Electromagnetic Theory

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Module - 2
Electrostatics
Lecture - 15
Solutions of Laplace Equation – II

We continue our discussion of solutions of Laplace's equation, which we talked about last time; last time we had discussed the solutions in Cartesian coordinates. Today what we plan to do is to first take up the solutions of the Laplace's equation, in spherical polar coordinates and then depending on how much of time we are left with, we will attempt a solution in the cylindrical coordinates as well.

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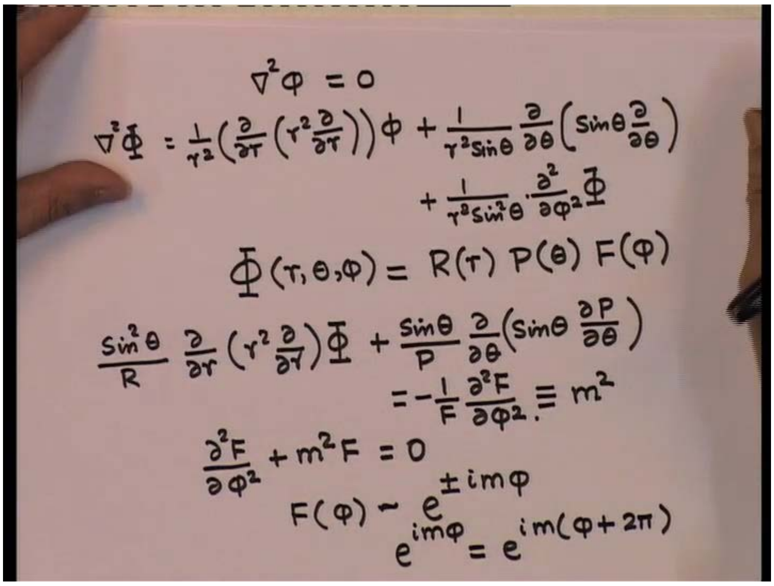

So let us look at our equation, if you recall the equation is del-square phi equal to 0 is what we are interested in, in spherical polar coordinate; the Laplacen is given by 1 over r square. So del square phi is given as, 1 over r square d by d r of r square d by d r of phi of course plus 1 over r square sin theta d by d theta of sin theta d by d theta plus 1 over r square sin square theta d square over d phi square of phi. I repeat again, that I am using azimuthal angle represented by small phi and the potential I am writing as capital phi; should not cause any confusion.

Now, it turns out that one can use almost the same trick as we used for the rectangular, rectangular coordinates; that is use the technique of separation of the variables. Thus technical separation of the variable is that write the function phi, which depends upon r theta and phi as product of a function of distance r only and a product of with a function of angle theta and a function a third function, which depends upon the azimuth phi.

Now if you substitute these into these equations and then divide the resulting equation by the product R P and F, this is fairly straight forward; then what you get is sin square theta divided by R d by d r of r square d by d r of phi plus sin theta by P d by d theta of sin theta d P by d theta and well actually plus 1 over F d square F over d phi square, but since that is equal to 0; I will write this as equal to minus 1 over F d square F over d phi square.

But this is my Laplace’s equation, now is fairly straight forward to work out; I am not going to spend time in doing this. Now notice as before, my left hand side is dependent only on the angle theta and the distance r; whereas right hand side of this equation depends upon the azimuthal angle phi only. Now since r theta phi are arbitrary, there is no way these two can be equal at all times; unless each one of them happens to be a constant. Now what will do is we will define that, this constant to be given by m square; what is m square? We will find out, but we have said each one of them must be constant and it is equal to m square.

Now you immediately notice that I can solve the phi equation without any problem, which gives me d square F over d phi square plus m square F equal to 0. Now this is of course, a very well known equation to us; the type that you find in solutions of simple harmonic motion and the solutions of that is F, which is a function of phi only; it goes as e to the power plus or minus i m phi.

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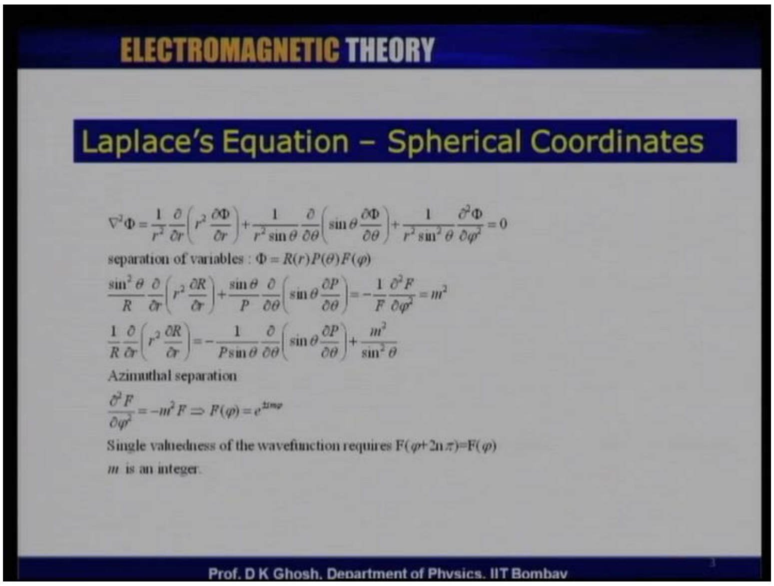

Now recall that my azimuthal angle, phi varies from 0 to 2 pi and since the function phi is single valued; the value of the potential when the azimuthal angle is 0 must be the same as its value when you, when it takes a complete turn; that is becomes 2 pi. In other words, I must have e to the power i m phi; plus or minus does not matter must be identical to e to the power i m phi plus 2 pi. In other words e to the power i m into 2 pi must be equal to 1; which restricts me to the values of m being integers only.

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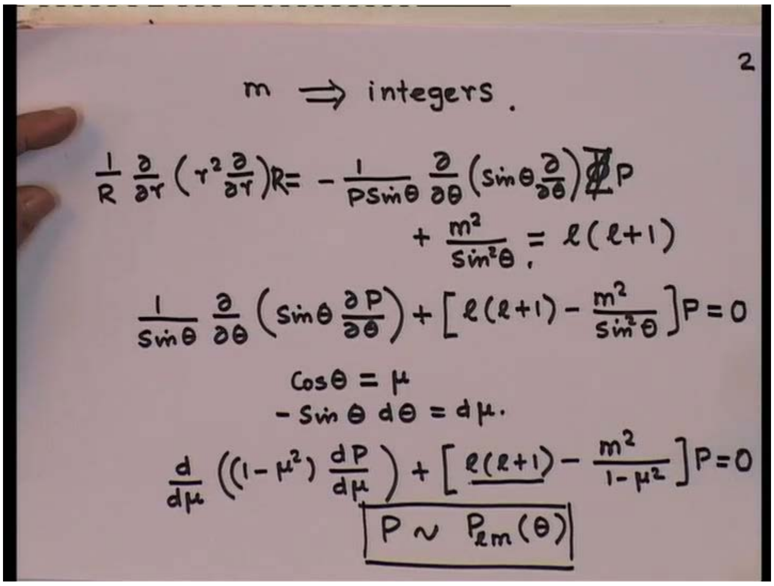

So, m must be integer in order that, the solutions as a function of phi or a single value.

