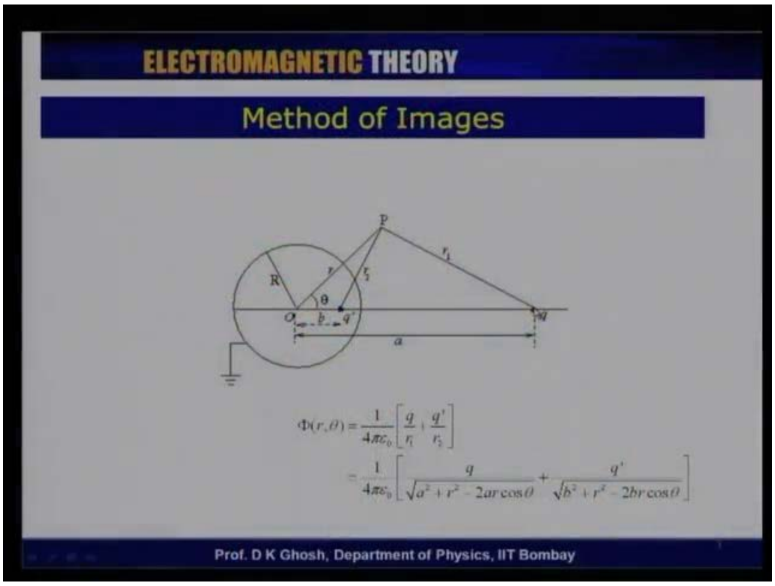

That if there is a charge q here, unlike in the plane case, the case of a plane conductor where the word image is readily understandable, because the position of the image charge is exactly equal to the object distance and the magnitude of the image charge is equal to that of the object charge though its sign is opposite.

Now, in this particular case I have a slightly different type of situation. So, I have this charge q here at a distance a from the centre of the sphere, and we said let the image charge be at a distance b and let it be q, prime and so we wrote down the potential at an arbitrary point r theta. Obviously, there is no phi dependence and this is given by q by r 1 and plus q prime by r 2. And since r 1 and r 2 are positive quantities, it means that q and q prime must have opposite sign and what we did is to write down r 1 and r 2 in terms of a or b and the radius capital R of the sphere and the angle theta that is there.

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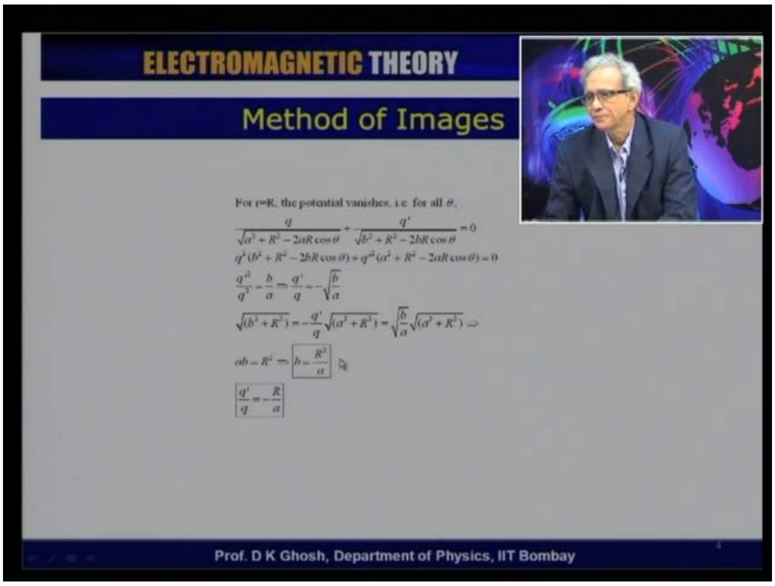

Having done that we said that since the potential at on the surface of the sphere that is at small r is equal to capital R must be equal to 0, I must have the expression for the potential that is q by r 1 plus q prime by r 2, which has been now expanded using triangle in equality to 0.

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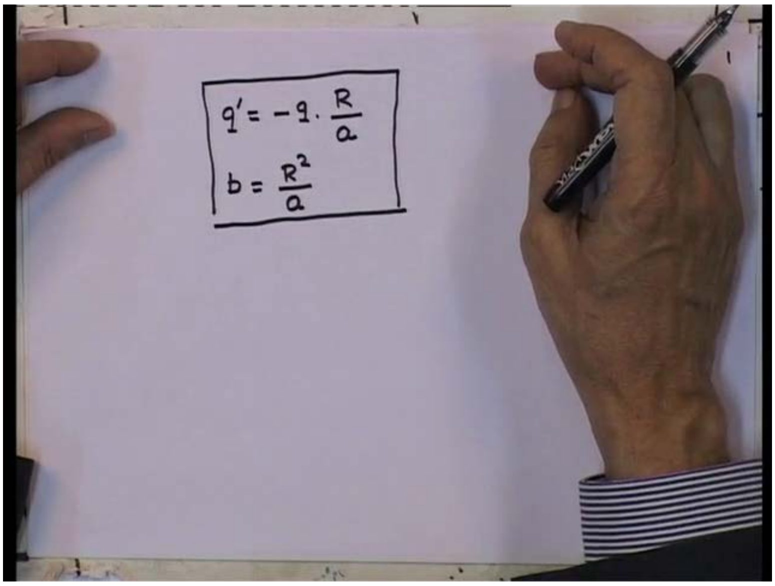

Having done that we could see that the image charge that is q prime must be equal to minus, that is it has an opposite sign q times the radius of the sphere divided by a where a is the object distance and the distance b is given by the R square divided by a. So, this is basically what happens that there is an image charge, which is located at a distance b which is given by r square by the object distance, and of course it has an opposite sign and the magnitude of that charge is q in to R by a. Now, this is what guarantees that the potential on the surface of the sphere equal to 0.

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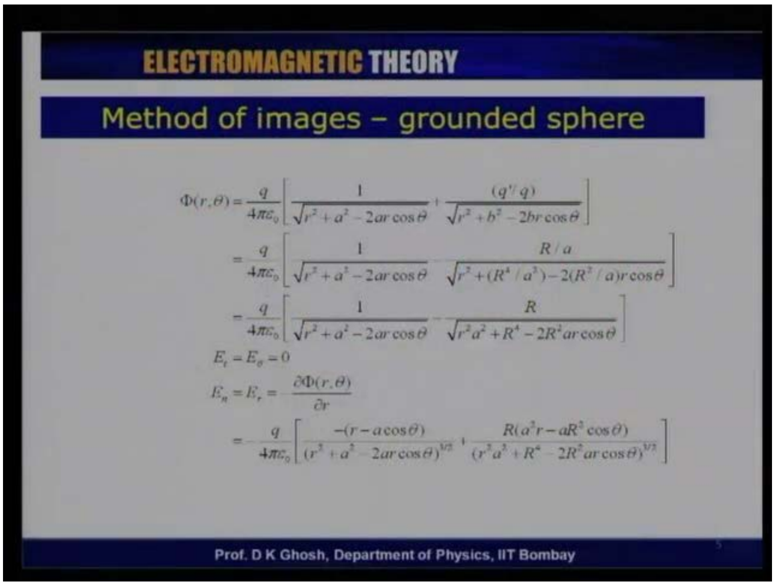

Now, let us look at what is an expression for the potential?

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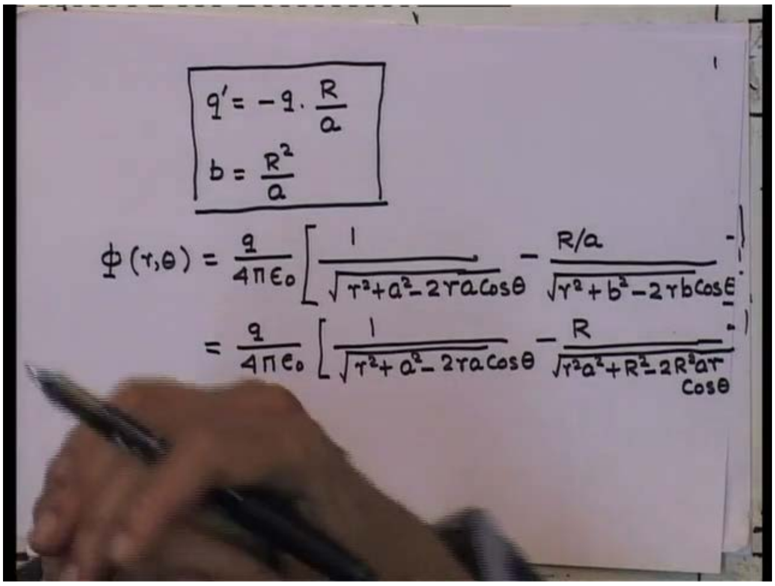

You look at this is the potential phi, which is a function of r theta and which you had seen other than q by 4 pi epsilon 0. So, this is well obviously r square plus a square I am just using triangle inequality minus 2 r into a into cos theta and a minus I have an R by a because q times R by a is the image charge and of course, I have got r square plus b square minus 2 r b cos theta. This is the expression for the potential at any arbitrary position and we can simply rewrite it by using these relationship b is equal to R square by a, q prime I have already written down and that gives me the bit of an algebra I have q by 4 pi epsilon 0.

