

Electromagnetic Theory
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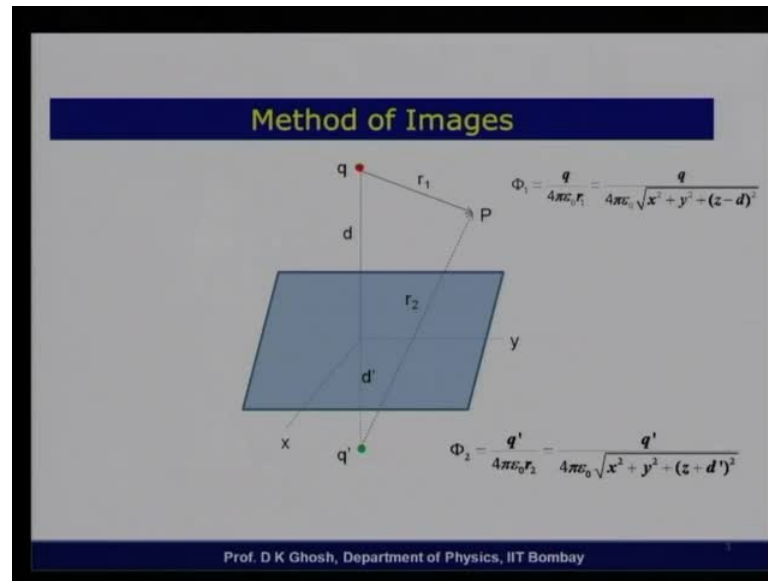
Module - 2
Electrostatics
Lecture - 17
Special Techniques

We have been talking about solutions of Laplace's equation and Poisson's equations, subject to a given boundary condition. One of the things that we learnt is, what is known as the uniqueness theorem. Uniqueness theorem implies that when we obtain a solution to Laplace's or Poisson's equation, which satisfies the given boundary condition, that solution is one and the only one solution. Now, this has some very interesting consequences, one of the consequences is that if by some means we are able to, let us say guess a solution.

So, we have seen there are ways of getting at the solutions analytically. These are not always visible and they also require certain use of high level mathematics, but supposing we appeal to our intuition, and for a given problem we are able to guess that this could have been the solution. Now, if we can show that that solution that we have guessed, satisfies the Laplace's or Poisson's equations as the case may be, and also it satisfies the boundary conditions for the given problem, then because of the uniqueness theorem, we are guaranteed that the solution that we have guessed is the only solution.

Today what we are going to do is to appeal to this sense of intuition that we have. And we will be talking about some a method which goes by the name of method of images. The name as you can see is borrowed from optics and we will see why such a name comes out.

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So, let me start with an example. Supposing I have a plane, a conducting infinite conducting plane, I have taken to be in the $x y$ plane. Now, this plane I will take it to be grounded. This is for convenience, we could also make it at a given constant potential, but let us say it is at 0 potential. Now, what we have is, we have a charge q located at a distance d above this conducting plane. So, that is for convenience, I will take this direction as the direction of my z axis.

So, I have an $x y$ plane which is at 0 potential and a charge q located at a distance d along the z axis. Now, notice everywhere in space other than at the location of charge q , I have Laplace's equation, because there are no sources there. So, I need to solve Laplace's equation everywhere in space subject, everywhere actually in the half space above the infinite conductor. Subject to the condition that the potential on the conducting plane is equal to 0.

Now, let us see what is the best way in which I can do it, but notice that supposing I am interested in calculating the potential at a distance r_1 from the charge q . I imagine that there is a charge, which I will name as the image charge, which is located at a distance d' below the conductor. Now, please try to understand, what is meant by below the conductor, in this context? Just as when we stand in front of a mirror, we know what is the distance of, let us say the object the us or the person who is standing in front of a mirror, from the mirror, now that is a real distance, from the mirror a real distance.

Now, image of this person is also visible to us and if it is a plane mirror, we know that the image is located at a distance, which is equal to the object distance, but behind the mirror. The behind the mirror also implies, that the whole process is the fictitious. The idea of a of the image is fictitious, because you cannot actually put a screen and catch that image there. So, in that sense what I am talking about, if a fictitious charge q' , which is located at a distance d' below the mirror, that is along the z axis. And let us look at that as a result of the real charge q , which is show here by a red dot and the image charge q' , which is shown by the green dot, what is the potential at the point r_1 from the point q .

Let us also assume, that the distance happens to be r_2 from the point q' , that is from the image charge. Now, it is easy to write down the potential equation, we of course, see that the potential at the point p due to the charge q is $\frac{q}{4\pi\epsilon_0 r_1}$. Well notice that, plane is the $x-y$ plane therefore, the point p 's coordinates can be written as $x-y$ and so the distance is what I am looking at. So, it is $x^2 + y^2 + z^2$ and plus d' whole square. So, this is this is the distance from the charge q to the point p .

Now, I can write down the expression for the potential ϕ_2 due to the image charge q' , from which the point p is located at a distance r_2 and that is then given by $\frac{1}{4\pi\epsilon_0} \frac{q'}{\sqrt{x^2 + y^2 + z^2 + d'^2}}$. So, this is the two potentials I have got. Now, my total potential at the point p , is thus given by $\phi_1 + \phi_2$. Now, suppose I now put in the condition, now this is a general expression for the point the potential at an arbitrary point p , which is $x-y-z$.

So, suppose I now impose the condition, the potential acts on the plane must be equal to 0, then my requirement is just to go back remember the plane is at $z = 0$, so the potential which is $\phi_1 + \phi_2$, I must just add them up and put $z = 0$ in those.

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ELECTROMAGNETIC THEORY

Method of Images

1. Electric field exists only on the upper half plane.
2. Potential of the grounded conductor is zero.

Potential is zero on the grounded plane, everywhere on $(x,y,0)$

$$q^2(x^2 + y^2 + d'^2) = q'^2(x^2 + y^2 + d^2)$$
$$(q^2 - q'^2)(x^2 + y^2) + (q^2 d'^2 - q'^2 d^2) = 0$$

⇓

$$q' = -q$$
$$d' = d$$

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And that gives me q square into x square plus y square plus d prime square, is equal to q prime square into x square plus y square plus d square. Now, notice that this equation I need to solve, but one thing I observe, that in order that ϕ_1 plus ϕ_2 should be equal to 0, then it is required that the potential ϕ_1 and the potential ϕ_2 must be of opposite sign. In other words q and q prime must have opposite sign. So, as a result though I have written here as q square, the two denominators I have equated and therefore, you notice that this gives me q minus q prime whole square into x square plus y square plus q square d prime square minus q prime square d square is equal to 0.