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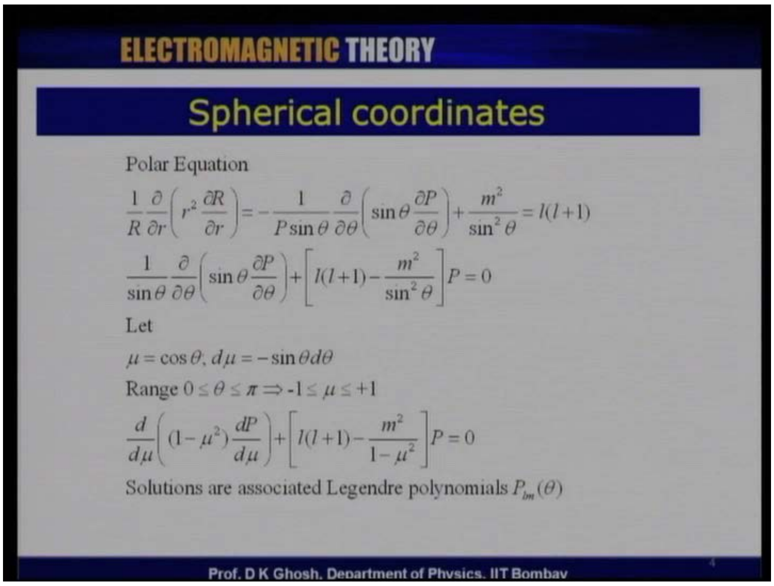

Now once you have done that, let us come back to our original equation. So what we have said is this, that if you recall let me flash it for a moment; I had sin square theta by R square, this whole thing was equal to m square. So what I could do is to divide this both sides of this equation by sin square theta. You notify do that, this term, this term becomes depends then only on capital R; the remaining two terms will depend only on theta. So let us write that down. So what is get is 1 over R d by d R of r square d by d r, this quantity is equal to let me take the other two terms to the right hand side; that is minus 1 over P sin theta d by d theta of sin theta d phi by d theta of phi of course.

This is actually on R and this depends only on P and plus m square by sin square theta. Now once again I repeat my argument; that is the left hand side is a function of r only, the right hand side depends only on theta. So therefore, for arbitrary r and theta, they can be equaled only if each one of the term is equated to a constant. Now for removes that you will appreciate little later; I will write this constant is equal to l into l plus 1. Now remember I have not said what is l? So what I use as a constant is totally immaterial.

Now once I have done that, let me first concentrate on the theta equation; which gives me 1 over sin theta d by d theta of sin theta d P by d theta; this is, this plus I have taken this term to the side, l into l plus 1 minus m square over sin square theta acting on P is equal to 0. So this is purely a theta depended equation. Now it turns out that, this equation can be simplified by making a variable transformation; that is take cos theta is equal mu.

Let me define cos theta is equal to mu, then you know that minus sin theta d theta becomes equal to d mu. It is a fairly straight forward exercise to show that, this gives me d by d mu of 1 minus mu square; which is nothing but this sin theta d P by d mu plus l into l plus 1 in the weight was minus m square, I had sin square theta which is 1 minus cos square theta.

So it is 1 minus mu square active on P is equal to 0. Now this looks like a difficult equation, but however it turns out that this equation is a very well known equation in mathematics and most of you would probably come across this equation, in your course on mathematics and the solution. Now this depends upon two things you notice, I have already said n is an integer; I have not quite said what is l, but this equation depends upon two parameters, namely l and m and the solutions are known to be polynomials in theta and these polynomials are known as associated Legendre polynomial P l n theta. So P is this is known as associated Legendre polynomial.

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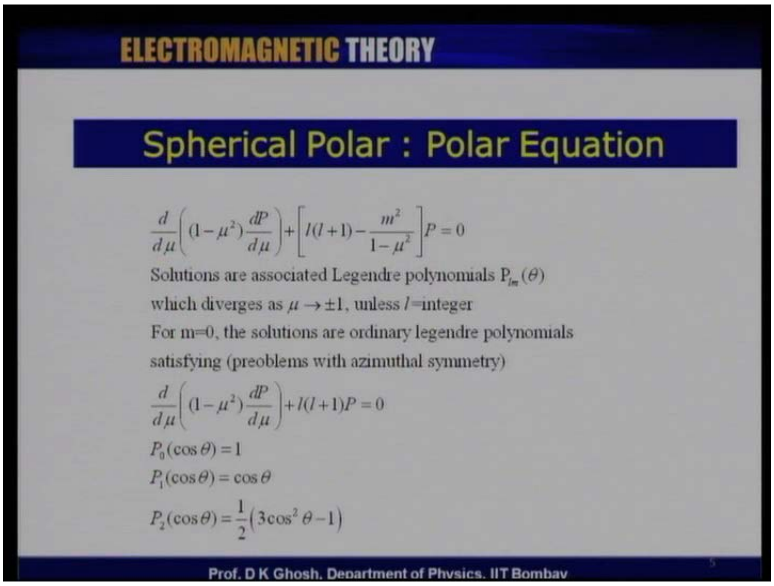

Now I will actually not attempt to solve this equation because it is somewhat time consuming, but never the less it turns out that the solutions can be easily guessed; and I will come back to some points regarding what happens, but before I talk about what the solutions are? In other words, what are associated Legendre polynomials? Let me tell you, that it is known that this equation the solutions of this equation will diverge as, mu goes to plus or minus 1. Unless l happens to be an integer, in other words physically meaningful solutions of these equations, this equation is known for or exists; when l happens to be an integer.

So what we have said is this? We have made the statement that, m is an integer that came from the single validness of the azimuthal equation and now what we are saying is that, l which also was included as a constant also must be an integer. We will later on see that, there is a relationship that must exist between the values of m and l. A specially simple class of solutions of this equation exits, when m is equal to 0. Now needed remember what was m, m is the integer parameter associated with the azimuthal equation solutions and what was the azimuthal equation solution?

The solution of the azimuthal equation was e to the power plus or minus I m phi. In other words, that part of the solution dependent was dependent on the azimuthal angle phi; there are some special cases where, the problem has azimuthal symmetry built in to it. What is meant by azimuthal symmetry? Azimuthal symmetry simply means that the problem looks the same, if you go around a cylinder; as the phi changes the nature of the solutions do not change.