The first term I have the same because I am not changing it r square plus a square minus 2 r a cos theta minus R divided by square root of r square a square plus r square minus 2 R square a r cos theta. This is this is simple, this is simply we use the fact that b is equal to R square by a into this expression. Now, notice that I need two things namely the tangential component of the electric field must be equal to 0 which is guaranteed here, if you take the differentiation with respect to theta of this one and put r is equal to capital R this these two terms will cancel out.

Now, I also need the normal component of the potential and which is simply d by d r of that and if you look at the normal component then of course, you realise that at this expression here a normal component is d by d r. So, this being 1 over square root I get to the power 3 by 2 in the denominator and I am differentiating with respect to r so I get a 2 r from here and 2 a cos theta and cos theta is differentiated gives you minus sin theta. So, this is differentiation with respect to r. So, I have got minus 2 a cos theta. So, this is what is written down here and a similar expression for this one. The reason I calculate the normal component of the electric field is because I know that the density of charges charge density on the surface is related to the normal component of the electric field.

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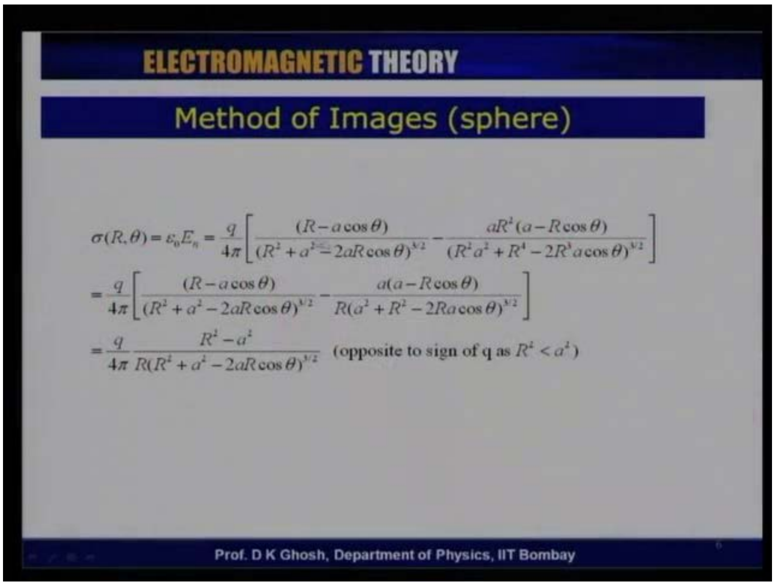

So, I simply find out what is the value of the normal component of the electric field at small r is equal to capital R and multiply it with an epsilon 0 and this is what you actually get fairly straight forward way to do it and we can combine these two terms and we get q by 4 pi R square minus a square divided by R into R square plus a square minus 2 a R cos theta to the power 3 by 2. Notice, immediately that since the radius of the sphere is less than the object distance, because the charge is outside this sphere. So, R square minus a square is negative. So, the charge density as expected is negative.

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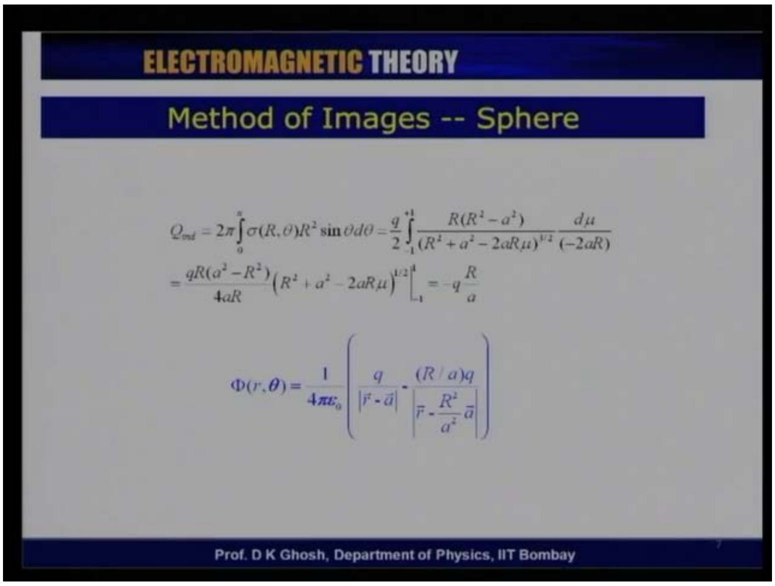

We, will use that expression as we go along, but let us look at how much is the net charge how much is the net charge that is introduced on to that sphere because I have a charge density now.

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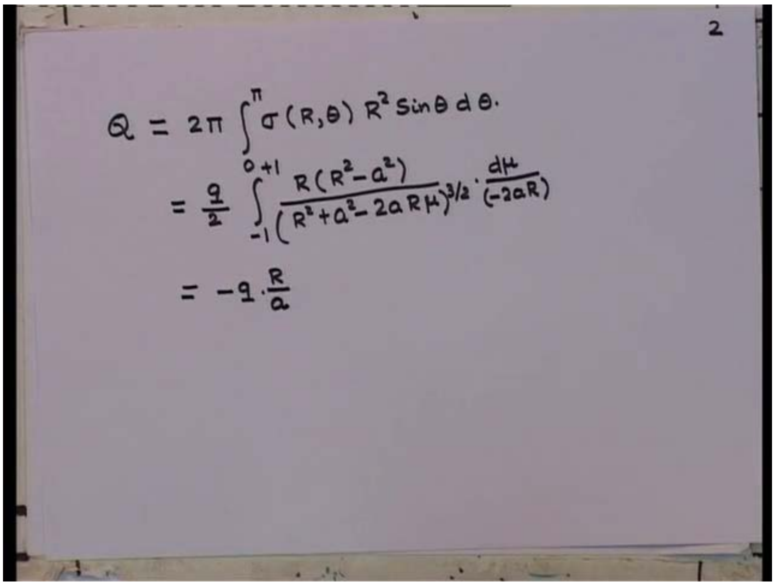

And this is very simple because that net charge that is induced is simply the surface integral and notice that I am calculating it on the surface. So, small r is equal to capital R and so the integration that I have to do has to be done with respect to the angle theta only. So, this gives me total there is a 2 pi from the phi integration on which it does not depend and I have got 0 to pi sigma at R theta then of course, the surface integral Jacobian R square sin theta d theta. So, this quantity if you look at that I have an expression for sigma earlier and you put that so 1 over 4 pi was there that will give me q by 2 and integral.

Now, this is the standard trick that we have been doing, using the fact that substitute cos theta is equal to mu so that sin theta d theta gives me minus mu and of course, the limits of the integration over theta is from 0 to pi that becomes from 1 to minus 1, but because of this minus sign, which is there it is from minus 1 to plus 1. And I have got R R square minus a square divided by R square plus a square minus 2 a R mu to the power 3 by 2 and of course, I need still my Jacobian which is minus 2 a R. This is this is fairly straight forward integration to be done and that gives you minus q R by a. So, notice that this is actually exactly equal to the image charge that we put in the location given by b equal to R square by a. So, that is the amount of charge that is induced on the surface of the sphere.

In terms of vectors well this is scalar, but in terms vector distances I write down the potential expression as 1 over 4 pi epsilon 0 q by r minus a and of course, this is the contribution due to the object charge and this is the contribution to the image charge.

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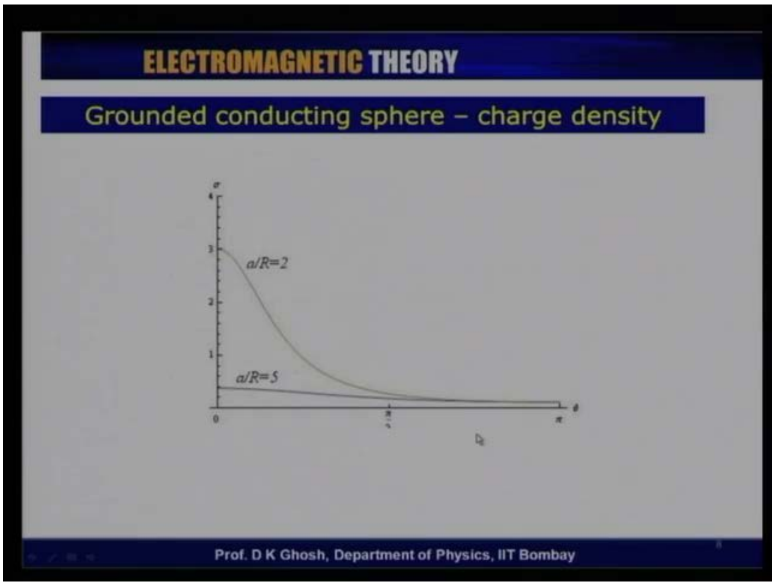

Look at the charge density expression. How does it go? As we are closer remember a is the distance from the sphere and so I have given you two graphs here. a by R is equal to

2 means the object charge is closer to the conducting surface as compared to this situation, where a by R is equal to 5. So, that is five times the distance and you can see that the as expected the angle theta, which is the direction directly opposite to the point where I have got the object charge, that is where the charge density is maximum.

