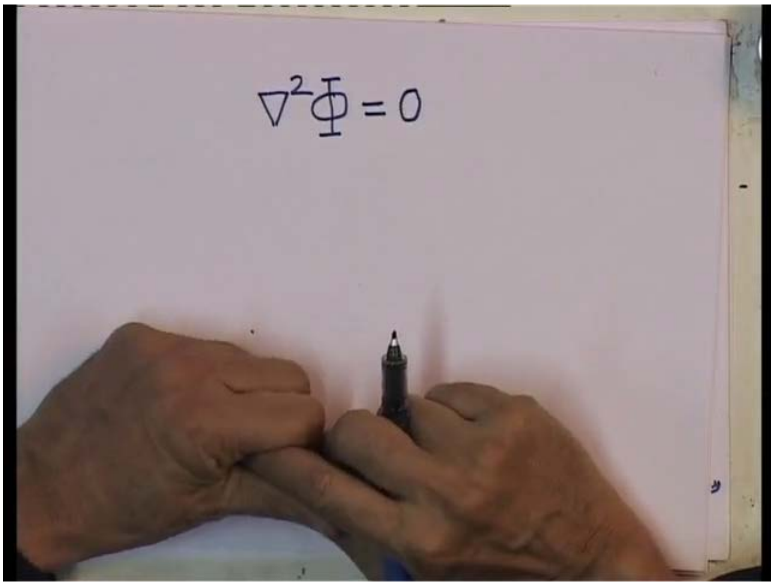
Electromagnetic Theory
Prof. D. K. Ghosh
Department of Physics
Indian Institute of Technology, Bombay

Module - 2
Electrostatics
Lecture - 14
Solutions of Laplace Equation

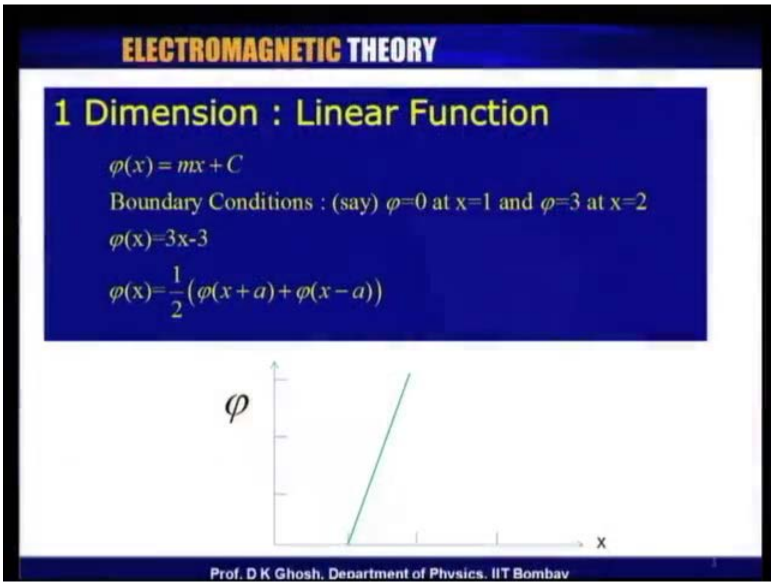
During the last lecture we have been talking about the Poisson's and the Laplace's equation. We also discussed how to obtain capacitances of few geometries starting from Poisson's and Laplace's equation rather than from the intuitive nature of the potential, which we had earlier in this lecture. We will continue with the formal solutions of Laplace's equation which is the equation, which is valid in regions where there are no charges you recall that Poisson's equation which is del square of the potential is equal to the charge density divided by epsilon 0 with a minus sign. That was valid in regions where there are charge sources of charge present, but we will be talking about source free region namely charge density is equal to 0 and therefore, the equation that we are looking for is the Laplace's equation which is del square of phi equal to 0 phi is my notation for the potential.

(Refer Slide Time: 01:28)



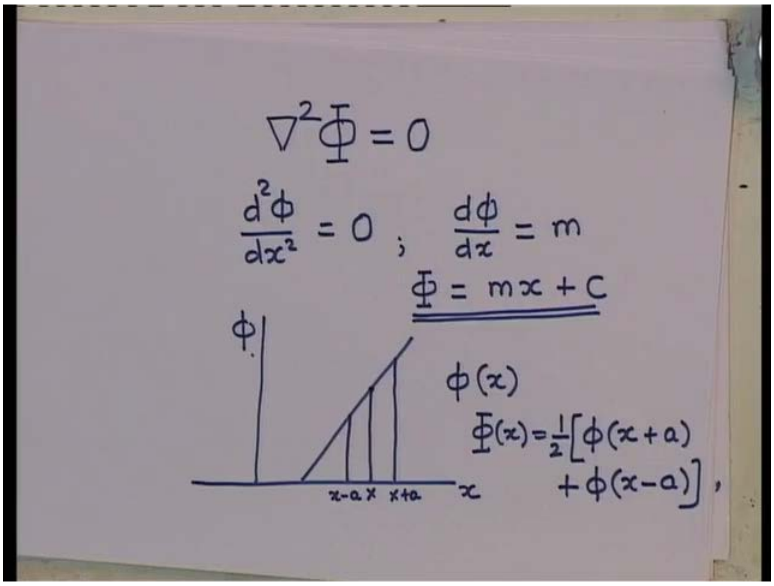
What we will do is to look at formal solutions of this equations in one, two and three dimension.