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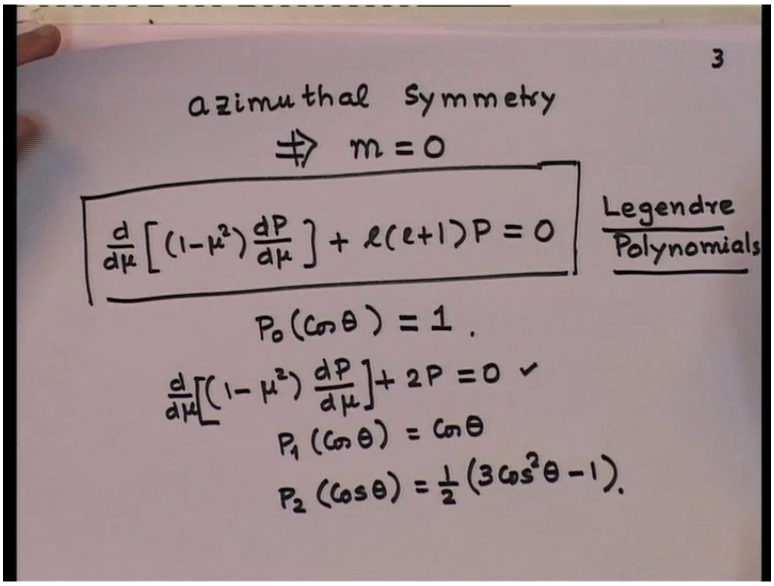

Now if that is true, it means that the only solution that we should consider in case of spherically symmetric situation; that is actually the solutions are azimuthally symmetric, I should say azimuthally symmetric solution is when the solutions is m independent.

So azimuthal symmetry implies that m is equal to 0, the solution does not depend upon phi. Now if you, if you look back on to this equation again, all that you need do is to then put m is equal to 0 in this equation. So this will give me d by d mu of 1 minus mu square d P by d mu plus l into l plus 1 P is equal to 0; this equation which is the equation special case of the equation, which we talked about little while back. Its solutions are also associated Legendre polynomial, but with m is equal to 0; these are just giving the name Legendre polynomials. The solution, the word associated is removed they are called Legendre polynomials.

Now turns out that the Legendre polynomials, the value of l if you recall must be an integer; the solutions are power series in cosine theta. First few of them I can actually show it to you for example, if I take l is equal to 0, which means this term does not exist. So d by d mu of 1 minus mu square d P by d mu; you can check that the one of the solutions would be P 0, that this I will call at a P 0 and P 0 depends upon mu or on cos theta and that happens to be the solutions is constant, which will take it to be 1, normalizing it.

Now let us take l is equal to 1 for example so notice that this becomes 1 into 1 plus 1 that is 2 times P. So what I have is d by d mu of 1 minus mu square, mu square d P by d mu plus 2 P is equal to 0. I came that a solution of this equation is P is equal to mu, that is P 1 of cos theta is simply equal to cos theta; you can check that. See P is equal to mu I get a 2 mu there and if P equal to mu then d P by d mu is 1.

So I am left with d by d mu of 1 minus mu square so d by d mu of minus mu square which is minus 2 mu plus 2 mu is equal to 0; it is trivially satisfied. P 2 of cos theta, all these you can actually inspect happens to be equal to 1 by 2 3 cos square theta minus 1. So as I go along P 3 cos theta would be a cubic in cos theta etcetera, etcetera; that is P l cos theta is a polynomial of degree l in cos theta and this I repeat again is valid where, I have azimuthal symmetry that is there is no phi dependence of the problem. When I go around changing the angle phi from everywhere, the physics looks exactly the same.

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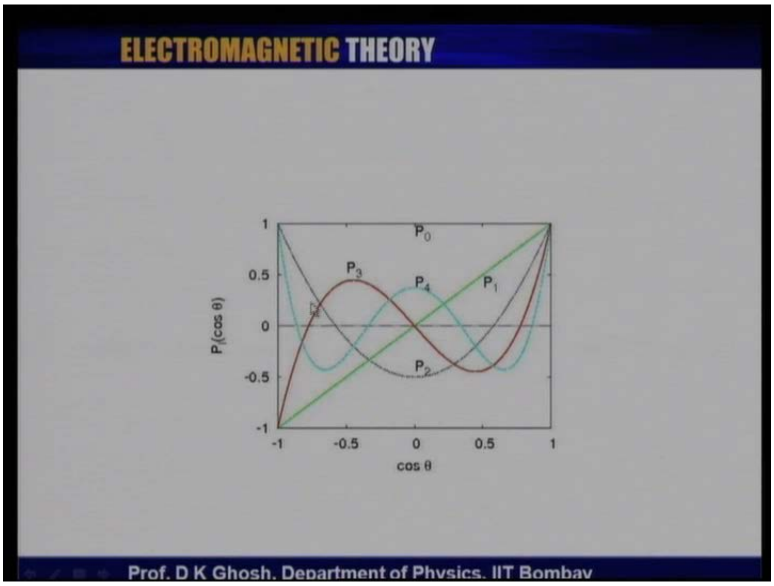

This is simply a picture of what P cos theta looks like so this is notice that P 0 is 1. So I am plotting P cos theta against cos theta, which is same as plotting P mu against mu. So when l is equal to 1; it is P P 0 of course constant, which is the value 1 which is parallel to the x axis and P 1 is cos theta which is same as mu. So which is simply a line and P 2 is this one, you notice that there function has as many nodes as are the orders of the equation.

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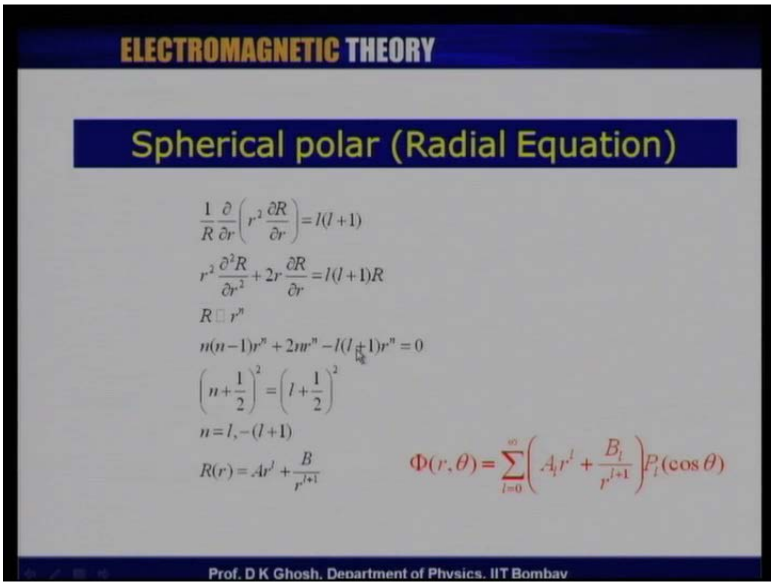

So we are left with now radial equation, once again we are talking right now about the azimuthal symmetry that is m is equal to 0 situations; where n does not appeared. Now if you look at the R equation, you find this is equal to 1 over R d by d r of r square d R by d r equal to l into l plus 1. Just expand this, what you get is r square d square R by d r square plus 2 r d R by d r plus l into l plus 1 into R equal to l into l plus 1 into R.