This is of course, you have to realise that I am just drawing it on the positive quadrant, this is the magnitude that has been drawn and as you go away. Then of course, as expected it goes on decreasing, but as you go further and further the spread out is lot more that is because from a distance the, you know directive opposite point theta equal to 0 and other angles are not that much to be differentiated.

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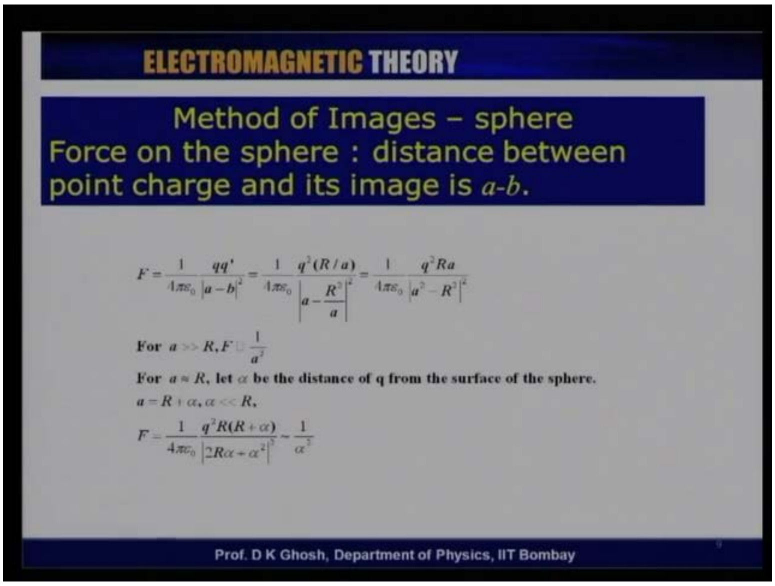

So, let us let us look at how much is the force that is exerted on the sphere. Now, this is simply done we have already explained in case of the object charge on a conducting plane that is the force is same as that between the charge and its image which is 1 over 4 pi epsilon 0 q q prime divided by a minus b whole square. And q prime of course, is so let us let us write down the expression for the force.

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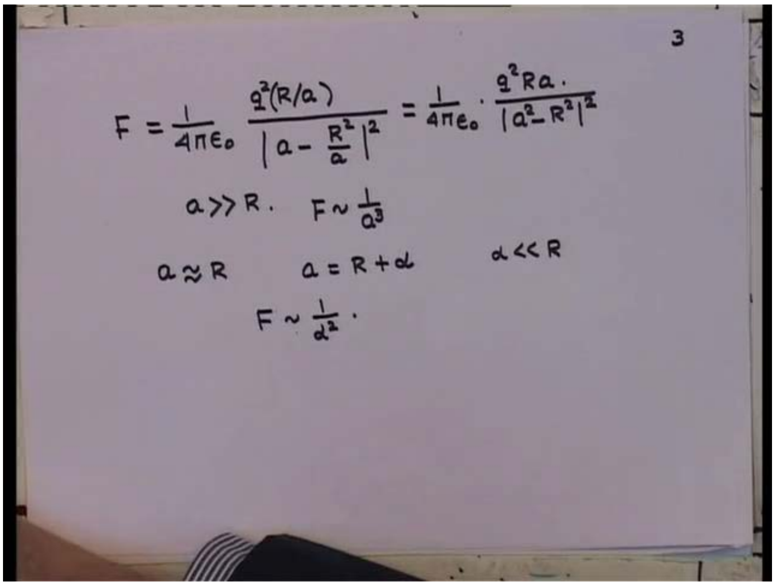

So, the force is q q prime 1 over 4 pi epsilon 0 of course, q and q prime is q times R by a. So, I got q times R by a and I am just writing down the magnitude of the charge or if you want because the q prime is negative. So, this an attractive force and this is the distance between them, which is a minus R square by a, which is my b and this quantity whole square. So, this you can of course, simplify and write it as 1 over 4 pi epsilon 0 times q square R a divided by a square minus R square.

The force is attractive because one of the charge is positive and the other charge is negative. Now, notice if I have a much greater than R, that is that is something which is fairly far off. Then if a is much greater than R this is square so I can neglect this R square and get this as a fourth and a there so my force would go as 1 over a cube. On the other hand if I say that the, you know it is the very close to the surface that is the distance a is approximately equal to the radius. So, let me write a is equal to the radius plus a small quantity alpha. So, alpha is much smaller than R, so that I will almost near that. Now, you can calculate how much is the force there and so that this force then goes as 1 over alpha square.

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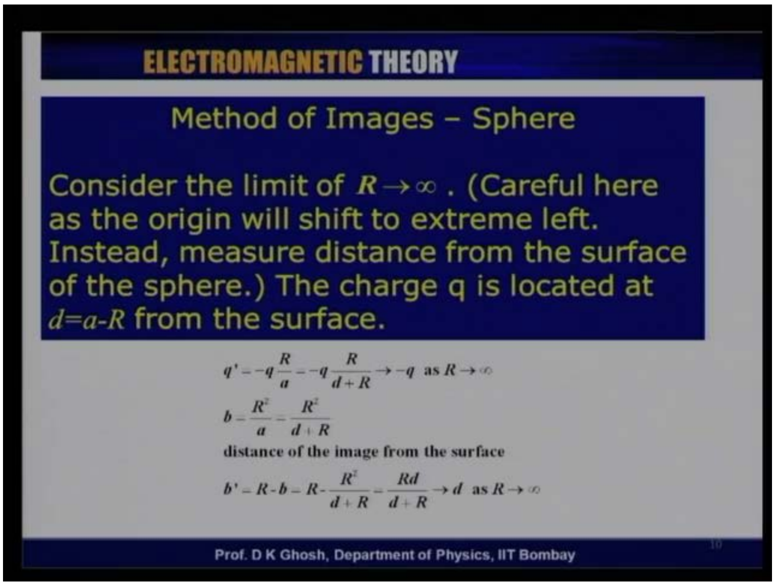

Now, one interesting thing I would like to point out here that suppose I let the limit of the radius become very large. Now, you realise that a sphere of radius, infinite radius is not a distinguishable from a plane. Therefore, what you would think is that all that I need to do is to pick up the expressions and simply put r going to infinity and I should recover the result that we had for the case of the plane conductor in front of which I have put a charge that is an image charge having a magnitude equal to that of the object charge and located at a distance equal to the object distance behind the conductor.

We have to slightly careful in the in doing this. The reason is that as you increase the radius the origin from which I measure all my distances actually shifts to the extreme left and as a result the distances all become very large. So, what we need to do is to convert this by measuring distances not from the origin, but from the surface of the sphere.

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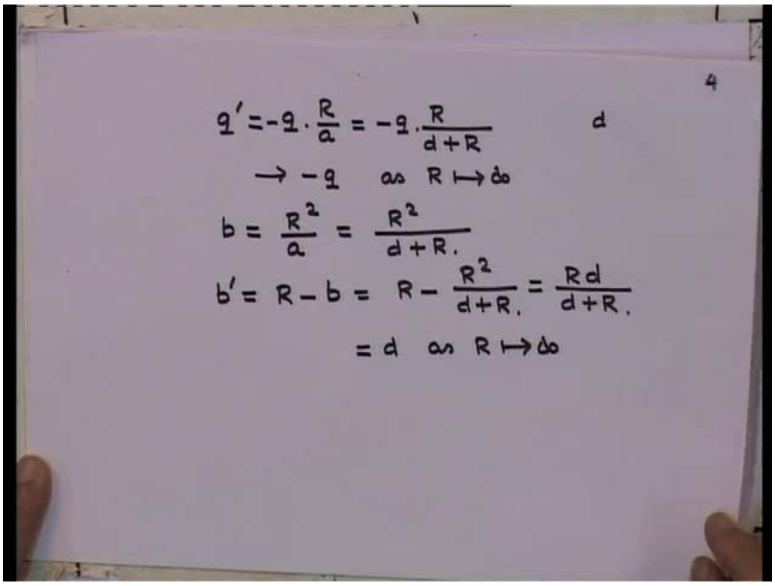

So, look at that. So, let me say q prime which we saw was q times R by a. Now, let me say that the distance of the object from the surface of the sphere is d. So, in other words I write this quantity as equal to, this is of course, minus q R divided by d plus R because a is equal to d plus R. Now, you should take the limit so R is very large. Therefore, this d is small compared to this R and can be neglected. So, as a result I get minus q as R goes to infinity as it odd to and similarly, the image distance b which we had seen as R square by a and we had just now seen that a is nothing but R square by d plus R.