Of course, I know q prime must have a sign which is opposite to that of q . Now, this equation should be satisfied at all points x y on the plane. Therefore, if you refer to this equation again, you notice q square must be equal to q prime square because this is, I require at all points x square plus y square, at all points x and y this equation must be satisfied. Therefore, these two terms must be separately equal to 0. Now, which implies q square must be q prime square, but we have already observed that q and q prime must be of opposite sign.

So, that tells me q prime should be equal to minus q . You substitute it into this equation, then you find that I must have d prime equal to d . Now, notice what it actually means? The image charge has the same magnitude, but has an opposite sign to that of the object charge and the image distance that is the distance at which the image charge is located is

equal to the object distance, but below the conducting plane. So, now notice the electric field only exists in upper half plane because there is this infinite plane, which is conducting plane, which is of course, screening out of the field below the plane.

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The image shows a handwritten derivation on a whiteboard. The first line is the potential function:
$$\phi = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{\rho^2 + (z-d)^2}} - \frac{1}{\sqrt{\rho^2 + (z+d)^2}} \right]$$
The second line is the electric field as the negative gradient of the potential:
$$\vec{E} = -\nabla\phi = \left(-\hat{\rho} \frac{\partial}{\partial \rho} - \hat{k} \frac{\partial}{\partial z} \right) \phi$$
The third line is the final expression for the electric field after differentiation:
$$= \frac{q}{4\pi\epsilon_0} \left\{ \left[\hat{\rho} \frac{\rho}{[\rho^2 + (z-d)^2]^{3/2}} - \hat{\rho} \frac{\rho}{[\rho^2 + (z+d)^2]^{3/2}} \right] + \hat{k} \cdot \left(\frac{z-d}{[\rho^2 + (z-d)^2]^{3/2}} - \frac{z+d}{[\rho^2 + (z+d)^2]^{3/2}} \right) \right\}$$

Now, for convenience let me write this total potential phi as equal to q by 4 pi epsilon 0. So, 1 over instead of carrying on that square root of x square plus y square, let me denote it by rho square x square plus y square is equal to rho square. So, I have got rho square plus z minus d whole square minus because of the opposite sign and I have got root of rho square plus z plus d whole square. Now, if this is the potential, then I know how to calculate the electric field? The electric field as you all know is minus the gradient of the potential phi.

And you must have observed that this is basically a cylindrical geometry. The, I have the x y plane, which is my polar plane the usual r theta plane, which rho theta plane in this case and of course, this z is the cylindrical z direction, so that my gradient is basically given by rho cap d by d rho plus, well there is a minus sign in front plus minus k d by d z. So, this phi therefore, if I look at the electric field, I have a component in the plane or component parallel to the plane and a component perpendicular to plane and these differentiation are extremely easy to work out.

So, this is equal to q over 4 pi epsilon 0. So, notice that I have a rho the minus sign will go because I am differentiating 1 over square root and therefore, this gives me rho square

plus z minus d whole square to the power 3 by 2. The half of the differentiation will go with that 2, that I will get from rho square differentiation therefore, I get a rho there. And minus rho this is from the second term and I get from here the same type of thing namely rho square plus z plus d whole square to the power 3 by 2. Now, that is the rho component and then of course, I have a k component.

So, the k component will be differentiation with respect to z and which is again very similar. So, I have once again rho square plus z minus d whole square to the power 3 by 2 and this term gives me z minus d and the other term gives me z plus d. It is fairly straight forward differentiation.

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ELECTROMAGNETIC THEORY

Method of Images

$$\Phi = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{\sqrt{\rho^2 + (z-d)^2}} - \frac{1}{\sqrt{\rho^2 + (z+d)^2}} \right); \rho^2 = x^2 + y^2$$

$$\vec{E} = -\nabla\Phi = \left(-\hat{\rho} \frac{\partial}{\partial \rho} - \hat{k} \frac{\partial}{\partial z} \right) \Phi$$

$$= \frac{q}{4\pi\epsilon_0} \left[\hat{\rho} \left(\frac{\rho}{(\rho^2 + (z-d)^2)^{3/2}} - \frac{\rho}{(\rho^2 + (z+d)^2)^{3/2}} \right) + \hat{k} \left(\frac{z-d}{(\rho^2 + (z-d)^2)^{3/2}} - \frac{z+d}{(\rho^2 + (z+d)^2)^{3/2}} \right) \right]$$

$$\sigma = \epsilon_0 E_{z=0} = -\frac{qd}{2\pi(\rho^2 + d^2)^{3/2}}$$

$$Q_{\text{ind}} = \frac{qd}{2\pi} \int_0^\infty \frac{2\pi\rho d\rho}{(\rho^2 + d^2)^{3/2}} = qd \frac{1}{(d^2 + d^2)^{3/2}} \Big|_0^\infty = -q$$

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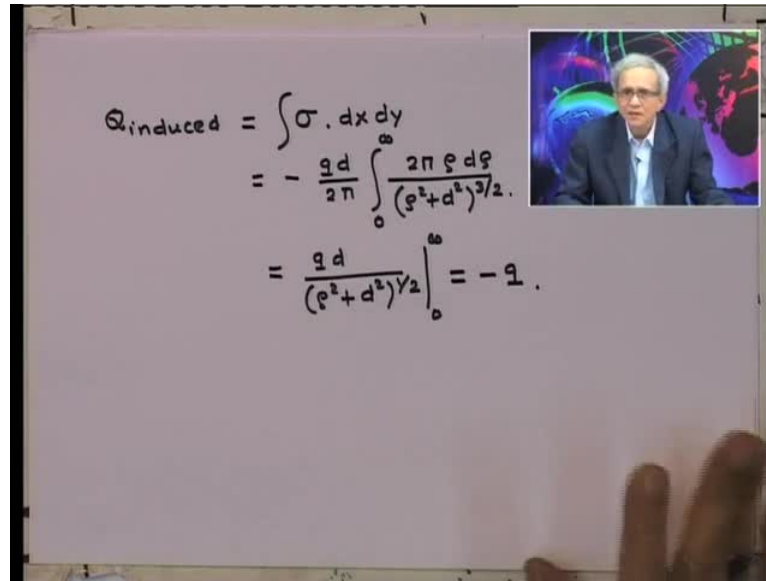
Now, I know that the charge density the charge density is induced on the plane and the charge density is nothing but epsilon 0 times the normal component of the electric field. The, since it is a conductor, the tangential component of the electric field must be 0 and the normal component of the electric field is nothing but the z component.