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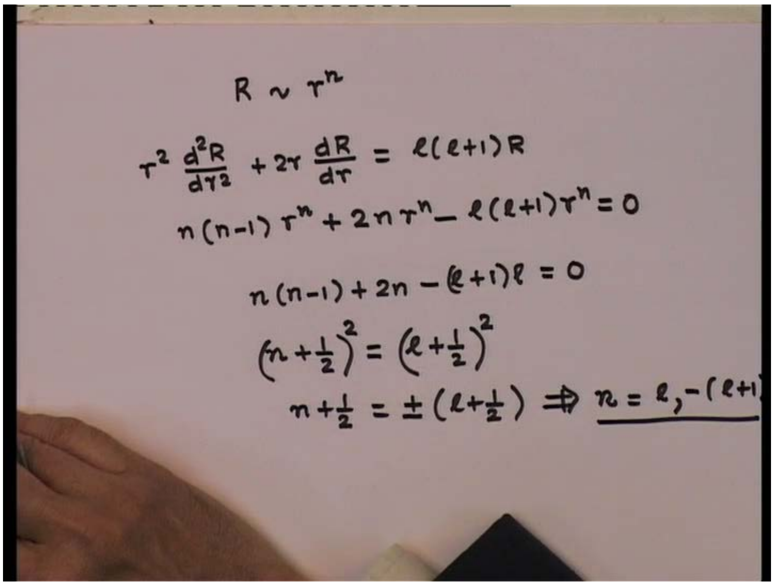

An inspection immediately tells you that, the solution must be a power series in r; that is R must go as r to the power n. Now let us look at rewrite the equation here so I get r square d square r by d R square plus 2 r d R by d r equal to l into l plus 1 R. So if you put n r to the power n, you notice this is a double differentiation. So I get n into n minus 1, if R goes as r goes r to the power n, i get n into n minus 1 r to the power n minus 2 that multiplied with this r square, gives me n into n minus 1 r to the power n plus 2 times d R by d r is n times r to the power n minus 1 there is an r there. So I get 2 n r to the power n minus l into l plus 1 into r to the power n is equal to 0. Now since this is homogeneous in r to the power n; it means the coefficient n into n minus 1 plus 2 n minus l into l plus 1 is equal to 0.

Now you can actually complete a square here, you notice this is n square minus n there is a plus 2 n so it is n square plus n. So I will write it as n plus a half whole square, which gives me n square plus n plus one-fourth and this quantity must be equal to l plus half whole square because that one-fourth is now taken care of and I have got l square which is here and 2 times l into l plus 2 times l, which is comes from there. Now you this gives me n plus a half is equal to plus or minus of l plus a half; which tells me that the value of n must be equal to l or minus l plus 1.

So directive possible values of n, one is n is equal to l the other value is n is equal to minus l plus 1; which tells me that R of r depends upon the radial distance as r to the power l and as 1 over r to the power l plus 1 so this is the solution that I have got. Now if I now put all them together for the case of Azimuthal symmetry, I get phi r theta is some over l is equal to 0 to infinity remember l is an integer and I have got l r to the power l plus B l by r to the power l plus 1 into P l cos P l of cos theta; where P l as we have stated r in Legendre polynomials.

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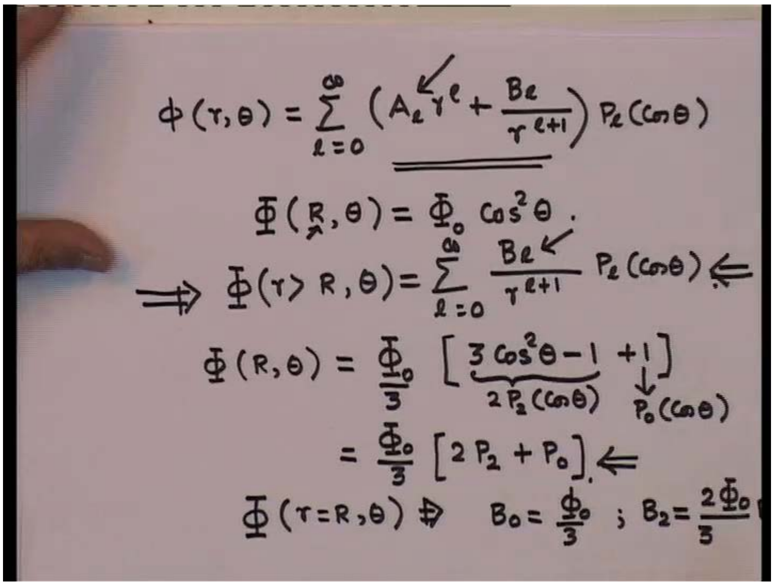

So this tells me, what is the general form of a potential function phi of r theta for the case of azimuthal symmetry? So it is l equal to 0 to infinity, A l r to the power l plus B l divide by r to the power l plus 1 into P l cos theta; repeat again that P l cos theta is the polynomial of degree l in cos theta.

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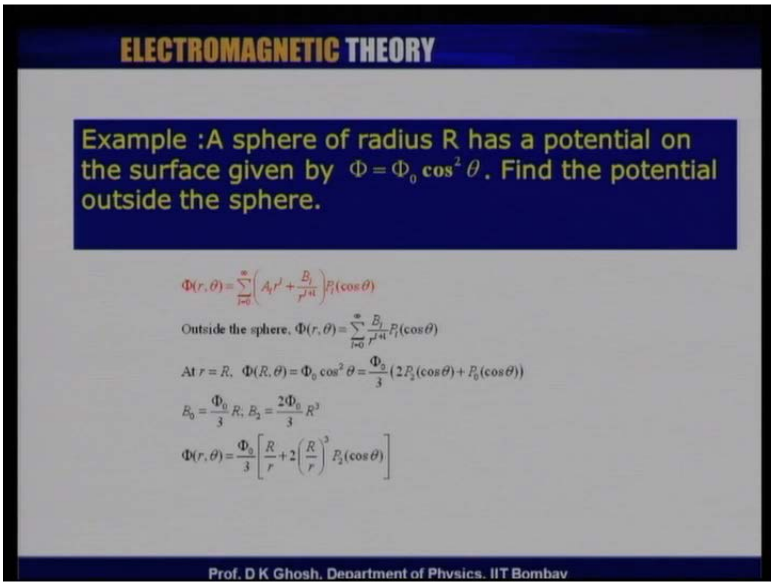

Now I will illustrate the solution by taking some specific examples. So the first example that I take is to consider as a sphere of radius R and I am given the potential on the surface of this sphere. I have said that the potential on the surface of the sphere of radius R, R is phi 0 cos square theta.

So take phi on the surface of the sphere of radius R, this R should not be confused with the R component of the potential that we wrote down earlier so phi R theta. So this only depends upon cos square theta, r is of course was given radius. So this is some constant phi 0 times cos square theta. Now I am required to find out the potential outside the sphere; obviously, the you know what I need to do? Then is to look a general solution here and I realize that potential must become 0 as r goes to infinity.

Now that tells me, that I need not worry about or in my solution such a term A l r to the power l cannot exist because whatever solution I get because I am doing it outside the sphere; r is greater than capital R. So my solution then for r greater than R theta is some over l is equal to 0 to infinity; the constants B l by r to the power l plus 1 into P l cos theta.

Now our job is now to determine what are these constants B r? Now this is done by realizing that on the surface of the sphere, this must boil down to phi 0 cos square theta, but this requires that I must express phi of r theta in terms of the Legendre polynomial. So I have got phi 0 cos square theta; now recall that cos square theta immediately tells me that, what I have is probably at P 2 because it is cos square theta as a degree 2; I could have of course, things which are lower than that, but nothing higher than it because anything higher degree for example, degree 3 will have cos cube theta.