Now, this image distance is from the origin, but I need the image distance from the surface, which would mean it is R minus b and that is equal to R minus R square by d plus R. And you can see it that, you can add it up you get R d minus R square plus R square, which is R d by d plus R and if R is very large then of course, it simply gives you equal to d. So, this is, this simply shows that one has to do some, take some care in writing these things when the radius of the sphere is taken to be infinitely large. So, in in working out this problem we had assumed that the surface is kept at a 0 potential.

Now, suppose I have, I am not really kept the potential at 0, but it is a conductor it has to be at constant potential. Suppose, I keep my constant potential to be phi naught instead of 0, well you can repeat the problem as we have done and replace the constant potential by an equivalent charge which gives rise to this potential, which is I am replacing it with a charge kept at the centre equal to 4 pi epsilon 0 times phi naught which is the constant potential in which the sphere is maintained times r.

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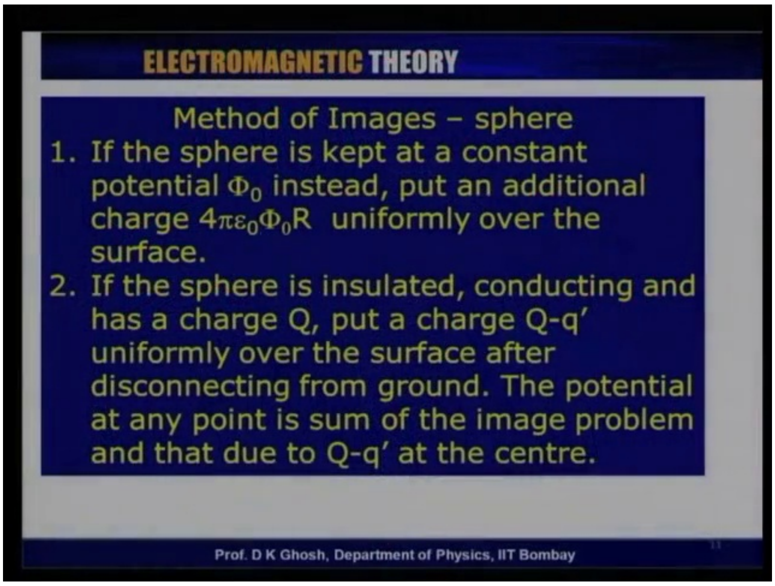

So, you spread this amount of charge uniformly at the surface which is equivalent to putting that amount of charge at the centre and sort of repeat that calculation. Now, suppose my sphere is insulated. If my sphere is insulated and the conducting of course, and has a charge plus q. Now, notice I have a charge minus q prime. I have a charge q prime, which is minus q minus q r by a at the image location. So, in order to maintain the charge at plus q I already have even when it was at a 0 potential the sphere had a charge equal to that of the image charge which is equal to q prime.

Now, if you want the total charge to be maintained at capital Q then what it means is that you must spread an amount Q minus q prime of charge. First you of course, disconnect the wire from the ground and then the potential at any point would then be a sum of the original image problem and that due to the charge Q minus q prime located at the centre. So, this this would be the situation if the sphere has a total charge capital Q.

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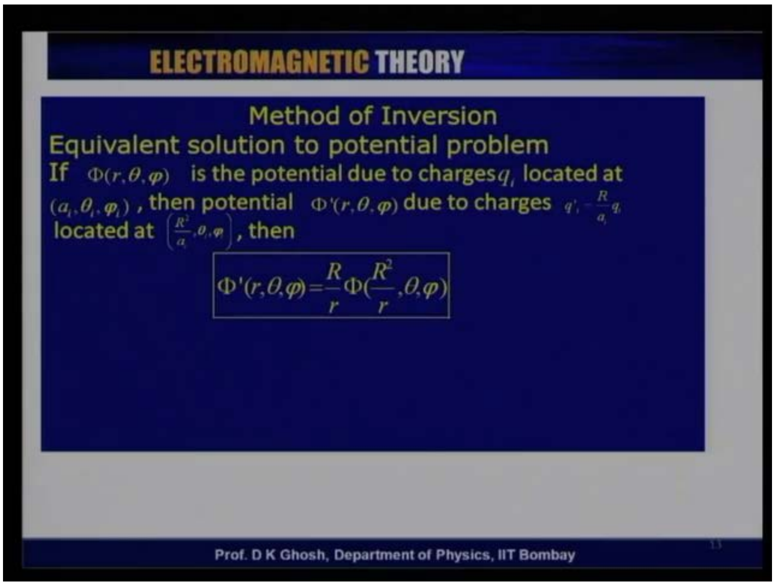

This these observations, I have made an interesting consequence that is what we notice is there is a one to one correspondence between the potential that is due to the charge q which is located at a distance a at any arbitrary point r theta phi. So, what we are, what we can observe, which I will show in a movement is that if phi r theta phi is the potential due to the charge q or it could be a series of charges q i, q located at the point a theta and phi or if it is a series of charges at a i theta i phi i then the potential due to the charges which are q i prime equal to r by a q i located at r square by a which is the inverse point, the image point theta phi. These potentials are the same. So, phi prime of r theta phi is r by r times phi of r square by r theta phi.

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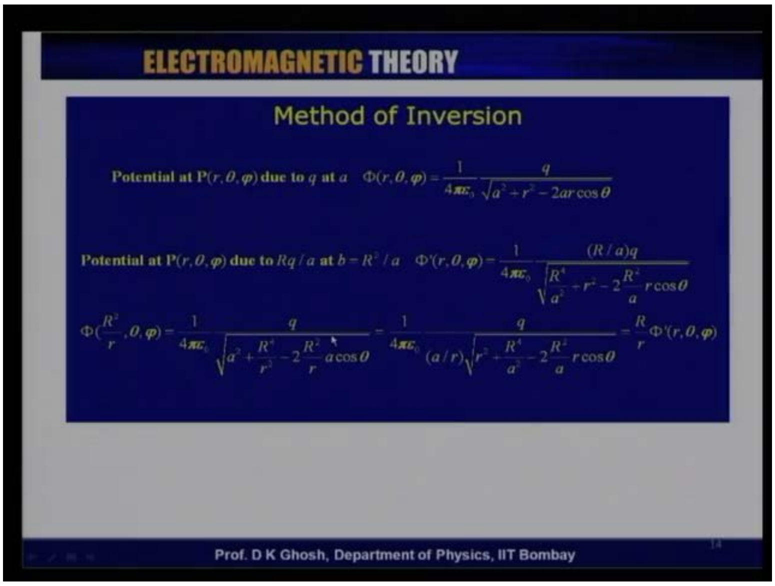

Now, one can easily show this relationship by making some observation.

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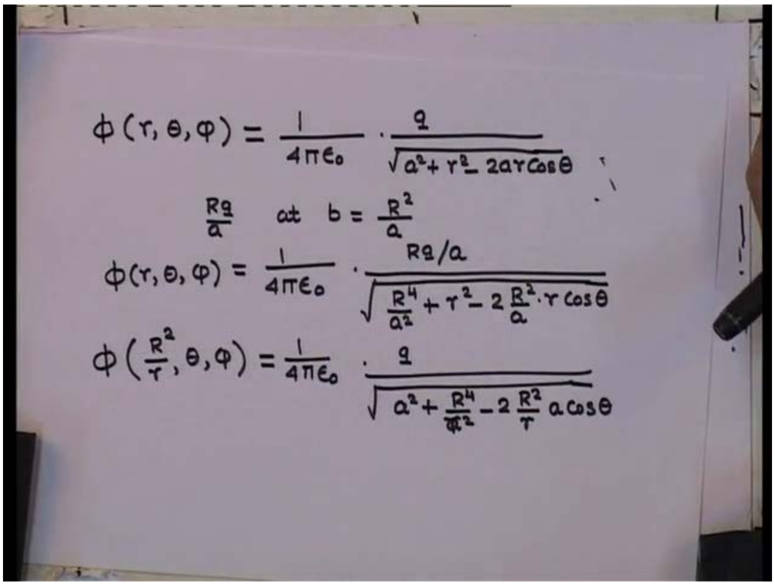

So, notice that due to the point charge at the distance a the potential at an arbitrary point p which coordinates r theta phi we had seen is given by 1 over 4 pi epsilon 0 q divided by square root of a square plus r square minus 2 a r cos theta. This is just the potential at a point p due to the charge q at the, at a distance a from the centre. Now, what we are saying is this. What is the potential at the same point by if I put a charge R q by a at b equal to R square by a.

Now, I can write down this same phi at r theta phi for this situation as equal to 1 over 4 pi epsilon 0 in place of q I have got R q by a over a square instead of a square I have a b square, which is actually R fourth by a square plus r square minus 2 instead of a I have a b so it is R square by a and of course, r cos theta. So, this is the expression for phi when I put a charge R q by a at b equal to R square by a. This is the expression which is simply obtained from this expression which is familiar to us.