(Refer Slide Time: 01:50)



So, let us look at first the simplest of the problem namely I look at the problem in one dimension for which this equation is d square phi by d x square is equal to 0 the simplest of the second order equation that you can think of.

(Refer Slide Time: 01:56)

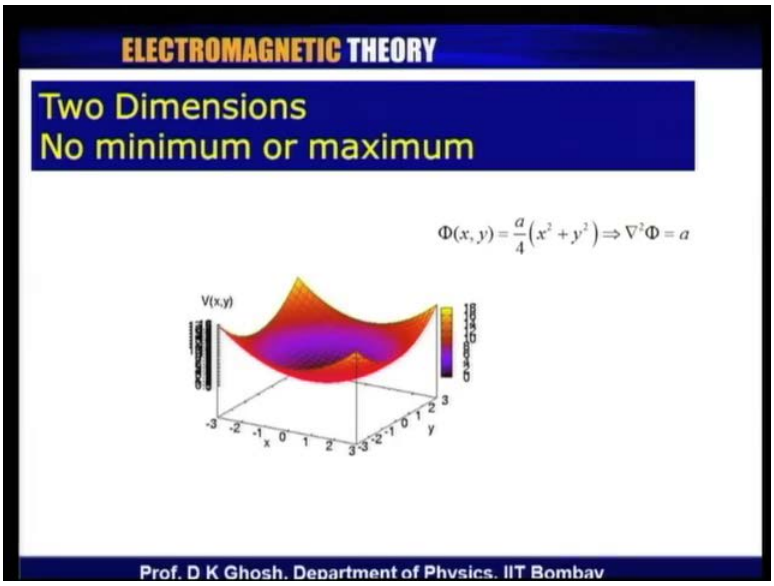


So, the second derivative is 0 the first derivative d phi by d x that is equal to a constant. So, let us call it some m and phi itself is one more integration which gives me m x plus a constant equal to c we all know that y equal to m x plus c is an equation to a straight line so as a result the equation that i have the graph of the potential versus the distance is something like this it is a straight line.

So, notice one thing, the solution of this equation as we will see it is a common feature with the Laplace's equation is essentially featureless the very monotonically increasing function. In this case or decreasing function and important point is that if you are looking at the value of the potential at any point x. Since, this is a straight line, what you could do is to find out. For example, what is the value of the so, this is the point x, you could find out what is the potential at. Let us say x plus a and at x minus a. You can easily show that the potential at the point x is the average of the value of the potential at a point x plus a and x minus a. a is absolutely arbitrary.

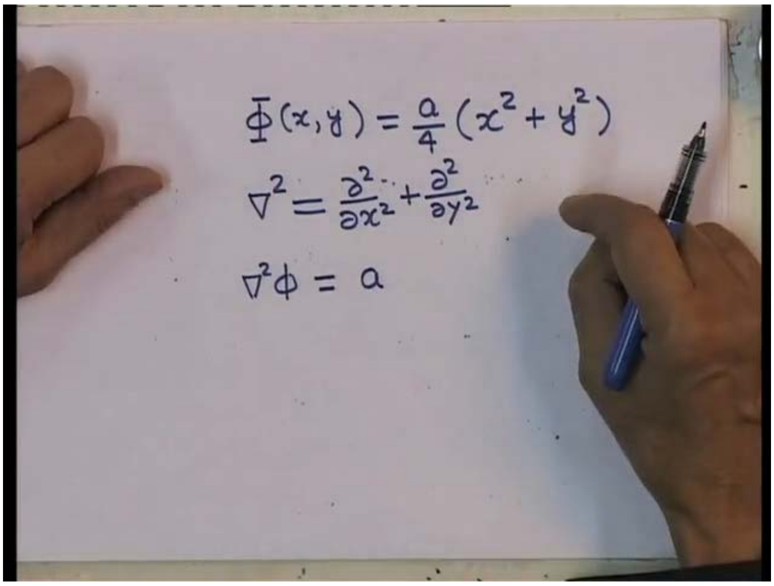
So, of course, it is very clear from the nature of the solution that there are no maxima or minima in the solution of this problem. We will see that this averaging is actually a common feature of the solutions. So, since it is an averaging, I cannot obviously have the minimum or a maximum because then the value at a midpoint between maximum and minimum cannot be actually an average. So, let us let us look at what happens in two dimensions.

(Refer Slide Time: 04:36)



Once again the same statement is true namely the function. So, which are solutions of Laplace's equations, they do not show any features. They do not show any maximum or minimum. Let me let me illustrate this with a very simple example. Let me look at a of an equation which satisfies the Poisson's equation for example, consider.

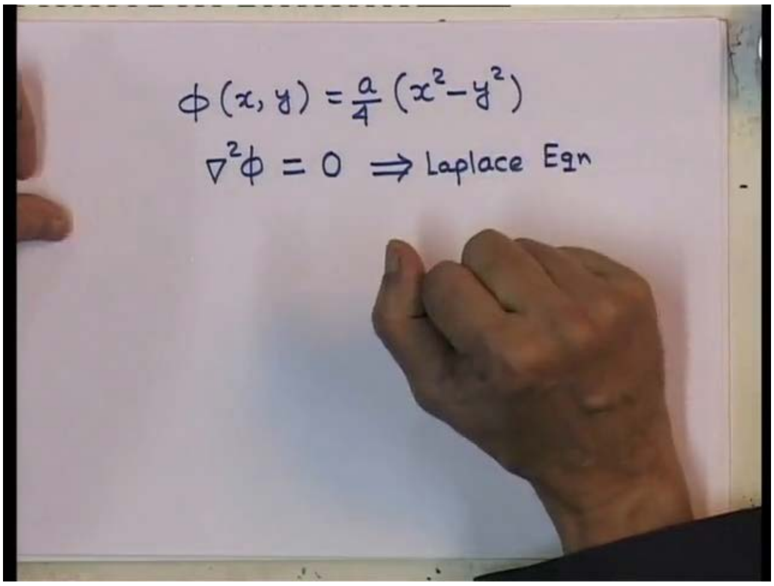
(Refer Slide Time: 05:08)