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$$\begin{aligned}\phi &= \frac{q}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{\rho^2 + (z-d)^2}} - \frac{1}{\sqrt{\rho^2 + (z+d)^2}} \right] \\ \vec{E} &= -\nabla\phi = \left(-\hat{\rho} \frac{\partial}{\partial \rho} - \hat{k} \frac{\partial}{\partial z} \right) \phi \\ &= \frac{q}{4\pi\epsilon_0} \left\{ \hat{\rho} \left[\frac{\rho}{[\rho^2 + (z-d)^2]^{3/2}} - \frac{\rho}{[\rho^2 + (z+d)^2]^{3/2}} \right] \right. \\ &\quad \left. + \hat{k} \cdot \left(\frac{z-d}{[\rho^2 + (z-d)^2]^{3/2}} - \frac{z+d}{[\rho^2 + (z+d)^2]^{3/2}} \right) \right\} \\ \sigma &= \epsilon_0 E_z = - \frac{q d}{2\pi (\rho^2 + d^2)^{3/2}}\end{aligned}$$

Therefore, the sigma which is my charge density, which is epsilon 0 times E z therefore, I have to simple worry about this component, but notice that the plane is at z is equal to 0. So, as a result both these denominators happens to be the same and I am left with z minus d by rho square plus d square to the power 3 by 2 and here z plus d, so z goes way and I will left with epsilon 0 times. So, there is a minus d minus d's are 2 minus d's come in and I will be left minus q is already there, I will write down a d, and that two d and 4 pi epsilon 0 gives me 2 pi times rho square plus d square to the power 3 by 2. So, this is my, this is my charge density that is induced on the plane. Now, let us look at what is the total charge that I get?

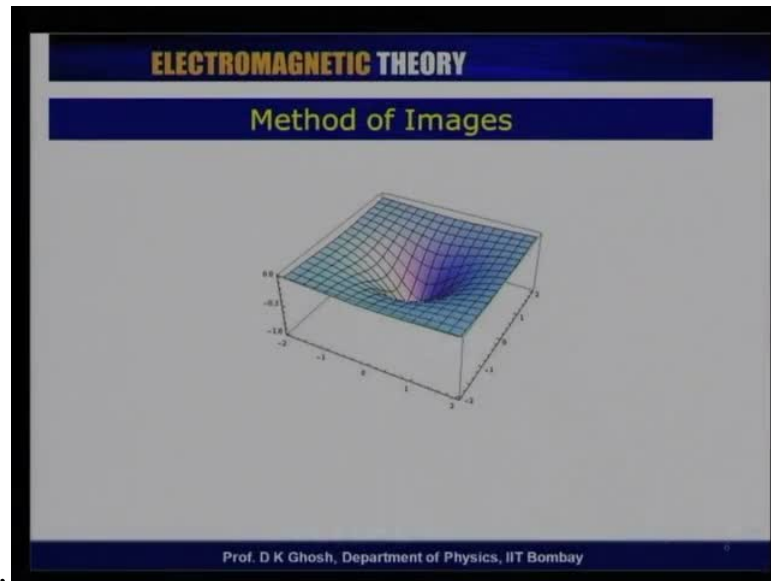
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$$\begin{aligned} Q_{\text{induced}} &= \int \sigma \cdot dx dy \\ &= -\frac{q d}{2\pi} \int_0^{\infty} \frac{2\pi \rho d \rho}{(\rho^2 + d^2)^{3/2}} \\ &= \frac{q d}{(\rho^2 + d^2)^{1/2}} \Big|_0^{\infty} = -q. \end{aligned}$$

Now, I know that in order to get the total charge, that is induced on the plane, I know to integrate the charge density sigma over the entire plane. So, sigma d x d y if you like, but we have written it in r theta coordinate. Therefore, this is equal to minus, I had q d over 2 pi which there and the charge density is given by 1 over rho square plus d square to the power 3 by 2. And we know that the angle integration gives me 2 pi because there is no angle dependence of my integral, and of course, rho d rho. And rho integral is from 0 to infinity.

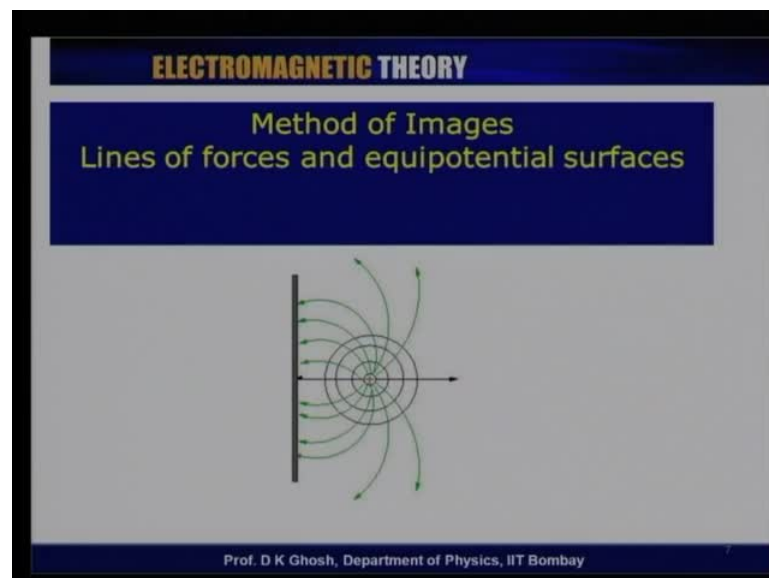
This is a fairly straight forward integral because rho d rho is the differentiation of rho square and what I am left with then is simply q d divided by rho square plus d square to the power half, and these limits are from 0 to infinity, which simply gives me minus q. So, you notice that the total charge induced on the surface is exactly equal to minus the charge that is there in front of the plane. So, that is the amount of charge that is induced.

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Now, here what I have done is to, sort of give me a picture of what does the charge density look like. Notice that charge density is negative, so close to the directly opposite to the point where the object charge is located, the charge density is large and of course, it sort of tapers off as we as we go to longer distances along the plane.

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You could physically look at what do the lines of forces look like. Let me assume, here of course, I have drawn the conducting plane on the site, but that should not matter. So, here is the charge positive charge with a red dot and lines are emanating from it shown

by this green arrows and I know that all the conductor they must approach normally. So, this is the way the lines of forces are and these circles, which are actually spheres, they are my equipotential surfaces. So, this is the way the geometry looks like.