Now, let us rewrite the potential phi not at r, but at R square by r theta phi which is simply saying that replace this expression by by putting in place of r R square by r and of course, these will remain the same. So, you put that equal to q divided by root of a square plus in place of R square I have got R fourth by a square minus 2, I have got R square by r R square by r into a cos theta. This should be R square by a r square. You can just manipulate this a little bit and as shown here and show that this is equal to r by r times phi prime of r theta phi.

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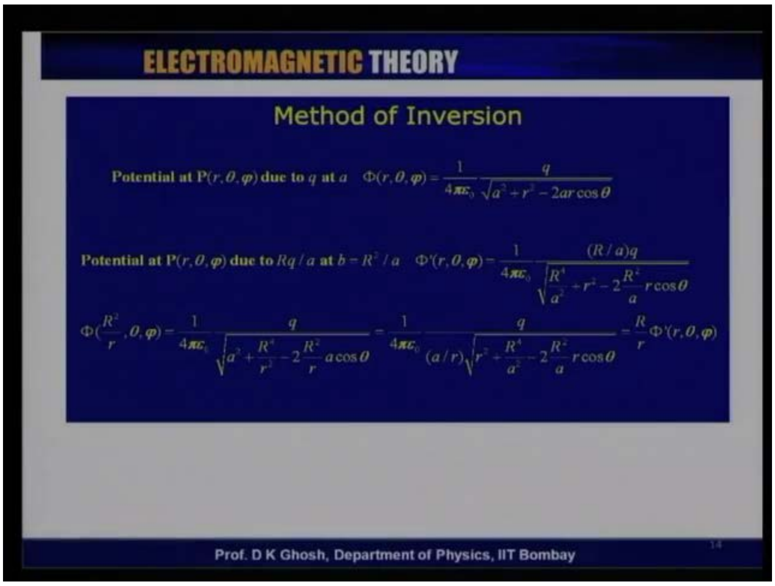

So, this observation that we had is what is known as the method of inversion that is relating the potential at a point due to a particular charge distribution to the inverse problem where the potential at a, you know the inverse problem the charges kept somewhere else to the potential at some other point. So, that is that is the observation of a inverse, it is called the inversion problem.

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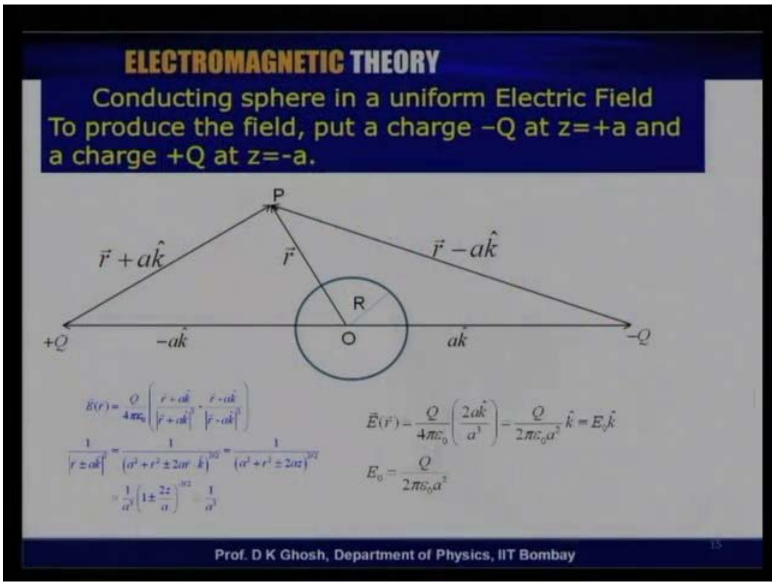

Some time back we had worked out the problem of the conducting sphere in a uniform electric field. The, this we did by straight forward method by doing a legendre polynomial expansion and things like that. It turns out it is a very interesting way in which you can do this by an image charge problem. For that we make an observation that if you consider two charges Q and minus Q located at minus r and plus r from the centre of the sphere and you ultimately will let these distances go to infinity.

Now, this reproduces a constant electric field. So, let us see why. Firstly, if you observe this expression this picture you notice that I have got a charge minus Q at let me take this as the z axis at a distance a, so this vector distance is a k and a plus Q at minus a k. So, if you look at a potential at a point P this distance vector distance is r minus a k and this is r plus a k and and of course, this is the vector r.

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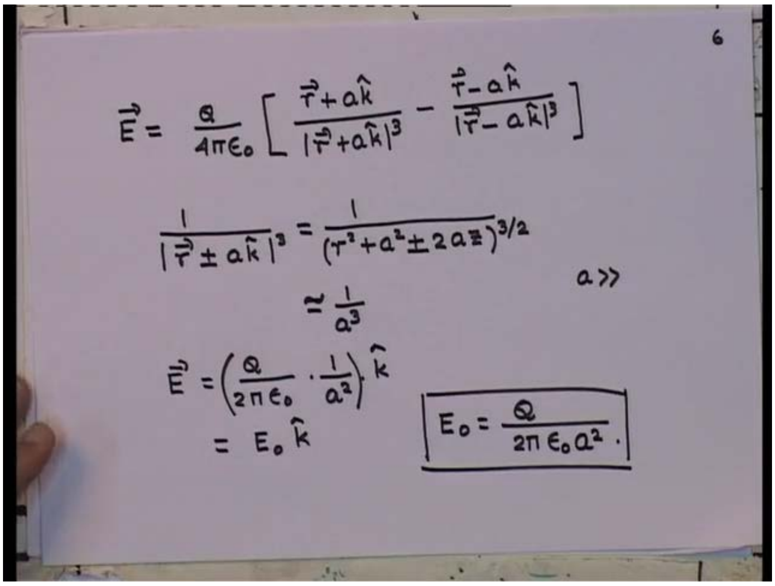

Now, let us look at let us look at what is the electric field at an arbitrary point like this, which we have noted there and that is equal to Q by 4 pi epsilon 0 into referring to the figure again is r minus a k actually r plus a k divided by r plus a k cube, this is just Coulomb's law and minus r minus a k by r minus a k cube.

Now, notice that 1 over r plus or minus a k cube I can write it as 1 over first let me square it, if I square it I get r square plus a square then plus or minus 2 times r, 2 times a times r dot k, but k is along the z direction. Therefore, I simply have plus or minus 2 times a times z, but this is squaring it. I need a cube so this is raised to the power 3 by 2.

And I need to take I need to take the distance a to be very large. So, if I do that this quantity here you can see is approximately equal to 1 over a cube. The remaining numbers are small compared to that. Now, if you replace in this expression both the denominators by 1 over a cube then you notice r plus a k minus r plus a k that adds up and I will be left with the electric field will be equal to Q by 4 pi epsilon 0, but there is a 2 a k there. So, Q by 2 pi epsilon 0 times well 1 over a cube there and a in the numerator so 1 over a square times along the z direction which tells me that this is equivalent to a constant electric field E 0 along the z direction where E 0 is Q divided by 2 pi epsilon 0 a square. Remember that I have kept these two charges at very large distances from the origin.

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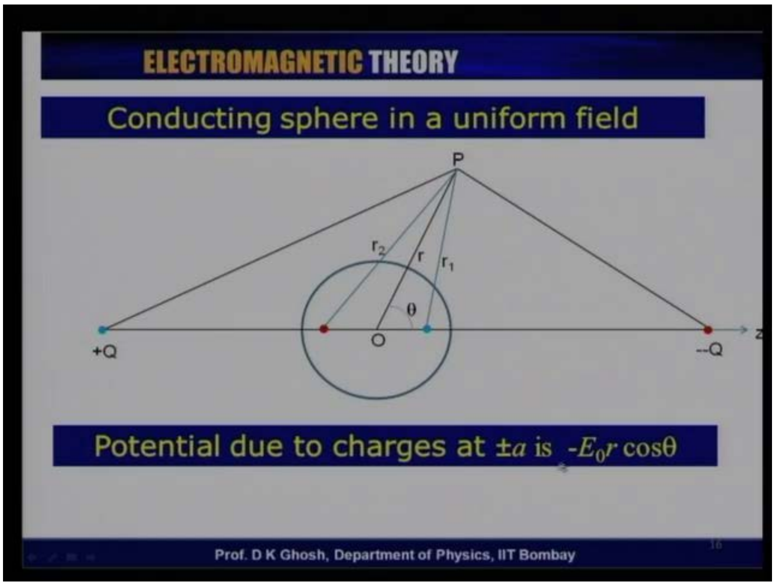

So, let me try to see how does one do this problem of a conducting sphere in a uniform field. So, I have got this a replace the original problem of a constant electric field with a charge minus Q there and a charge plus Q there and of course, I have got now this sphere there, the centre is slightly looking displaced, but should not worry.

So, I am interested in calculating the potential at an arbitrary point P which is given by r theta and so I have got this charge minus Q which is at this distance and this is this is the point from the corresponding image charge and this is the charge plus Q and its image which is being shown there. Therefore, what I have are two objects and two images and let us look at what happens to the potential due to the charges at plus or minus a. So, you see these two we have just now shown as giving us the constant electric field E, which means at an arbitrary point P given by r theta the potential due to these two charges gives me simply minus E naught r cos theta.