A function of x y which is a simply a by 4 a is a constant into x square plus y square. Now, let us look at what is del square remember we are in two dimension. So, del square implies d square by d x square partial derivatives plus d square by d y square partial derivatives. Therefore, my del square of phi in two dimension is simply equal to a. You can see it first derivative gives to 2 x a by 4 that is a x by 2. The second derivative will give me another it would reduce the power by 1.

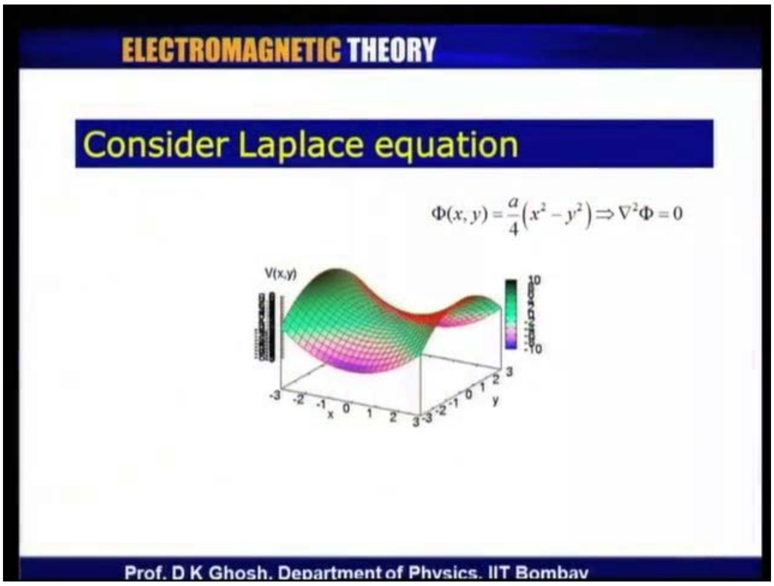
So, I have a by 2 into x just a by 2. I am sorry just a by 2 and similarly, the second derivative of this will be another a by 2 and as a result the total second derivative is equal to a. Now, this is of course, not a Laplace's equation. This looks like a Poisson's equation and of this solution. If you plot here is a genuine plot of the function, you notice this function that the it looks like a cup. So, there is a minimum at the point x equal to 0, y equal to 0 and the that is very obvious because this is a positive function everywhere, and since the functions is positive it is a absolute minimum value can be 0. So, once you are at that the once you are at the bottom then of course, the value of the function as to raise no matter which direction you go to. So, this a function with a minimum. Now, consider for example, a slightly different function namely phi of x y the phi of x y is.

(Refer Slide Time: 07:09)



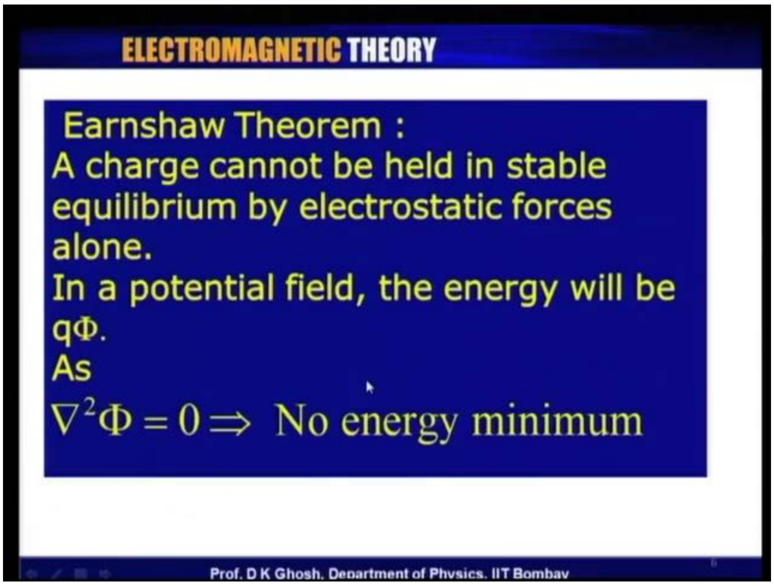
Let us say instead of a by 4 x square my plus y square, I have x square minus y square. Now, you can immediately check that del square of phi, works out to be equal to 0. Which means this is Laplace's equation. Now, supposing you have are plot this function. This is del square phi is equal to 0 a by 4 x square minus y square. Now, what happens at the point x equal to 0, y equal to 0, is that a minimum? the answer is no. Look at this structure here.

(Refer Slide Time: 07:55)



Now, the point x equal to 0, y equal to 0, is what is known as a saddle point. You all know what is saddle. A saddle is a seat on the back of a horse. Now, if you have looked at a saddle, you will realize that in one direction, the if you go in one direction that is along the back of the horse this function will go down, but on the hand if you go in the other direction the function will go up. It is a saddle, it is not a maximum because there are regions which where the value of the functions is greater than the value at x equal to 0. On the other hand it is not a minimum because there are regions for example, in this part where the value of the function is smaller than its value at x equal to 0, y equal to 0.