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ELECTROMAGNETIC THEORY

Method of Images
Field strength at the position of the charge
At $z=d, \rho=0$

$$\vec{E}(0, d) = \frac{q}{4\pi\epsilon_0} \left[\hat{k} \left(\frac{z+d}{(\rho^2 + (z+d)^2)^{3/2}} \right) \right]_{\rho=0, z=d} = -\hat{k} \frac{q}{4\pi\epsilon_0} \frac{1}{4d^2}$$

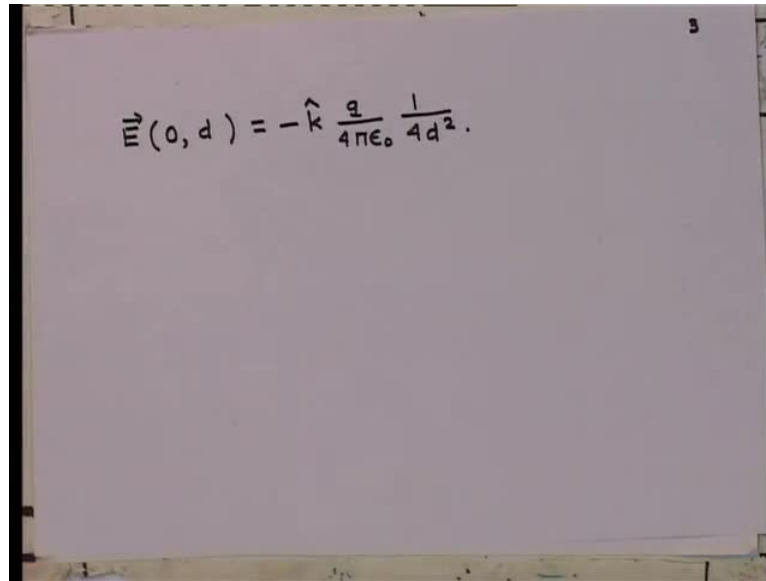
Force on the charge q due to surface charge

$$\vec{F} = -\hat{k} \frac{q^2}{4\pi\epsilon_0} \frac{1}{4d^2} = \text{Force between the charge q and its image}$$

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Now, notice that, so what we have done so far, is to say that my original problem is replaced by taking a charge of magnitude equal to but sign opposite to that of the object charge located at a distance d below the conducting plane. Now, let us look at, what is the field that is generated at the position of the charge? Now, where does it come from? There you see there is an induced charge on the conductor, so this charge density will be giving a field at the location but remember we have calculated the electric field everywhere, due to the problem. Therefore, what we are interested in is to look at what is the electric field?

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$$\vec{E}(0, d) = -\hat{k} \frac{q}{4\pi\epsilon_0} \frac{1}{4d^2}$$

The electric field everywhere it had been done, near the position of the charge and remember the position of the charge is ρ is equal to 0 and the distance is equal to d . So, if you recall the original expression, which I given back here. So, I am looking at ρ is equal to 0, now once you put ρ is equal to 0 and z is equal to d . So, I have an electric field, which since ρ is equal to 0 both this numerators vanish and only the k component will remain, and you put here ρ is equal to 0, you notice that this term, this is report ρ is equal to 0 and also z is equal to 0, both these terms become the same.

And I am left with this term here, which is q by $4\pi\epsilon_0 k$ and this we have written down, minus z plus d divided by ρ square plus z plus d whole square. The first term which is z minus d in the numerator having vanished because I I am having it at the location, which means at z is equal to d . Therefore, only one term remain and this term is as you can see from here, I have to put z is equal to d , so that is a $2d$ there and ρ is equal to 0, so I have left with the $4d^2$ there.

So, $E(0, d)$ is along minus k direction and q by $4\pi\epsilon_0$ times 1 over $4d^2$. Now, if the electric field is this at the location of the charge q , then the force that the charge q experiences due to the induced charges on the surface is simply obtained by multiplying the electric field at that point, with the charge q . And as a result the electric field, the force between the charge, the the force on the charge due to the induced charges is given by this expression. And this is nothing but if I replace that conducting

surface and I just has a charge and its image, which are now separated by a distance 2 d. So, this is nothing but the force as dictated by Coulomb's law.

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ELECTROMAGNETIC THEORY

Method of Images

The field at the surface due to q is $E_z = \frac{E_z}{2}$ as the other half is due to image charge.

$$E_z = \frac{q}{8\pi\epsilon_0} \left[\left(\frac{z-d}{(\rho^2 + (z-d)^2)^{3/2}} - \frac{z+d}{(\rho^2 + (z+d)^2)^{3/2}} \right) \right]_{z=0} = \frac{q}{8\pi\epsilon_0} \frac{-2d}{(\rho^2 + d^2)^{3/2}}$$

Force exerted by q on the surface

$$F = \int \sigma E_z d\rho = \frac{q^2 d^2}{8\pi^2 \epsilon_0} \int \frac{1}{(\rho^2 + d^2)^3} 2\pi\rho d\rho = \frac{q^2 d^2}{4\pi\epsilon_0} \left[\frac{1}{(\rho^2 + d^2)^2} \right]_0^\infty = \frac{q^2}{16\pi\epsilon_0 d^2}$$

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Let me look at a slightly different thing, what is the field on the surface due to the charge q? Now, remember I have calculated that total electric field, but I must only consider half of this electric field. If I am looking at how much is the force that a charge q on the surface we will experience, because the image charge is actually a fictitious charge. Therefore, on the field that I have calculated, I must look at my my conducting plane is at z is equal to 0.

So, I must look at half of the field that we have calculated and this is again, the is prime, if you look back the expressions that we had once again, this is the expression that survives, and I must put here z is equal to 0, but remember rho can be anything now. And as a result the expression that I get is q by 8 pi epsilon 0 minus 2 d by rho square plus d square to the power 3 by 2. So, I I I have got the electric field.

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The image shows a whiteboard with handwritten mathematical derivations. At the top, the electric field vector is given as $\vec{E}(0, d) = -\hat{k} \frac{q}{4\pi\epsilon_0} \frac{1}{4d^2}$. Below this, the text 'Force exerted on the surface' is written. The force vector \vec{F} is then calculated as $\vec{F} = \int \sigma E_z d\rho$, which is further simplified to $= \frac{q^2 d^2}{8\pi^2 \epsilon_0} \int_0^\infty \frac{2\pi \rho d\rho}{(\rho^2 + d^2)^3} = \frac{q^2}{16\pi\epsilon_0 d^2}$.

On the surface and I know that the surface has a charge density sigma. Therefore, force exerted on the surface that is f, that is nothing but sigma times the electric field at the surface times d rho. That is my actually d square rho because it is integration is over the surface and this integration is fairly straight forward to do. Look at this, that I already had an electric field which is rho square plus d square to the power 3 by 2, the charge density also has a very similar variation.

So, if you plugged in charge density there, you will find this is equal to q square d square divided by 8 pi square epsilon 0 and rho square plus d square to the power 3 by 2 being the expression for both E z and sigma. So, I have a q, 2 pi is my angle integration and of course, rho d rho. And this is from 0 to infinity. Calculate this and you will get this to be equal to q square over 16 pi epsilon 0 d square. So, which is of course, the same expression as what we got earlier, that is the force, that is exerted by the plane on the charge. Obviously by Newton's third law these must be equal and opposite, but notice one thing, there is a bit of a hand weaving came in because it is important to realize that the force that is there on the surface is not the total field due to the charge and its image, but we must take half of that.