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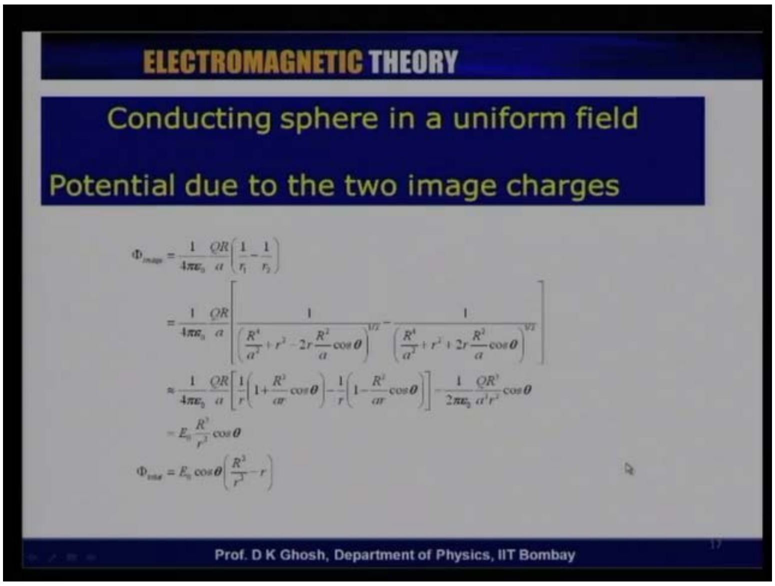

So, I need to now worry about what is the potential due to the image charges.

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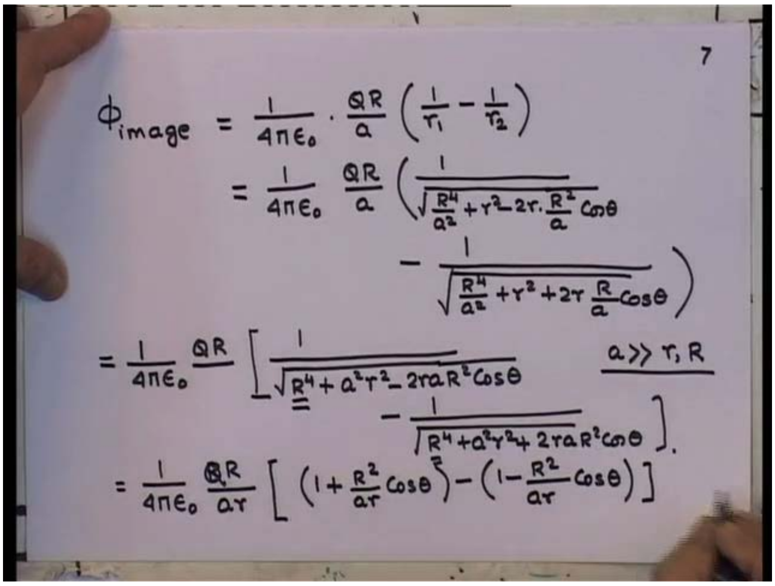

So, let us look at just images and I will later on simply add that minus E naught r cos theta to that. So, potential due to images that is 1 over 4 pi epsilon 0. Remember, the magnitude is Q R by a thus the and I have got 1 over r 1 minus 1 over r 2. So, this is equal to 1 over 4 pi epsilon 0 Q R by a. If you refer back to the picture so you notice that r 1 since this angle is theta just use triangle law here. So, r 1 is for instance b square plus r square minus 2 b r cos theta and and a very similar expression for this, but because the angle between this and that is pi minus theta so I have got a expression with a plus cos theta there. So, this is what you see here.

So, that is root of R fourth by a square which is my b square plus r square minus 2 r R square by a which is b times cos theta minus 1 by square root of very similar expression, but plus 2 r. So, I have said already that my a is much greater than anything else in the problem r capital R etcetera. So, these quantities that I have got there we do a binomial expansion of the numbers. So, let us see how we can work it out. So, this is I got a an expression here. I can simply notice that if you pull out this a, a square from there this will go to the numerator and will cancel with this.

So, the expression will be 1 over 4 pi epsilon 0 Q R. Let me keep this blank for a movement so I have got 1 over square root of R fourth, I am multiplying the entire denominator by a square plus a square r square minus 2 r a R square cos theta and a very similar expression with a plus sign there. So, what we do is this that we of course, we realise that these two terms have a in it and this term does not have an a. So, I first, I had a first approximation, I ignore that and then I pull out a square r square out.

So, I will be left with 1 over 4 pi epsilon 0 Q R divided by a r I am pulling out and I have got then 1 minus so a square r square I am dividing. Therefore, I have got 1 minus 2 times r square by a r cos theta raised to the power half which if I do a binomial expansion raised to the power minus half so I get 1 plus R square over a r cos theta and minus 1 minus R square over a r cos theta.

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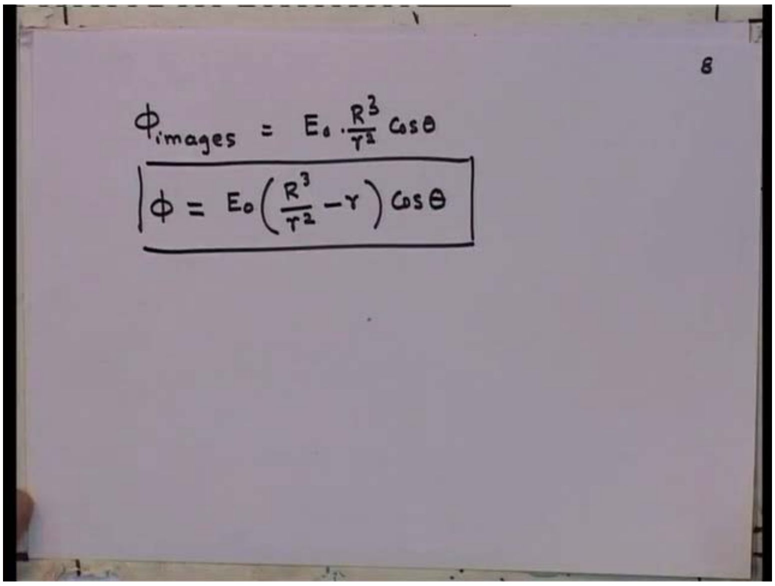

Just open it up and you can find that this is given by the the potential due to the images is given by E 0 R square by r square times cos theta and you recall that sorry this should have been R cube by r square by cos theta and you recall that E 0 r cos theta was my potential due to the uniform field. So, as a result my net potential is E naught R cube by r square minus r cos theta, and if you go back and look at what we did earlier, this was precisely this expression that we had.

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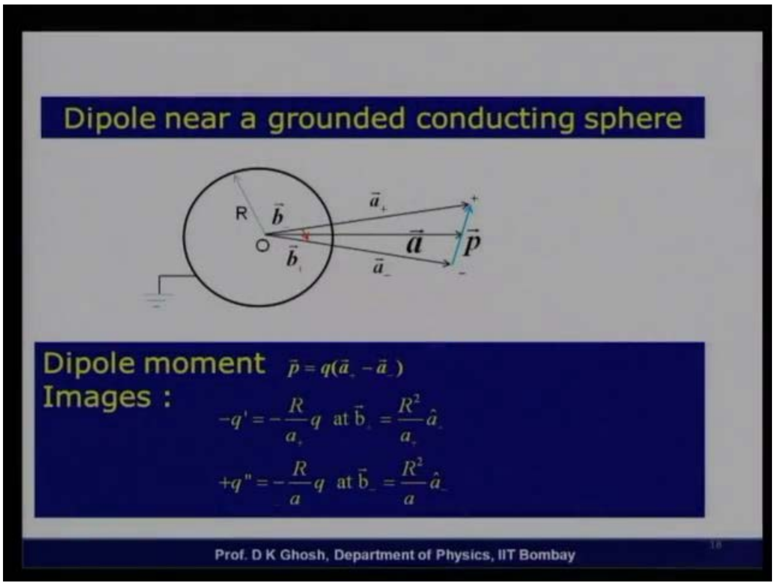

Let me let me go for a slightly different and a little more difficult question. Instead of putting a single charge in front of a conducting sphere grounded conducting sphere let me put a small dipole there. Now, my dipole in the, will be first approximated as a charge plus q and a minus q which has been separated by a small distance d. Now, notice this picture. What happens is that this plus q will have a charge minus q there, image charge minus q there and the minus q will have a charge plus sorry not q it should be q prime there and it will be another charge there. But as we have seen that the magnitude of the charges depend upon the distance of the object from the centre, and my distances are not exactly the same if I take this to be a finite dipole.

So, as a result two things happen. One is that there is an image charge here so let me look at what are the various charges. So, let me call this as a minus q prime because this is a positive charge and this is a charge here and so minus q prime is minus R by a plus times q where a plus is the distance to the positive charge and this is located at R square by a plus and q double prime I am using and this is a plus q double prime that is equal to minus R by a minus times q which is at b minus equal to R square by a minus a. So, first thing that you notice is that though the dipole is electrically neutral my collection after two image charges are not exactly neutral. So, that is one of the things which means that the image charges will actually leave a net charge on to this sphere.