(Refer Slide Time: 08:54)

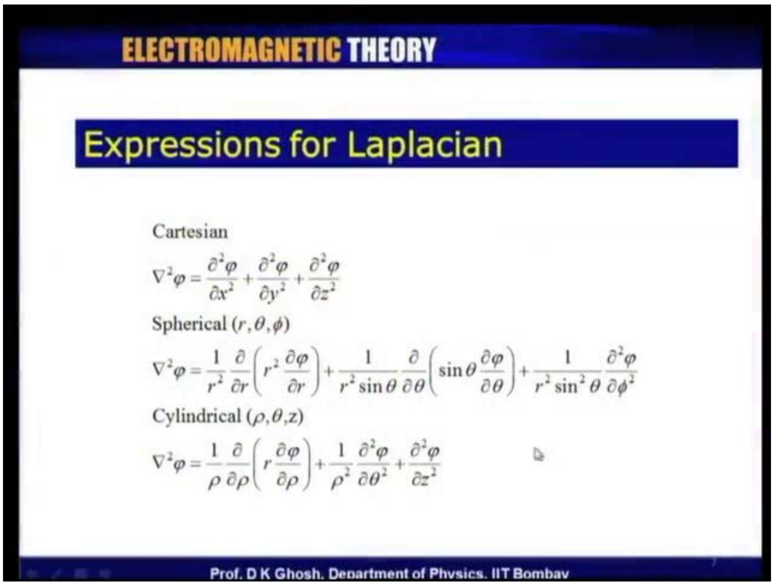


So, once again that tells me that the Laplace's, we assumes at least from the examples that we have pointed out they do not show any features. It does not have a minimum, it does not have a maximum end. So, we have seen that the averaging property works and things like that a consequence of this type things that we are doing is what is known as an Alonso theorem. Now, Alonso theorem states that supposing, you have a charge particle the theorem states that it is not possible for a charge particle to be in equilibrium, static equilibrium only by action of electro statics forces. The other ways of keeping it in the equilibrium, but if the only force that is there is an electrostatic force you cannot keep the charge in equilibrium the y.

Now, look at this. I know that if I have a charge q in a potential field v the energy of that will be equal to q times V. That the energy is q times V and the charge is in a region which satisfies Laplace's equation because there are no sources in that region. So, I have del square V equal to 0 which implies that the del square of the energy function which is q times, the potential that is also equal to 0. Now, if del square of V equal to 0 del square of q V, we I am using the symbol V for both potential and phi also for potential should not cause any confusion at this moment. Now, this solution does not have a minimum at least we have seen just now.

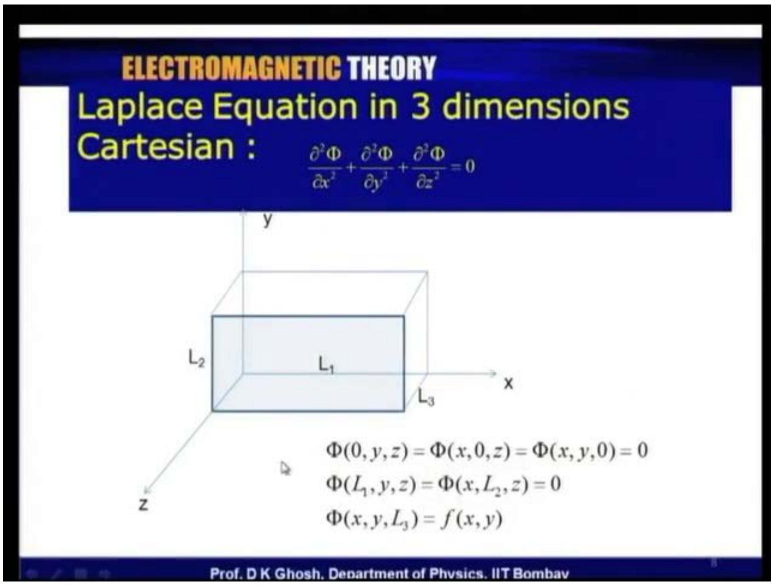
Now, everything energy function does not have a minimum, then clearly the configuration cannot be stable. It cannot be in stable equilibrium. So, this is known as Alonso theorem, follows from the fact there is no minimum in energy which in turn follows from the fact that the potential function satisfies the Poisson's equation. Now, we will come back to solutions of Laplace's equation in two dimension, a little while later, but at this moment we will jump to this solutions in three dimensions in different types of geometries.

(Refer Slide Time: 11:50)