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ELECTROMAGNETIC THEORY

Method of Images
Electrostatic Energy : Work done to bring the charge q from infinity (bring it along the z -axis). This is half the energy of the charge q and its image.

$$\vec{F}(0, z) = -\frac{q^2}{16\pi\epsilon_0 z^2} \hat{k}$$
$$E = +\int_{\infty}^d \frac{q^2}{16\pi\epsilon_0 z^2} dz = -\frac{q^2}{16\pi\epsilon_0 d}$$

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Now, I can look at the electro static energy to the problem. How do I calculate the electro static energy? I simply try to bring in, supposing initially I just had that plane and then a charge q is to be brought from infinity to the point where it is to be located, which is at a distance d from the conducting plane. I know that electric forces are conservative and as a result, I can bring it the charge from infinity any way I like and its very convenient because I know the force, to bring it from infinity along the z axis because just little while back I have calculated what is the force, that is exerted on the charge, when it is at a distance d .

And I can use the same expression and say that when it is at a distance z what is the force that is exerted? And that force as we have seen when it was a distance d it was q square by $16 \pi \epsilon_0 d$ square. So, when it is at a distance z it is obviously q square by $16 \pi \epsilon_0 z$ square and the force is along the minus k direction. So, since the force is minus q square by $16 \pi \epsilon_0 z$ square along the negative z direction, the total energy the E is not be confused with the electric field but the total energy is simply this quantity integrated from infinity to d , and that is equal to minus q square by $16 \pi \epsilon_0 d$. So, that is my total energy of the problem.

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ELECTROMAGNETIC THEORY

Method of Images

Electrostatic Energy : Work done to bring the charge q from infinity (bring it along the z -axis). This is half the energy of the charge q and its image.

$$\vec{F}(0, z) = -\frac{q^2}{16\pi\epsilon_0 z^2} \hat{k}$$

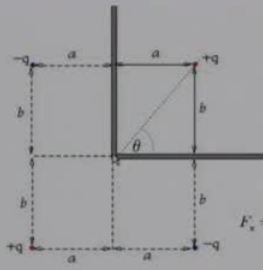
$$E = +\int_{\infty}^d \frac{q^2}{16\pi\epsilon_0 z^2} dz = -\frac{q^2}{16\pi\epsilon_0 d}$$

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ELECTROMAGNETIC THEORY

Method of images



$$F_x = \frac{q^2}{4\pi\epsilon_0} \left[\frac{1}{4a^2} + \frac{1}{(2\sqrt{a^2 + b^2})^2} \cos \theta \right] = \frac{q^2}{4\pi\epsilon_0} \left[\frac{1}{4a^2} + \frac{a}{(a^2 + b^2)^{3/2}} \right]$$

$$F_y = \frac{q^2}{4\pi\epsilon_0} \left[\frac{1}{4b^2} + \frac{b}{(a^2 + b^2)^{3/2}} \right]$$

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Now, this method of images can also be applied for many other problem. Let me give you an example of when I have got two conducting surfaces, making a right angle with each other. So, look at this situation, I have one conducting plane, semi infinite conducting plane at y is equal to 0 and another semi infinite conducting plane at x equal to 0. And the, I have got a charge here, which is located in other words this is in xz plane, this is in yz plane z axis is out of the screen. And I have charge plus q , which is located here. Now, let us look at, what would be the image problem corresponding to

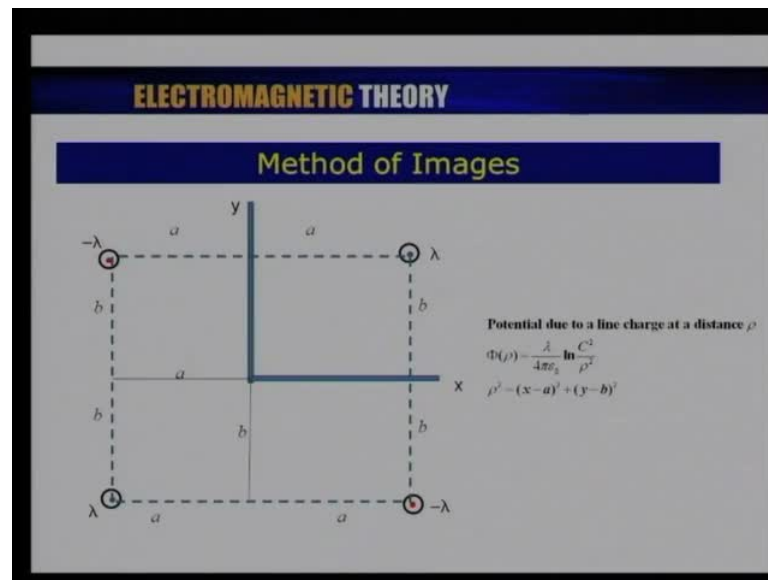
this? The, as I know that if I had two plane mirrors at right angle to each other, I will first of course, have an image of this at a distance along this mirror at a distance this, which is minus q and another image on the other screen, which is at a distance again, say if this is at a distance b it is at minus b which is also minus q .

But these images for example, these images, which is due to this will have its image here and similarly, this will have an image here and the whole process will be completed, if I have a plus q here, minus q there, plus q there, and a minus q there. Now, this is of course, I did my hang way. So, what you do is to write down an as an exercise. The potential due to these charge distribution and show that that by simple symmetry, if you want the total potential on the two plane surfaces to be equal to 0, then these must be the image charges, that is plus 1 plus q at a diagonally opposite corner and a minus q there and a minus q on other side.

Now, let us look at, I will not repeat the potential expression, they all work out fairly straight forward, but let us look at what is the force, that is exerted on this charge, due to the charges that are induced on these two planes? The charges that are induced on these two planes, we have seen equivalently we can consider the force between these charge and the image charges. Now, notice that this image is between a charge plus q and a minus q located at a distance to a along the x axis. Therefore, this gives me a force along the positive x axis. This of course, that will give me any force along the x axis because along the y axis.

So, this force in minus q by $4\pi\epsilon_0$ minus 1 over $4a^2$ minus because it is along the minus x axis. Now, let us look at these two points, these are similar charges. The distance between them is 2 times square root of a^2 plus b^2 and since this is along the diagonal, what I need is to take the x component of it, which is a divided by the square root again. So, as a result my net force is q^2 by $4\pi\epsilon_0$ 1 over minus 1 over $4\pi\epsilon_0 4a^2$ plus, because this is between 2 positive charges, a by square root of a^2 plus b^2 to the power 3 by 2. And F_y similarly, is obtained by replacing a with b by symmetry, so these are the forces that are exerted.