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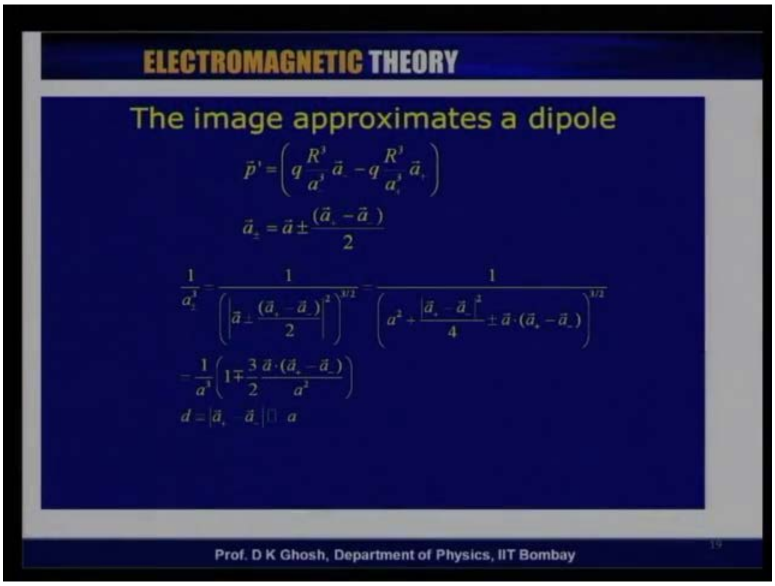

So, I can do the following that, but let us let us do the, a small approximation, reasonable approximation. Let me write the dipole vector remember the dipole vector is always from the negative charge to the positive charge. So, the image dipole is given by q times R cube by a cube vector a minus, minus q times R cube by a plus cube in to a plus where this a plus minus vector is the distance from the origin to the two. So, let me repeat that picture again so this is a plus for instance. So, a plus vector is vector a plus this much which is a plus minus a minus divided by 2.

Now, you can do a simple calculation and rewrite this expression, I mean p prime expression by approximating this 1 over a plus minus q and you take the assumption that the length a plus minus a minus of the dipole is much smaller than the length a.

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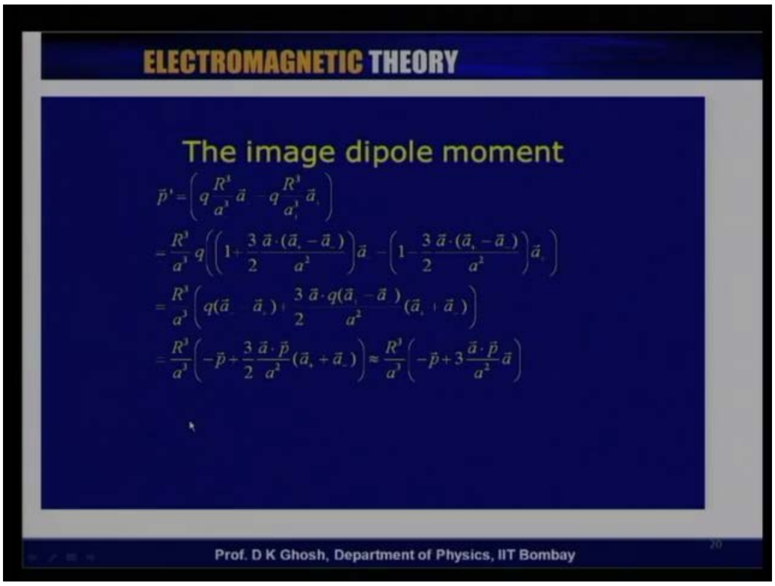

Now, if you do that then one can show that in, there is an image dipole moment which is related to the algebra I am not repeating it, fairly straight forward algebra. The image dipole movement which is given by R cube by a cube times the negative of the object dipole movement, but there is an additional term that comes there which is proportional to 1 over a to the power 5 and of course, is 1 over a to the power 4.

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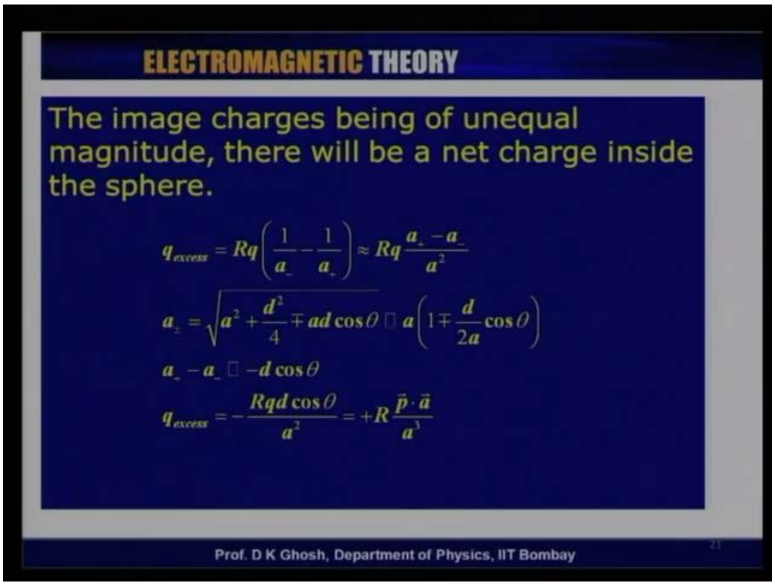

So, what we have said is since the image charges are unequal there will be a net charge that is left and we can find out how much is this excess charge that is there and I know it is R q by a minus minus R cube by a plus, and which is simply written as equal to R q times a plus minus a minus by a square. And you can find out simply write this expression approximate expression taking a to be much larger than d, and so that the excess charge is given by R times p dot a by a cube.

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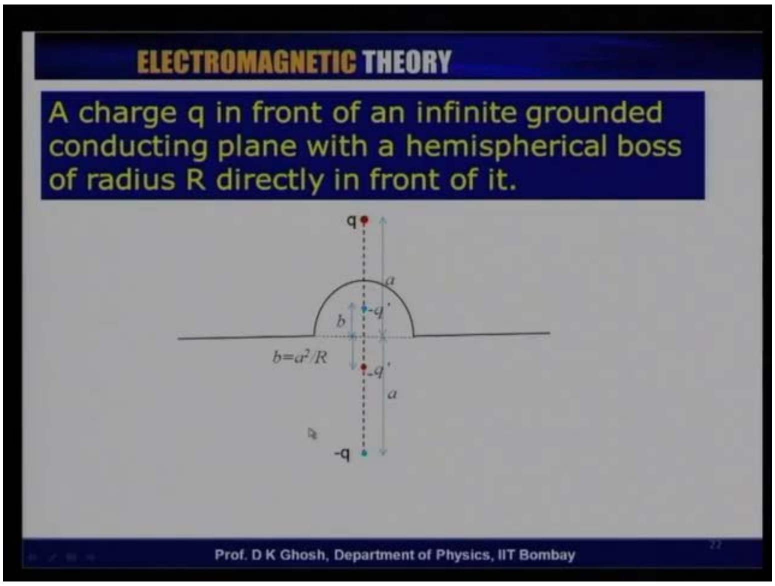

I continue with a a a problem which sort of become famous because it appears in J D Jacksons famous book on electro dynamics and I have a charge q in front of a conducting plane. It is infinite plane excepting that right opposite the charge there is an hemispherical bowls, that is actually hemispherical bowl shaped things protrudes out. Now, how does one do this problem? This problem is can be solved by the method of images and that is done by realising the following.

I have a charge q at a distance from the centre and so far as this hemisphere is concerned it can be kept at a 0 potential by putting in a charge minus q prime which is R cube by a at a distance b equal to R square by a, but remember that I have to also have to keep the infinite conductor as 0 potential. So, I need two more image charges. One is the image charge of q due to minus q this is at a equal distance a and a below and in place of minus q prime another not in place of, but for minus q prime another image which is plus q prime here. So, my total image problem consists of four charges q at a minus q at minus a minus q prime at r square over a plus q prime at minus r square over a. You you can convince yourself that these four charges would be keeping the potential at 0.