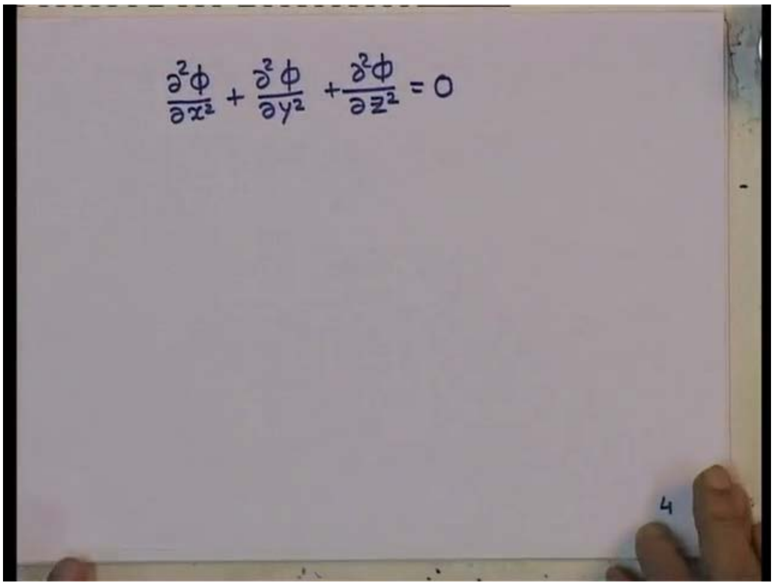
The three types of geometry which we are concerned with primarily are the rectangular Cartesian for which the Laplacian equation is simply del square phi by d x square del square phi by d y square and del square phi by d z square. This is Cartesian form. The spherical polar form depending upon distance r polar and angle theta and azimuth phi is given by this expression here we will be talking later about this equation in detail in cylindrical rho theta z coordinate as you know that the z coordinate is same as that of the Cartesian. Therefore, the Cartesian expression remains the same and the other two are basically the polar in the x y plane and that is the way it looks like.

(Refer Slide Time: 12:38)



So, let us look at the solution of Laplace's equation in three dimension. First we will take up the rectangular for Cartesian coordinates and then we will look at the spherical polar in Cartesian coordinates we have seen.

(Refer Slide Time: 13:01)

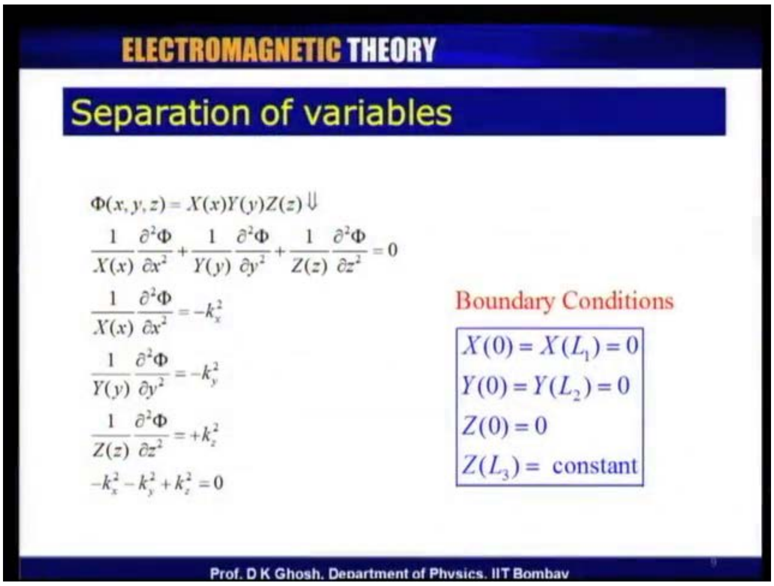


That I have got del square phi by d x square plus del square phi by d y square plus del square phi by d z square equal to 0. We know that we would like to solve this problem subject to some boundary conditions. Now, let me let me say, I have chosen a region of a rectangular parallelepiped whose dimensions are length breadth height are L 1 along the x direction L 2 along the y direction and L 3 along the z direction, on the six surfaces. I have certain values of the potential specified. So, these are my boundary condition. Now, what we will do however is use the fact that the solutions obey the position principle.

So, instead of giving all the six conditions, I will work out in detail the solution for a problem where on the five sides of the rectangular parallelepiped the potential will be taken to be 0, if their conductors it means there will be grounded. On the sixth side I will say that the potential function is known to be so, let me state that in this way. So, I have this this is my x y and the z direction, coming out of the plane of the picture and on five of these that is when x is equal to 0 and x is equal to L 1, y is equal to 0, y is equal to L 2 and z is equal to 0. These five surfaces the potential value is 0 on the sixth surface which is at z is equal to L 3, the potential distribution as a function of x y is known to me.

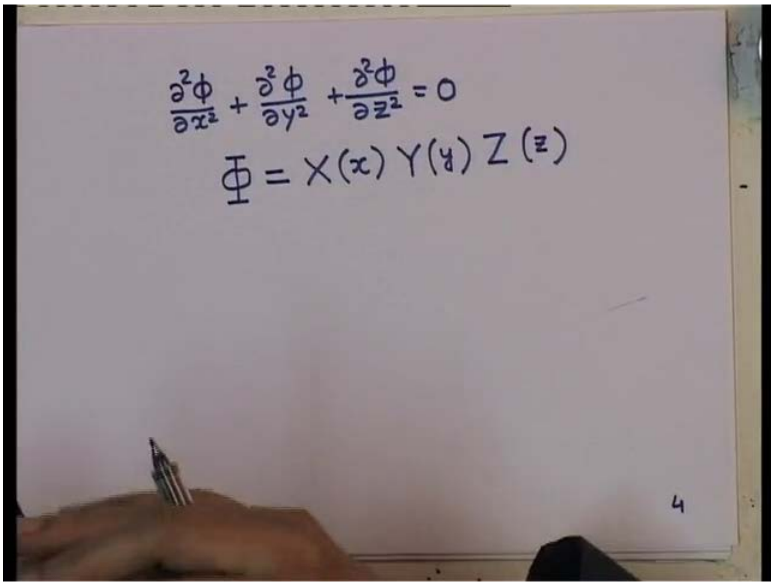
So, that is f x y how does it solve this problem? Now, supposing you have six different conditions. Now, I could actually solve these subject to a given condition on each one of them. Then sort of make a super position of these to get a general solution, but at this moment I am looking at a specific problem, where on five surfaces the potential value is given to be 0. On the sixth surface which is at z equal to L 3, the potential function is known to be some given function f which is of course, a function of x and y because the z is constant there.