Now if you look at the solutions that I showed little while back; I told you that P 2 of cos theta is half of 3 cos square theta minus 1. So let me write this down so I have got a cos square theta; let me write it as 3 cos square theta by 3, then minus 1 plus 1. Now this is half of this, half of this is P 2 cos theta so therefore, this is nothing but 2 times P 2 cos theta and 1 is by definition P 0 cos theta. So this is can be written as phi 0 by 3 2 P 2, I am ignoring the functional dependence plus P 0.

Now what I need to do is to compare this expression with this expression here; there is another statement that I want to make use, that these polynomials are orthogonal; that is if you take the integral of the product of a polynomial of a particular degree with a polynomial of a different degree, it becomes 0. So if you now look at what should be, since this is my general expression for the potential outside this sphere. So we since the potential must give me this, this value; when r becomes is equal to R.

So what I require is P 2 and P 0, which tells me that only B 2 and B 0 are going to be there because if it is B 0; then it is r to the power 0 plus 1 will be there. So from here, I can make out that my B 0 should be equal to phi 0 by 3 and B 2 must be equal to 2 phi 0 divided by 3 and I must have an R cube there because when l is equal to 2; I get a r to the power 3 and I am putting r, r is equal to R. So therefore, this has to be equated to this and I get 2 pi 0 by 3 r cube. These are the only non-vanishing coefficients in that equation.

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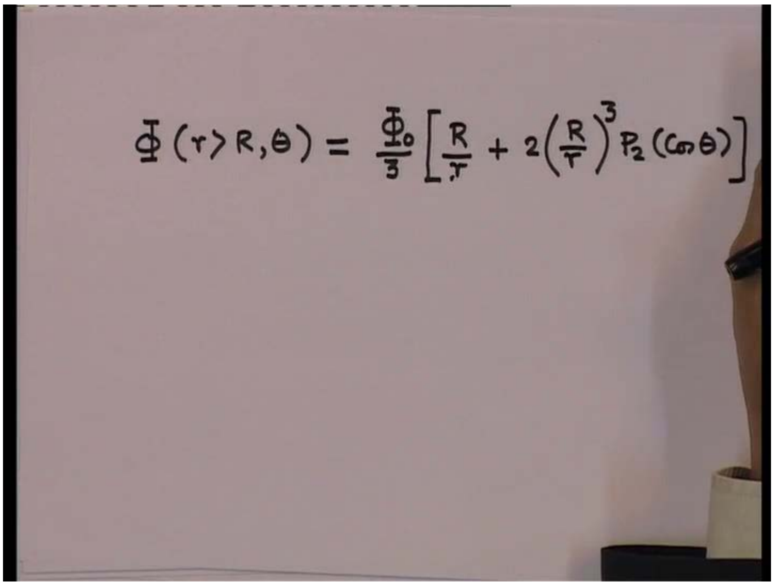

So therefore, my potential at an arbitrary r greater than R, as a function of theta will only have phi 0 by 3; so I get here, I think B 0 should have had a R there. So R divided by r plus 2 times R by r cube times P 2 cos theta. So this is my solution of for the potential outside the sphere.

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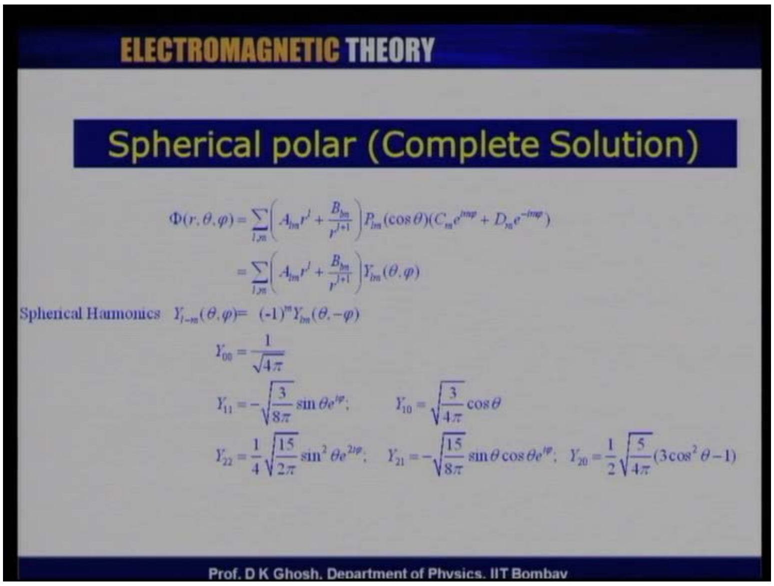

Let me take a second example, but this second example I will not do with a azimuthal symmetry, but I will do it without azimuthal symmetry, but before that let me sort of summarize, what happens when I do not have an azimuthal symmetry.

This is actually fairly straight forward, if you recall my r equation; solutions were constant times r to the power l plus a constant divided by r to the power l plus 1. Now if there is no azimuthal symmetry, these constants in principle can depend upon l and m and so I write it as some over l m, A l m r to the power l l plus B l m divided by r to the power l plus 1; times P l m cos theta, I told you already p l m are called the associated Legendre polynomial into C m e to the power i m phi plus d m e to the power minus i m phi; these were the solutions of the azimuthal equation.

Now these two angle parts P l m cos theta and e to the power plus or minus i m phi; they are combined into a different function, they are called spherical harmonics; Y l m theta phi. Now I have written here some solutions for Y l m theta phi; for example, Y 0 0 remember P 0 was 1, but Y 0 0 is 1 over square root of 4 pi. This is required for normalization and like P l m, like the Legendre polynomial the spherical harmonics are also orthogonal functions; orthogonal polynomials in both l and m.

Now it turns out that corresponding to given l; m can take value from minus l to plus l in steps of one. What it means is that if l is equal to 1, then m can be 1 0 or minus 1; if l is equal to 2 then m can be 2 1 0 minus 1 and minus 2. Now between Y l m and Y l minus n there is a very simple relationship; that is Y l minus m is simply minus 1 to the power m of Y l Y l m of minus phi and these are the expansions, you notice slight changes Y 0 0 is of course a constant; Y 1 1 is sin theta times e to the power i phi. Now because of this relationship I have not separately written down Y 1 minus 1; is obviously, it will be simply plus square root of 3 by 8 phi sin theta into e to the power minus i phi.

Y n 1 0 is root of 3 by 4 pi cos theta; this constants come out from normalization and these are orthogonal polynomial, but nevertheless other than for the phi dependence, you notice one thing that corresponding to a given value of l; it is a polynomial of degree l in sin theta or cos theta or their combination. So like for example, this is Y 2 1; now Y 2 because l is equal to 2, it the permissible things would have been sin square theta cos square theta, but it turns out that it happens to be sin theta cos theta here and so these are also the same degree, in sin theta cos theta.