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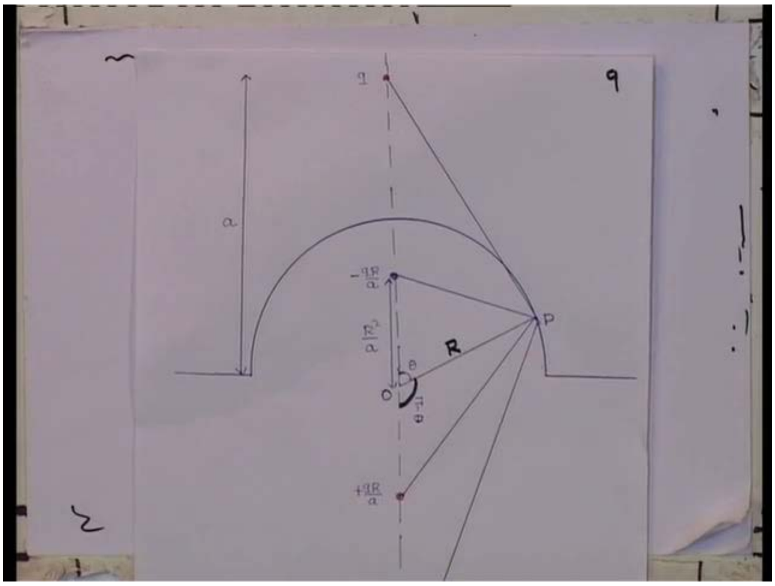

So, let me let me illustrate this by drawing the same picture in a slightly different fashion. So, this is this is the boss that we have talking about and I have a charge q here and the corresponding image is at minus q R by a, it has this has an image which is here and that had a image which is well below which I will come to little later. Now, let us calculate we are interested in calculating how much charge is on the surface of this hemispherical loss. So, take any arbitrary point P which is at obvious distance is radius R and an angle theta being made with this line.

Now, this distance for instance take the distance from q to P. Now, if I use the triangle law this is R square plus a square minus 2 a R cos theta and an identical statement I can make regarding this one also that is b square plus R square minus 2 b r cos theta. But look at this, this also I can use the triangle law, but because this angle is pi minus theta this will be, this distance will be b square plus r square plus 2 b a cos theta, b r cos theta and of course, you could have a similar thing here.

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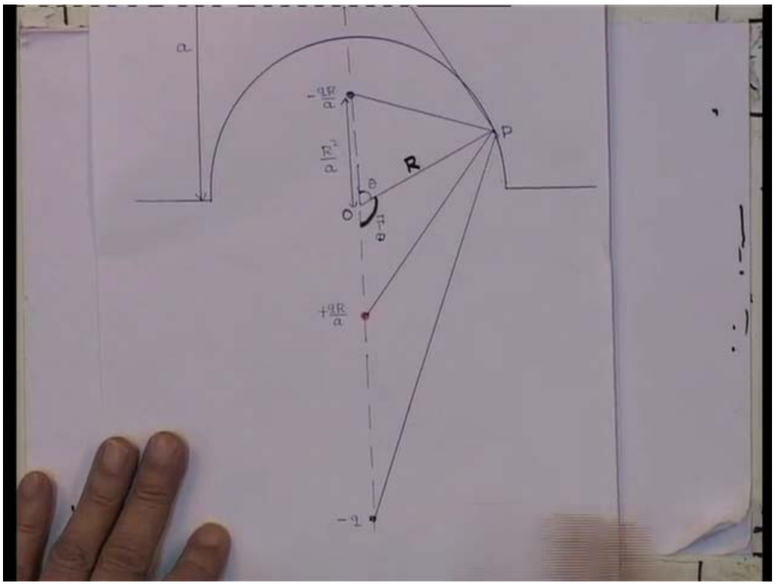

So, I have got two terms with plus cos theta term, and two terms with minus cos theta term.

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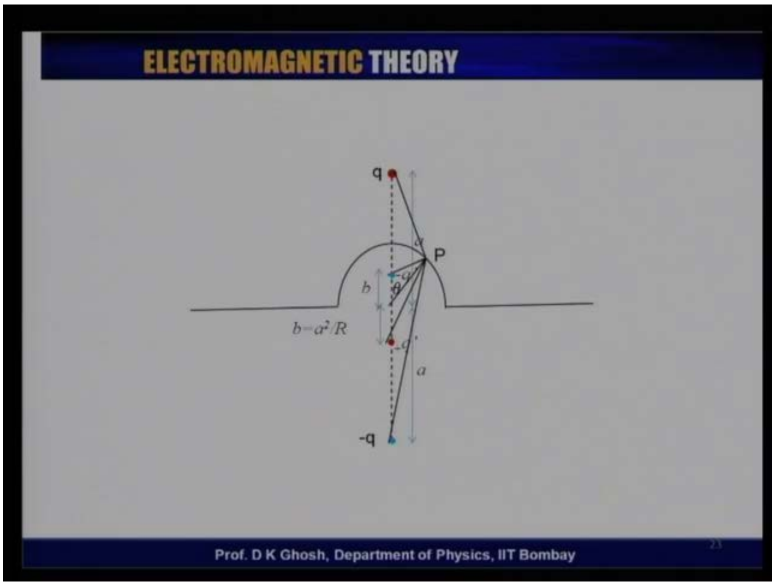

And this is the same picture which was shown in bigger thing there.

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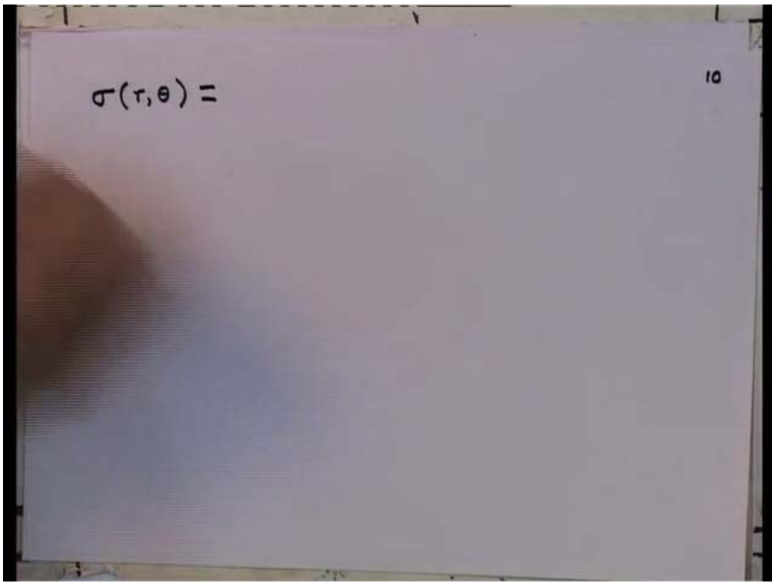

But let us borrow the expression for the charge density which is the function of r and theta, we had obtained this expression earlier.

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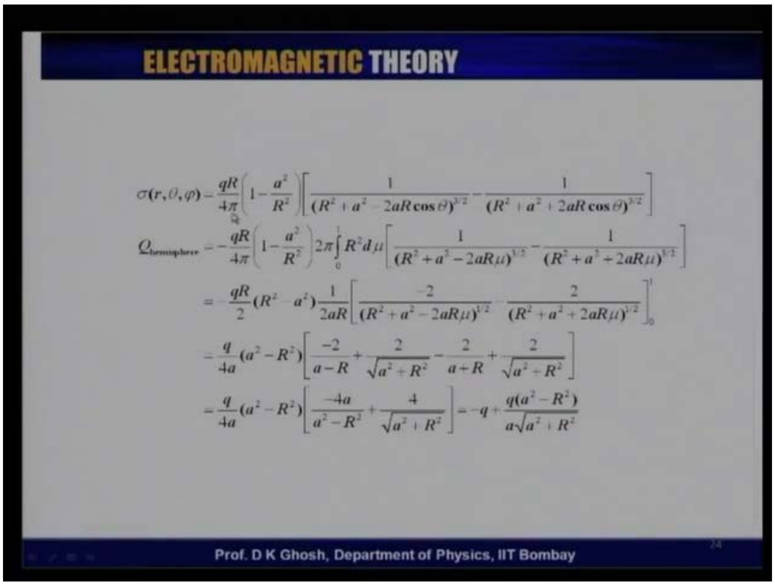

Namely for a sphere we had seen it is q R by 4 pi 1 minus a square by R square into only the first term 1 over R square plus a square minus 2 a R cos theta to the power 3 by 2. Now, I have two more charges there because this is the surface charge density due to the original charge and its image.

Now, I have another pair that is the images of these two and this we have seen is simply there is a minus sign because the charges are opposite and instead of a minus sign I get a plus sign there. So, this is my total sigma of R theta due to four image charges. The two charges a charge and its image charges. So, if you now want to find out how much is the total charge on the hemisphere all that you need to do is to take this expression and integrate it over the surface of the sphere.

So, this is actually fairly straight forward. So, basic idea is that you integrate so most of these are constant. The, there is no phi dependence so I get a 2 pi as a angle and R square sin theta d theta so I substitute cos theta is equal to mu. So, I get R square d mu and these two expressions are very similar and the integrations are straight forward because it is just a power law and if you do the integration you find that the net charge on hemisphere is given by minus q plus q times a square minus r square by a into root of a square plus r square. This is precisely the expression given for the Jacksons problem

So, so far we have been talking about the the application to spherical geometry. We will spend a little more time in the next lecture on the image problem, but this time taking up a problem in cylindrical geometry, and the next what we want to do is another powerful special technique, which is applicable purely for two dimensional problems and which is known as the method of conformal mapping. So, with this two our special techniques will be generally done.