(Refer Slide Time: 16:02)



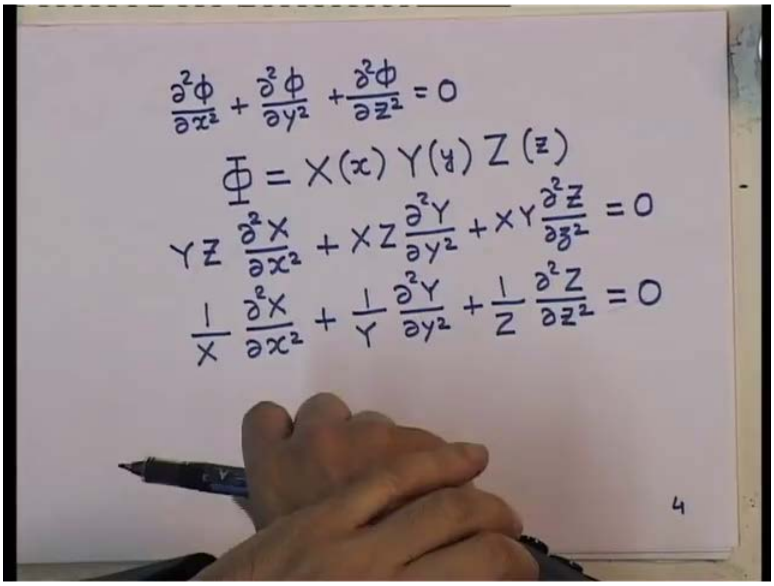
Now, so let us look at this equation the method that one follows is what is known as separation of variables.

(Refer Slide Time: 16:15)



The separation of variable work this way that this function phi is taken to be a product of three functions. Function capital x, which dependence upon x only, function capital y dependence upon y only and function capital z dependence upon z only. Recall at this moment, I do not know whether this trick will work out or not? But if this trick works then I will fall back on the uniqueness of the solutions and say that look by this trick I have found the solutions and therefore, that must be the only solution. So, let us look at this. So, you notice this that, this so there what will happen is look at this term. d square pi by d x square.

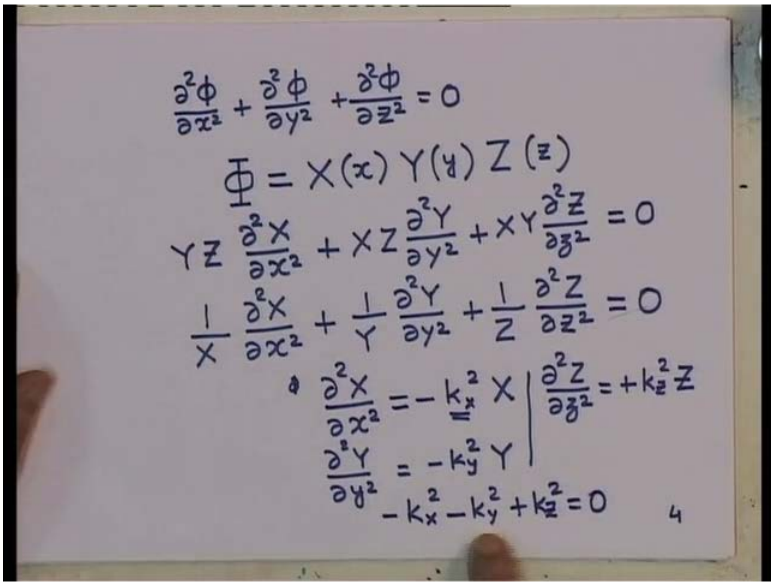
(Refer Slide Time: 17:20)



So, since capital Y and capital Z are constant, I will get from this term Y Z times d square capital X by d x square plus here, I will get X Z d square capital Y by d y square plus X Y d square capital Z by d z square, that is equal to 0. Let me divide this equation by capital X Y Z that will give me 1 over capital X d square x by d x square plus 1 over capital Y d square y by d y square plus 1 over capital Z d square z by d z square this equal to 0.

Now, look at this equation. In this equation I have got three terms. The first term depending on on x only, second term depending upon y only, third term depending upon z only. There are variations here with respect to x here with respect to y and here with respect to z. I am making a big demand, that sum of these three must be equal to 0. now since x y z they vary arbitrarily in the space, this is too tall an order and the only way we can do that is that if each one of these terms is a constant, subject to the fact that three of those constants coming from three equations become equal to 0. So, this is purely because of the fact that the variation of each is independent.