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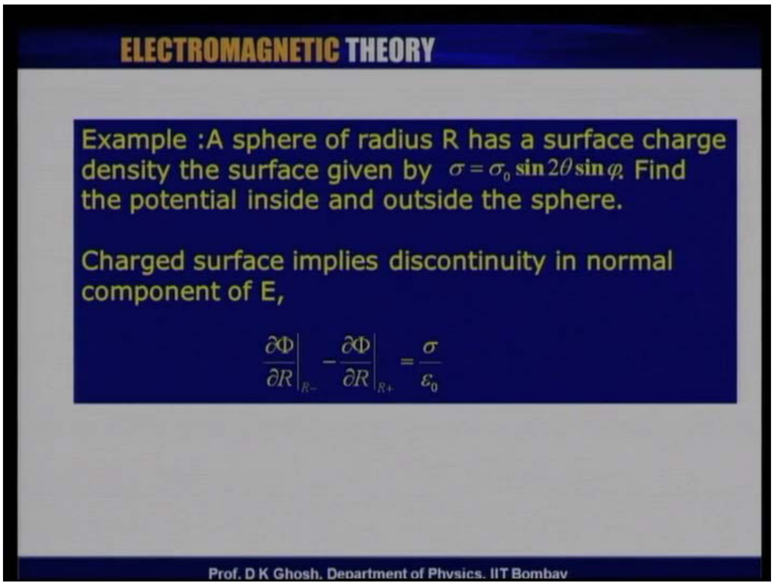

So let me take a slightly different example and this example is that, I have A sphere of radius R; which has the surface charge density given by sigma equal to sigma 0 sin 2 theta into sin phi. So in other words, the problem does not have azimuthal symmetry because the charge density depends upon phi; we need to find potential both inside and outside the sphere. Now remember that, if I have a charged surface might (( )) it implies that, the normal component of the electric field must be discontinuous across the surface. We have seen the tangential component of the electric field is continuous. Now normal component of the electric field since it is we had talking about sphere; the normal direction is just the radial direction.

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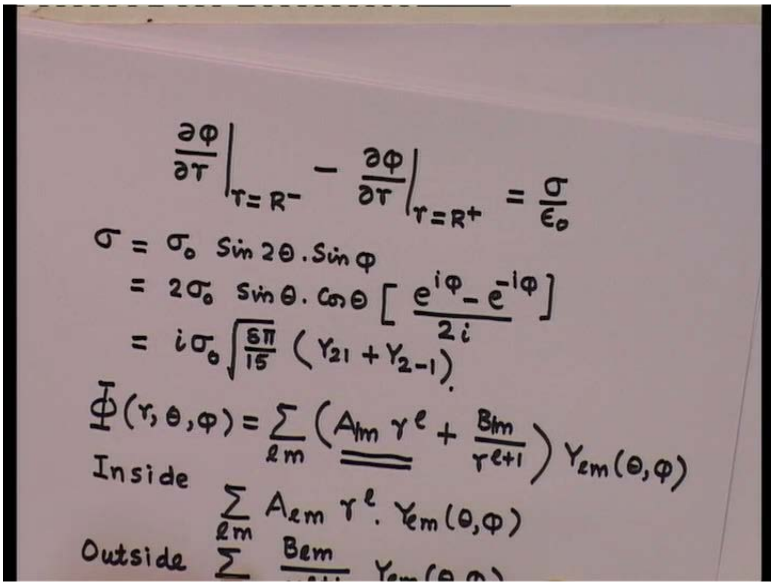

So this requires the d phi by d r, where r is slightly less than r; I write it as r is equal to R minus; minus d phi by d r, where r is slightly greater than r we write it as r plus. This we had seen is given by sigma by epsilon 0 remember; this is nothing but the normal component of the electric field, just inside the sphere and that is just the normal component of the electric field; just outside this sphere.

Now I have an expression for phi, which I had written down in general we will come back to it, but let us look at what is this sigma? Now sigma is on the surface. So sigma on the surface is given as sigma 0 sin 2 theta times sin phi. Now you can immediately rewrite it as 2 sigma 0 sin theta cos theta and let me expand the sin phi as e to the power i phi minus e to the power minus i phi divided by 2 i. So this is i sigma 0, 2 and 2 cancels out; I am actually I have a minus i, but I am coming back to it.

Now recall the expressions for the Y l m; you notice here that, since I have got sin theta cos theta; so obviously, l must be equal to 2, but I have got only e to the power i phi there so the only functions involved in sigma are for l is equal to 2, but m is equal to 1 and minus 1. So, let us check that.

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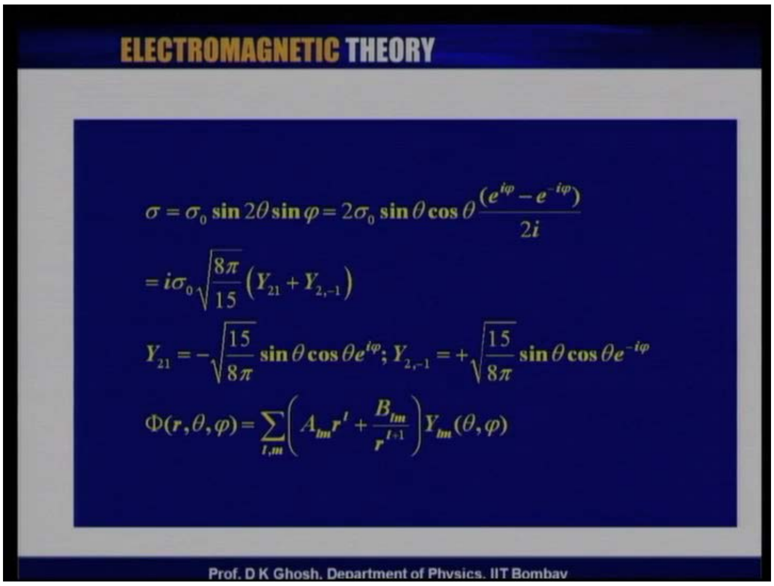

Now this works out to if you look at those expressions; here I given you rewritten the expression for Y 2 1 and Y 2 minus 1. With this it turns out that these are square root of 8 pi by 15 Y 2 1 plus Y 2 minus 1. So this is the charge density that I have got; how do I use it? Look at this general expression for the phi; the sum is our l, which goes from 0 to infinity and m corresponding to the l remember for a given l, m can take values from minus l to plus l in steps of one. So this is my general expression phi r theta phi is given by that now.

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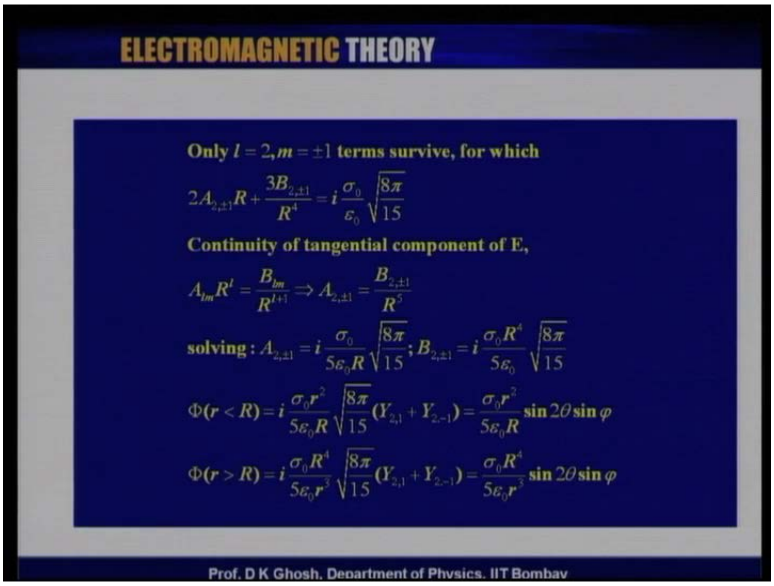

So if I look at it now and I have got this d phi by d r. So look at this expression again so let me rewrite it for you; phi of r theta phi is equal sum over l m A m r to the power l plus B m, actually B l m A l m by r to the power l plus 1 and multiplied by Y l m theta phi. Now what am I going to do? I am going to take the derivative of this function inside and outside; that is for r less than R and R greater than r and then equate it to sigma by epsilon 0.