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Let me take a little more complicated problem and this problem is the problem of... Again the, I have the same two conductors intersecting at a right angle, one in the xz plane the other in the yz plane, but instead of a single charge, what I have is at the location a, b , I have a line charge, line charge of charge density λ . Now, clearly now remember that this line charge is parallel to z axis. It is an infinite line charge, now we know the electric field and the potential due to infinite line charge.

By simple intuition it follows, that I must have a line charge object charge plus λ here, a image line charge minus λ there, image line charge minus λ there, and an image line charge plus λ here. If you combined these four, then the potential condition on both the surfaces will be satisfied.

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ELECTROMAGNETIC THEORY

Method of Images

Potential at (x,y) (θ is independent of x)

$$\Phi(x,y) = \frac{\lambda}{4\pi\epsilon_0} \left[\ln \frac{C^2}{(x-a)^2 + (y-b)^2} + \ln \frac{C^2}{(x+a)^2 + (y+b)^2} - \ln \frac{C^2}{(x-a)^2 + (y+b)^2} - \ln \frac{C^2}{(x+a)^2 + (y-b)^2} \right]$$

$\Phi(0,y) = \Phi(x,0) = 0$

$$E_x|_{y=0} = -\frac{\partial\Phi}{\partial x}|_{y=0} \quad E_y|_{x=0} = -\frac{\partial\Phi}{\partial y}|_{x=0} = 0$$

$$E_x|_{y=0} = -\frac{\partial\Phi}{\partial x}|_{y=0} = -\frac{4\lambda abx}{\pi_0 [(x-a)^2 + b^2][(x+a)^2 + b^2]} = \frac{\sigma|_{y=0}}{\epsilon_0}$$

$$E_y|_{x=0} = -\frac{\partial\Phi}{\partial y}|_{x=0} = -\frac{4\lambda aby}{\pi_0 [(y-b)^2 + a^2][(y+b)^2 + a^2]} = \frac{\sigma|_{x=0}}{\epsilon_0}$$

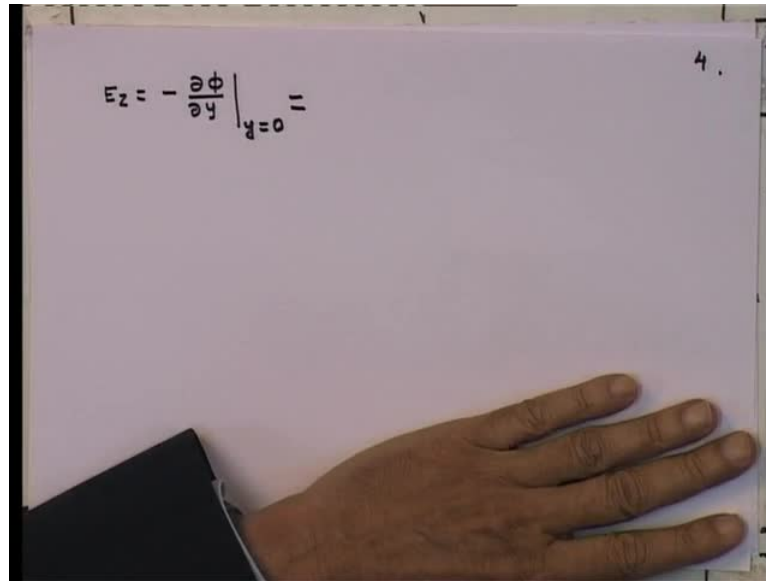
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We all know, how to write down the potential at an arbitrary point x y due to a line charge and that was the lambda by 4 epsilon 0 logarithm of a constant, which I have written as C square there divided by x minus a square plus y minus b square. Now, this is the potential, this term is the potential due to your original real line charge. I have now let us look at the pictures again, I have a line charge here at a distance a, that is along the x axis at a location minus a at a location minus b along the y axis and this at a minus a minus b.

So, look at that, so I have got at a b that is my original line charge, at minus a minus b, which is of the same sign as this, because that is the one along the diagonal corner. Then minus because the line charge has changed from lambda to minus lambda and this is at a minus b, and this is at this is at a minus b, this is at minus a plus b. Now, you can check that this expression if you take, the tangential component of the electric field, remember that the two conductors are in the x z plane and the y z plane.

So, the tangential component on the y equal to 0 plane is d by d x of the electric field and tangential component on the x equal to 0 plane is the d by d y of their potential. So, these automatically are satisfied, if you take y equal to 0 and x equal to 0 respectively. Now, let us look at what happens to the normal component. So, the normal component on the this plane, is along the y direction. Therefore, look at this expression.

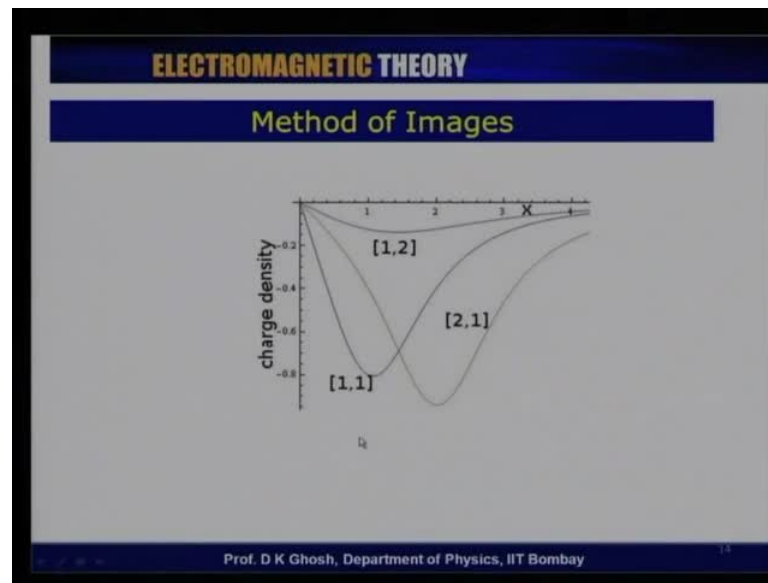
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$$E_z = - \frac{\partial \phi}{\partial y} \Big|_{y=0} = 4 \lambda a b x \pi \epsilon_0 x \sqrt{x^2 + a^2 + b^2}$$

So, let me calculate what is the normal component namely E_z , that is equal to minus $d\phi$ by dy , but I must take this at y equal to 0. Now, this is, this will have two terms. So, I need $d\phi$ by dy and at y equal to 0 and this is all these logarithms, I know $\log a$ by $\log b$ \log of a by b is $\log a$ minus $\log b$, so open these up. And logarithms of x has a differentiation 1 over x . So, straight forward differentiation substitute y is equal to 0 and you find that this is given by $4 \lambda a b x \pi \epsilon_0 x \sqrt{x^2 + a^2 + b^2}$. And this quantity is the expression for charge density on the y equal to 0 plane.