(Refer Slide Time: 19:22)

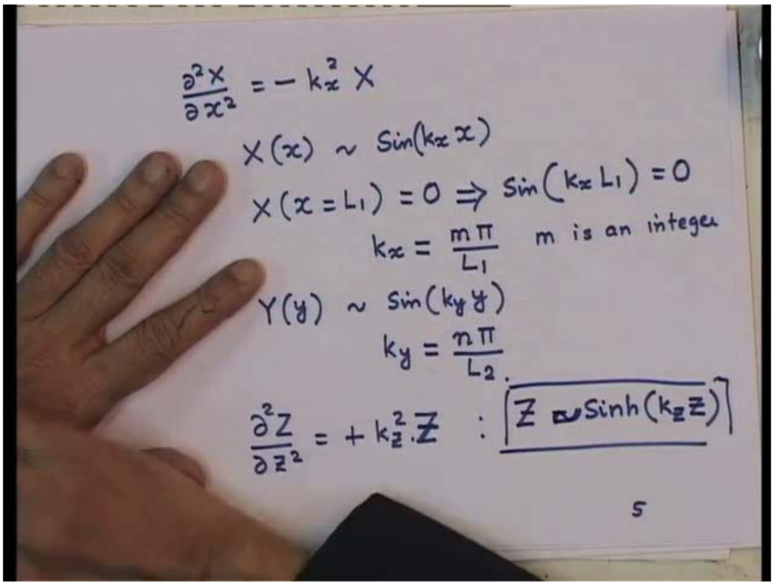


So, so what do I get? So, let me write then the first equation will be 1 over x d square X by d x square is a constant. So let me take this to be equal to, some let us take minus k square times constant. This k x square the, this x is because I am getting this constant related in associated with this x.

So, this is constant value. Let me take d square Y over d y square, is equal to minus k y square into Y, but d square Z by d z square then has to be related to these two constant, in such a way that the corresponding k z square if I take it so I will take d square Z by d z square is equal to plus k z square I will take z subject to be fact that k x square minus k y square plus k z square is equal to 0. Where all that we know about k x k y and k z are that they are constants.

Now, let us look at what happens to the corresponding boundary conditions. Remember on five of the faces the potentials were 0. So, I can translate that and say that look since X the depends only on X so, this implies that the capital X function at X equal to 0 must be 0 and it also has to be 0 at the value L 1. The sign is true for Y. Y must be 0 at 0 and at L 2 about Z of course, I have a slightly different story because the value of capital Z must be equal to 0 at Z is equal to 0, but at Z is equal to L 3 it must be equal to constant the solutions of these equations.

(Refer Slide Time: 22:01)



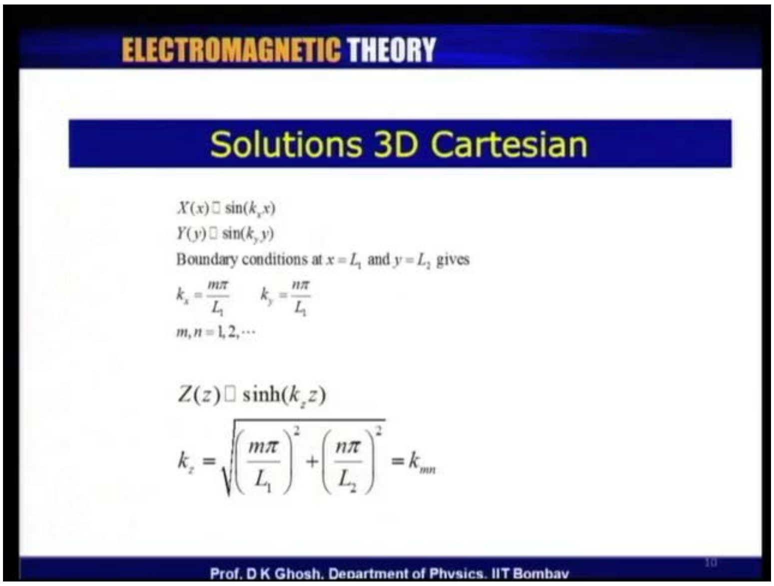
For example, the first equation which I have taken as d square X by d x square is equal to minus k x square X and also well known to all of us and the solutions of course, are capital X as a function of x is a sin k x x. Now, notice that I have chosen these two surfaces the at x equal to zero I have got the value of the capital X equal to zero, it is automatically satisfied because I have taken a sin function. But there is a one more demand, we say that X at x equal to L 1 also equal to 0, this implies that sin of k x L 1 is equal to 0.

Which immediately tells me that the k x values must be such that it must be multiple of pi over L 1, m is an integer. m is an integer It is plus or minus 1 plus or minus 2 etc. I cannot take m to be equal to 0 because that would mean that the function itself is 0. So, which is of course, not what we are looking for. Now, identical statement is true for the Y function so Y of y is equal to sin k y y, the boundary condition at y equal to 0, is automatically satisfied and the boundary condition at L 2 tells me that this should be an integer n divided by L 2.

So, also good therefore, on four sides the solutions have been satisfied. Now, once I have taken, once I have taken k x square and k y square to be positive or k x square and k y square to be positive, the k z square term which is associated with the Z equation is a negative. That it solution cannot be trigonometry function, because that equation now is d square Z by d z square is equal to plus k z square Z and since, this is the positive values I know its solution will be exponential with real arguments.