But recall that, if I am inside this sphere the origin is a part which is included there. Since origin is included, I cannot have functions of the form which is 1 over r to the power l plus 1; because the minimum value of l is 0 so the minimum variation there is 1 over r and the function will grow up at the origin. So therefore, inside the sphere this is the solution. So let us look at it, inside the sphere I have got sum over l m A l m r to the power l Y l m theta phi.

But outside this sphere since infinity is included I must only have the other one, that namely B l m by r to the power l plus 1 and of course, Y l m theta phi. I differentiate this one, d by d r so I get l times r to the power l minus 1; Y l m depends only on theta phi. So it does not bother me; differentiate this one, I get minus l plus 1 divided by r to the power l plus 2 and again I have to put r is equal to R and this difference I will take.

When I take this difference, that is to be equated to sigma by epsilon 0. So let me look at what it means. So, so this is the part that comes from inside; let me just illustrate that only by at least with one term.

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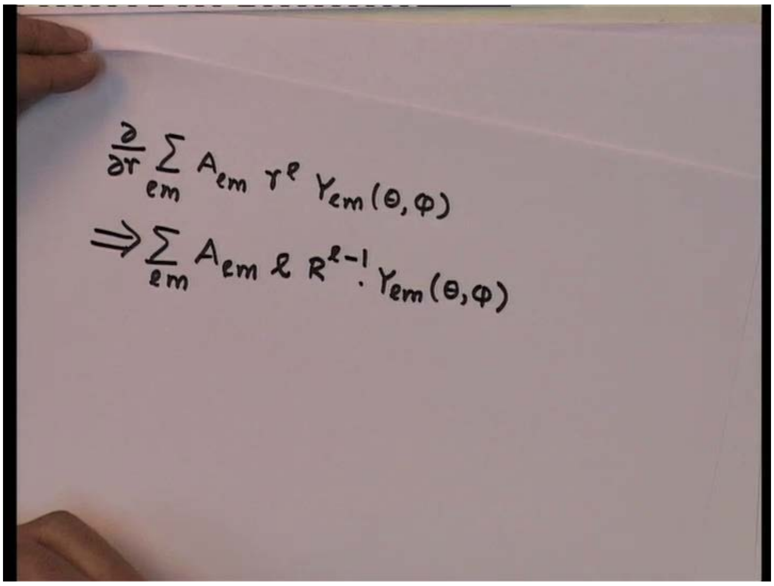

So inside I have got A l m r to the power l y l m theta phi, sum over l m and I am doing d by d r of that. So that will give me sum over l m A l m l times r to the power l, but my R will be put as l times R to the power l minus 1, but my R will be R so it is this into Y l m theta phi.

From this, so this is what I have written down that; now remember that the what we have in sigma? What we have in sigma are just l is equal to 2. So in other words, in these coefficients that I have got; the only l that I need to worry about should be l is equal to 2 and if you take these things, then this is what you get? A 2 plus or minus 1 r plus 3 B 2 plus or minus 1, this is obtained by subtracting the inside part, outside part from the inside part and equating this two what I have got there and the Y l m functions cancel from both sides, because of orthogonality.`

So this is, this is what one relationship we have; the second relationship comes from the continuity of the tangential component of e. Now tangential component of e means, derivative with respect to the theta phi directions; I need not really write it down because those will be identical on both sides of that equation. So that keeps me a much simpler job, that A l m r to the power l is equal to B l m r to the power l plus 1 for all l and m and in particular for 2 plus or minus 1. Now these two equations we have to solve simultaneously and if you do that, you find that A 2 plus or minus 1 is obtained as this and B 2 plus or minus 1 is obtained as this.

Now what is my job? I simply substitute these A's and B's in the general expression, that I have by just taking out the terms which are non 0 there and you find that the expression for the potential for r less than R is given by this expression. This is done by rewriting the spherical harmonics in terms of their expansions and r greater than R is given like this. So r greater than R goes as 1 over r cube and r less than R goes as r square.

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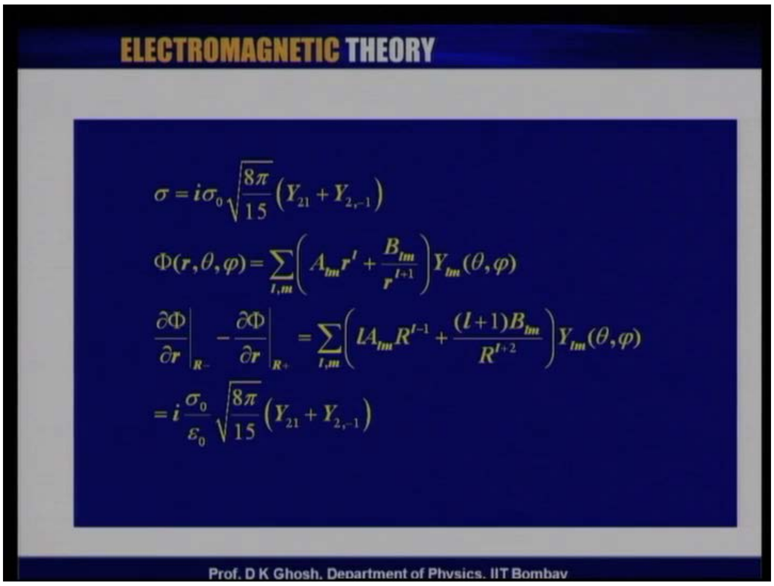

So this is what we get from here.

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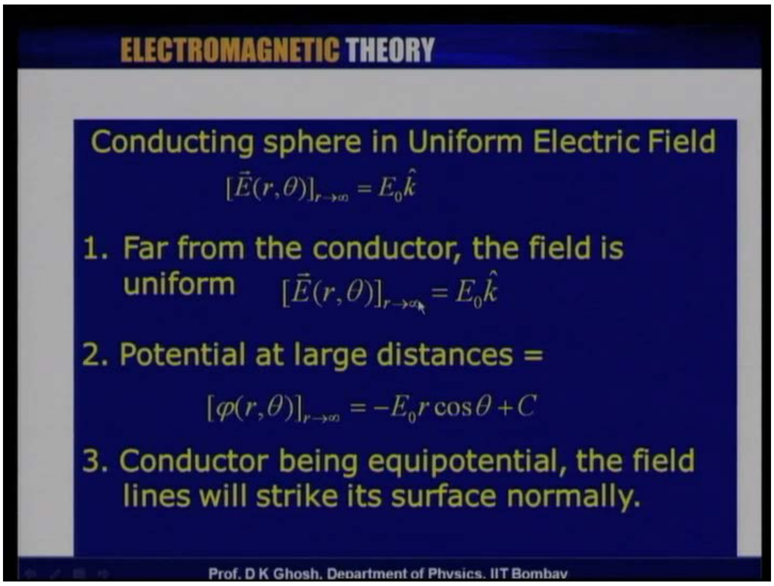

Now let me let me come to what can be considered as a classic problem in electrostatics and that is what happens? If I put a conducting sphere in a uniform field; the field will be taken to be along z direction. So what happens is this? Far from the sphere, what the sphere is doing we will see it later, but there is an uniform field in the z direction; E 0 and I put a conducting sphere in that field.