You do not actually have to work out the electric normal component of the electric field on the x equal to 0 plane. The symmetry is obvious, you simply put you know a going to b and there you get the result. Therefore, I got and this and this is the result for the expression and of course, x going to y .

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So, I get these two. What I have done here, is to plot the charge density for different location of the original image, the original line charge. These are only x y coordinates and so notice this that for example, this graph is the charge density that I have got, and this is only along the, this distance is along x. Remember that one of the planes is at y is equal to 0, x is equal to 0, so obviously I am looking at the other plane. Now, so if I have the x distance equal to 2 and y distance equal to 1.

This is just arbitrary location x distance equal to 2 and y distance equal to 1, how much is the charge density that, is generated on that plane? So, let us suppose it is given by this expression, you can plot it using any plotting. Now, let us reduce the x distance. So, you notice as x distance is reduced the charge density spreads out much more and here for example, I am increasing the, for a correspondingly I am increasing the y distance, and this is of course, become much more shallow.

And now remember in all cases that total charge that is there would still be the same, but obviously if I am closer to one of the conducting plane, then the charge density directly opposite that point should be much more. This is what you see here, because here the line charge is at 2, 1 therefore, the peak is at the point 2. Here it is at 1, 1 so the peak is at the point 1.

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ELECTROMAGNETIC THEORY

Method of Images

on $y=0$ plane, the charge density

$$\sigma = \frac{\lambda}{\pi} \left[\frac{1}{(x-a)^2 + b^2} - \frac{1}{(x+a)^2 + b^2} \right]$$

$$Q'_{y=0} = \frac{\lambda}{\pi} \left[\int_0^{\infty} \frac{dx}{(x-a)^2 + b^2} - \int_0^{\infty} \frac{1}{(x+a)^2 + b^2} \right]$$

$$= -\frac{2\lambda}{\pi} \tan^{-1} \frac{a}{b}$$

$$Q'_{x=0} = -\frac{2\lambda}{\pi} \tan^{-1} \frac{b}{a}$$

$$Q_{\text{total}} = -\lambda$$

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Now, let me just pick up y is equal to 0 plane and calculate how much is the total charge induced on that? We had seen that the, all that you need to do is to calculate the electric field, the normal component of the electric field, on the surface and multiply it with epsilon 0 and that gives me the total charge density. So, the charge density is obtained by these expression on the y is equal to 0 plane. Now, obviously on the x equal to 0 plane, all that you need to do is to replace a with b and x with y .

So, I am not repeating that argument. Now, how much is the total charge induced on the y is equal to 0 plane? So, this is simply an integration over x , remember these are semi infinite plane, so it goes from 0 to infinity. Now, these are standard 1 over x square plus a square type of integral giving me \tan inverse function. So, this works out to minus 2 lambda by pi \tan inverse of a by b . By symmetry the total charge on the x equal to 0 plane is given by 2 lambda by pi minus 2 lambda by pi times \tan inverse of d by a . Now, what is the total charge?

Remember \tan inverse x plus \tan inverse 1 over x is equal to pi by 2. So, if you just add of these expressions, you get that the total charge, that is induced on these two surfaces is just equal to minus lambda. So, once again, the net charge that is induced on all these plane problems happened to be equal and opposite to the, opposite to equal in magnitude, but opposite to the charge or the charge density, that you have put in. What I am going to

do is this, that... Now, is it restricted to only the planes, planar conductor, it is my easier. The name method of images came from there.

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ELECTROMAGNETIC THEORY

Method of Images

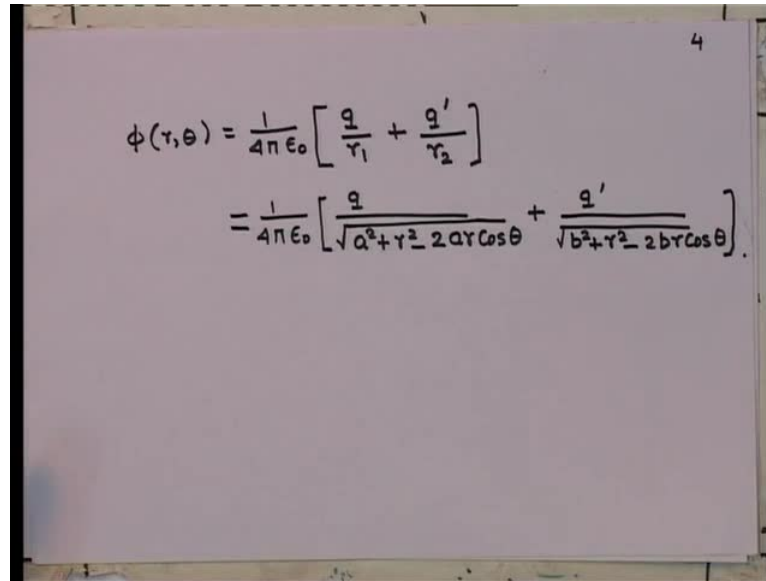
$$\Phi(r, \theta) = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{r_1} + \frac{q'}{r_2} \right]$$

$$= \frac{1}{4\pi\epsilon_0} \left[\frac{q}{\sqrt{a^2 + r^2 - 2ar\cos\theta}} + \frac{q'}{\sqrt{b^2 + r^2 - 2br\cos\theta}} \right]$$

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However, by suitable adaptation of this technique, you can also use the method of image for for instance a spherical conductor. Now, let us look at how this works out. So, what you see here is a spherical conductor which is grounded, I have a charge q, which i have taken at a distance a for the center of the spherical conductor. Now, I do not know where is its image charge, but let me state, that let the image charge be located at a distance b let its the charge d cube prime from the center again, and let us calculate the potential at the point P, due to the charge q. And I still call it is it is image charge q prime at an arbitrary point. Let the distance of that arbitrary point P the r 1 from q and r 2 from the, its image.

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$$\begin{aligned}\phi(r, \theta) &= \frac{1}{4\pi\epsilon_0} \left[\frac{q}{r_1} + \frac{q'}{r_2} \right] \\ &= \frac{1}{4\pi\epsilon_0} \left[\frac{q}{\sqrt{a^2 + r^2 - 2ar\cos\theta}} + \frac{q'}{\sqrt{b^2 + r^2 - 2br\cos\theta}} \right].\end{aligned}$$

The net potential is $\phi(r, \theta)$ is equal to $\frac{1}{4\pi\epsilon_0} \left[\frac{q}{r_1} + \frac{q'}{r_2} \right]$. Remember r_2 is my image charge, I am not quite said, how much is q' , I have not even stated where is the q' ? However, if you look at the picture again, you notice that, I can use the triangle law. If this angle, that is the point P is at a location at a distance R from O, if this angle between the line joining the center and the location of the charge q is theta, then normal triangle law tells me, that this r_1 is $r^2 + a^2 - 2ar\cos\theta$.

So, let us write it down $\frac{1}{4\pi\epsilon_0} \left[\frac{q}{\sqrt{a^2 + r^2 - 2ar\cos\theta}} + \frac{q'}{\sqrt{b^2 + r^2 - 2br\cos\theta}} \right]$. Then plus q' divided by well suppose, the image is at a distance d from the center, I will write it as $b^2 + r^2 - 2br\cos\theta$. So, this is the general expression for the potential.

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ELECTROMAGNETIC THEORY

Method of Images

For $r=R$, the potential vanishes, i.e. for all θ ,

$$\frac{q}{\sqrt{a^2 + R^2 - 2aR \cos \theta}} + \frac{q'}{\sqrt{b^2 + R^2 - 2bR \cos \theta}} = 0$$

$$q^2 (a^2 + R^2 - 2aR \cos \theta) + q'^2 (b^2 + R^2 - 2bR \cos \theta) = 0$$

$$\frac{q^2}{a} = \frac{q'^2}{b} = \frac{b}{a}$$

$$\sqrt{(b^2 + R^2)} = -\frac{q'}{q} \sqrt{(a^2 + R^2)} = \frac{\sqrt{b}}{\sqrt{a}} \sqrt{(a^2 + R^2)} \Rightarrow$$

$$ab - R^2 = \frac{b - R^2}{a}$$

$$\frac{q'}{q} = \frac{R}{a}$$

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Now, remember on the surface of the sphere, that is at r is equal to R my potential must vanish.

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4

$$\phi(r, \theta) = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{r_1} + \frac{q'}{r_2} \right]$$

$$= \frac{1}{4\pi\epsilon_0} \left[\frac{q}{\sqrt{a^2 + r^2 - 2ar \cos \theta}} + \frac{q'}{\sqrt{b^2 + r^2 - 2br \cos \theta}} \right]$$

$r = R \quad \phi(r, \theta) = 0$

$$q^2 (b^2 + r^2 - 2br \cos \theta) = q'^2 (a^2 + r^2 - 2ar \cos \theta)$$

$$\frac{q^2}{q'^2} = \frac{a}{b} \quad \therefore q' = -\sqrt{\frac{b}{a}} \cdot q$$

$$b^2 + r^2 = \frac{q'^2}{q^2} (a^2 + r^2) = \frac{b}{a} (a^2 + r^2) \Big|_{r=R}$$

$$\boxed{ab = R^2}$$

So, first thing that you observe is, these two must have opposite sign, otherwise I cannot make them vanish. So, q prime must have a sign opposite to that of q , but let us calculate its magnitude. The way to calculate the magnitude is, to say this plus this is equal to 0. So, I square them and I find that q square into b square plus r square minus $2 b r \cos \theta$ is equal to q prime square into a square plus r square minus $2 a r \cos \theta$. Now,

this condition is to be satisfied at all values of theta. Now, if you want to satisfy it for all values of theta, this means these two terms must be good.

So, for any theta this has to be true. Therefore, I write down that q square by q prime square is equal to q square divided by q prime square is nothing but a by b that is q prime is equal to... Now, I put in a minus by hand because I know it has to be opposite, square root of b by a into q. Now, this is really not at a determination because I still do not know a is a distance given to me, but I need to know, where is the image located? In other words I need to calculate the distance b. Now, that is done by equating the remaining two terms that I have.

Therefore, what I require is b square plus r square is equal to q prime square by q square into a square plus r square. And we have already seen that q prime square by q square is b by a, so this times a square plus r square. So, this tells me that a times b must be equal to R square and this R is just the radius of... I have to put r is equal to R because it is on the surface of the sphere. Now, notice what I have done, I have said q prime is equal to minus square root b by a q and a b is equal to R square. So, that tells me that the, image charge magnitude.

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The image shows a whiteboard with handwritten mathematical equations. The equations are:

$$q' = -\sqrt{\frac{b}{a}} \cdot q$$

$$= -\sqrt{\frac{R^2}{a^2}} \cdot q = -\frac{R}{a} q$$

$$b = \frac{R^2}{a}$$

The number '5' is written in the top right corner of the whiteboard.

So, the image charge magnitude which is q prime, which is equal to minus square root of b by a into q and b is root of R square by a, so that mean a square into q. So, which is minus R by a into q and b is, since a b is equal to R square b is equal to R square by a.

So, this is the net solution that is, I have an image charge, which is located at a distance equal to R^2/a . Now, notice $a \cdot b$ product is R^2 therefore, if the object charge is outside the sphere, the image must be inside the sphere, at a distance b equal to R^2/a and the image charge magnitude is simply R/a into q .

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The slide contains the following mathematical derivations:

$$\Phi(r, \theta) = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{r^2 + a^2 - 2ar \cos \theta}} + \frac{(q'/q)}{\sqrt{r^2 + b^2 - 2br \cos \theta}} \right]$$

$$= \frac{q}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{r^2 + a^2 - 2ar \cos \theta}} + \frac{R/a}{\sqrt{r^2 + (R^2/a^2) - 2(R^2/a)r \cos \theta}} \right]$$

$$= \frac{q}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{r^2 + a^2 - 2ar \cos \theta}} + \frac{R}{\sqrt{r^2 a^2 + R^4 - 2R^2 ar \cos \theta}} \right]$$

Since the sphere is grounded, the potential on its surface is zero:

$$E_r = E_\theta = 0$$

$$E_r = E_\theta = -\frac{\partial \Phi(r, \theta)}{\partial r}$$

$$= -\frac{q}{4\pi\epsilon_0} \left[\frac{-(r - a \cos \theta)}{(r^2 + a^2 - 2ar \cos \theta)^{3/2}} + \frac{R(a^2 r - aR^2 \cos \theta)}{(r^2 a^2 + R^4 - 2R^2 ar \cos \theta)^{3/2}} \right]$$

At the bottom of the slide, it reads: Prof. D K Ghosh, Department of Physics, IIT Bombay

Now, I can continue with this exactly the way we did, that is write down an expression. Now, now that I know exactly, what is the image charge location as well as the magnitude, I can write down an expression for the potential. And then calculate the charge density on the surface of the sphere, by calculating how much is the normal component of the electric field. We will take up this problem and solve the complete sphere problem, as well as some other application of the image problems next time.