So this what I have written, that far from the sphere; that is E r theta as r goes to infinity is E 0 times unit vector k, unit vector k is along the z direction. Now what it tells me is this? I use spherical coordinates again so that the potential at large distances; is the one which gives rise to this electric field, remember minus the gradient of the potential is the electric field and it is a uniform filed in the z direction. If it the uniform field in the z direction, the potential must depend only on z; it should be linear in z. So minus E 0 z is what I am writing as minus E 0 r cos theta; this plus of course, a constant.

Now I know that my conductor is an equipotential, since the conductor is an equi-potential; when the field lines come on the conductor, they will strike the conductor normally.

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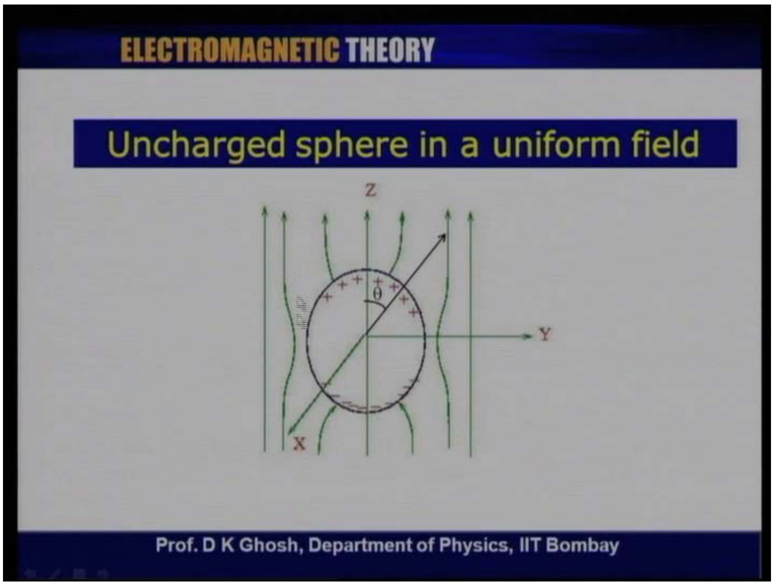

So, this is the picture that you have, far from this is not really far, but on the other hand I have drawn it like this; far from the sphere the lines are the electric field lines are parallel to the z direction. Now as the field's approaches sphere, they enter the sphere like this and they diverge from the other side.

In other words there would be a charged separation, the positive charges will go towards this side, positive z and the other side will become negatively charged. So this is, this is the way the field lines will and the field lines will strike the sphere in a normal fashion.

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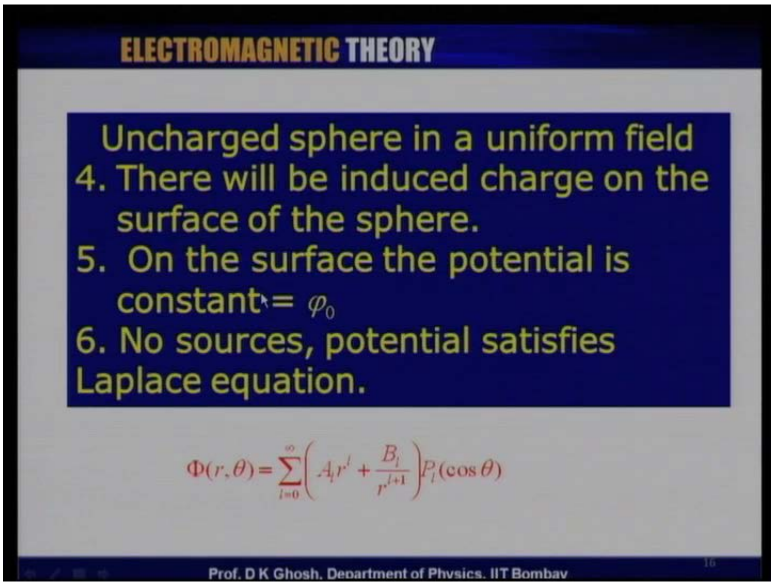

So, we have seen that the, there would be charge separation; now on the surface of this sphere, I know that the potential is constant. Now since there are no sources in the problem; I have also azimuthal symmetry because there is no phi dependence. So potential satisfies Laplace's equation and my phi r theta is given by the familiar expression; that is A r to the power l B by r to the power l plus 1 P l cos theta.

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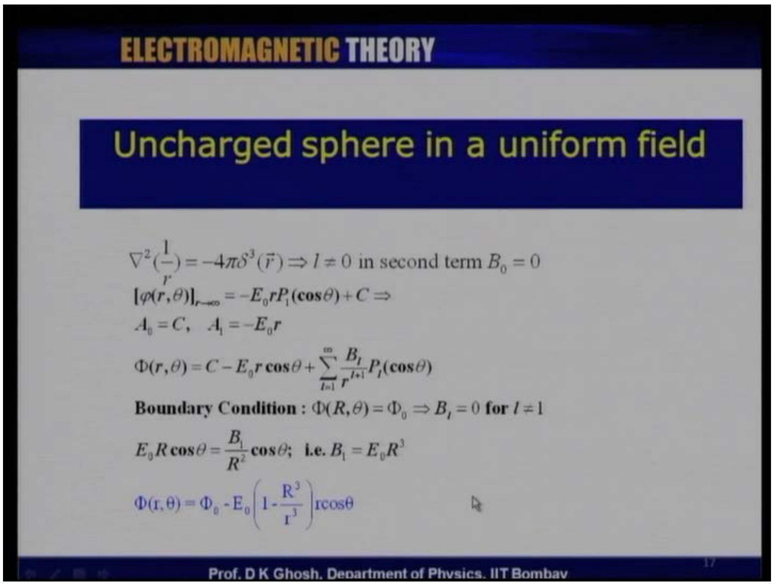

Now I need to look at the solution; I will continue it in the next lecture, but let us let me make a few comments. Now notice that, in this expression I cannot have the term which is l is equal to 0 in this expression because l is equal to 0 means it is one over r and I know that del square of 1 over r is a delta function; this we have talked about several times. In other words, there should have been a singularity in my charge distribution at the origin; which I do not have, I have just a neutral conducting sphere. So therefore, my l is not equal to 0 and my B 0 term becomes equal to 0. Now what I will I do is this, I will take it from here in the next lecture and complete the solution of this problem as I go along.