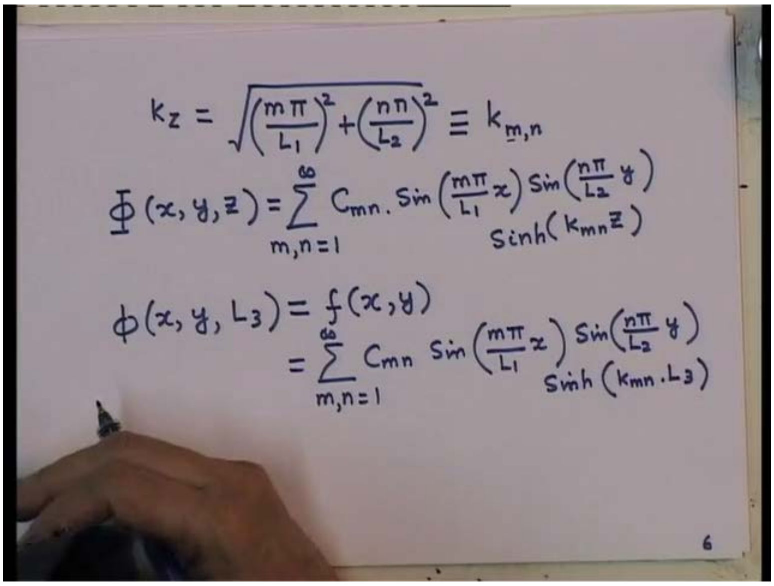
However, I also have one more condition to be satisfied. That is the value of capital Z at z is equal to 0 is also 0. Now, I know they are not trigonometric function, they are exponential function. Therefore, what I do is to choose the solution to be a hyperbolic sin function. So, we will take capital Z to be given by hyperbolic sin of k z Z. Remember sin hyperbolic of z at z equal to 0 is 0. So, this is this is the solution I take this is actually not equal to, but should be constant times z.

(Refer Slide Time: 26:01)



Now, let us look the k x k y and k z. We have seen that k z square is equal to k x square plus k y square.

(Refer Slide Time: 26:15)

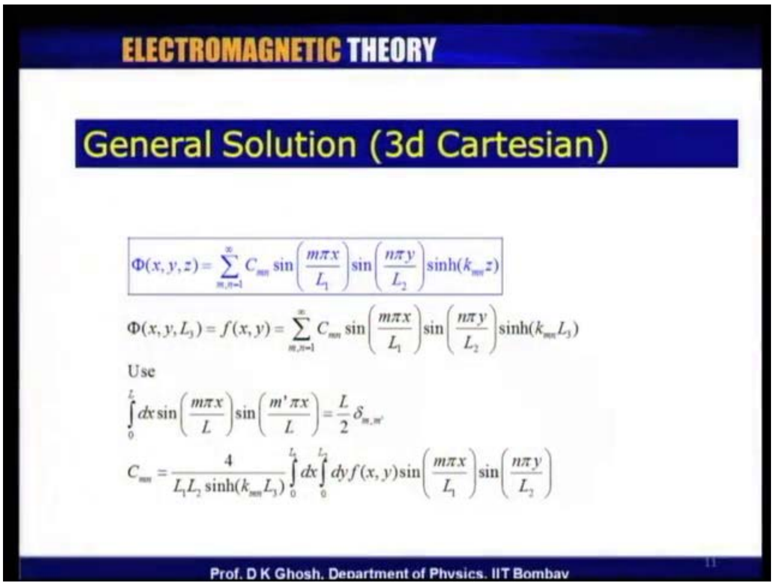


So, that tells me that this k z is given by square root of k x square plus k y square and since k x is m pi over L 1 square and k y is n pi over L 2 square, square root. So, this quantity k z can be also written as k m n. It tells me that in order to specify k z you need two indices m and n. So, therefore my solution which is phi x y z is now, a linear combination. Linear combination because m and n can vary from one to infinity, each one of them.

Coefficient C m n, sin function of m pi over L 1 into x another sin function of n pi over L 2 into y and of course, sin hyperbolic of k m n z. So, this is my solution, but there is still a sixth boundary condition to be satisfied. This sixth boundary condition is that when z is equal to L 3, the value of the potential at every point on that plane that plane you remember z is given, but x and y varies and that is a known function f of x y.

So, if you look at this then we get phi of x y and z is equal to L 3. This is simply substituting z is equal to L 3 in to this equation, but this value is a function given function of x and y. So, this is equal to just to rewrite m n equal to 1 to infinity C m n sin of m pi over L 1 x sin n pi over L 2 y sin hyperbolic k m n times L 3, that is your f x y.

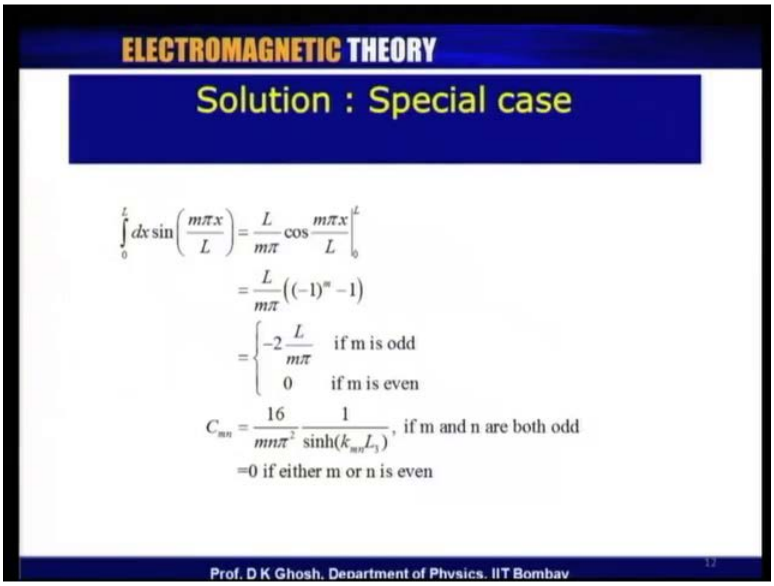
(Refer Slide Time: 29:09)



Now, I can determine this constants C m n by using the properties, the orthogonal properties of the integrals. You notice that here, I have given you an integral this is d x of sin m pi x over l sine m prime pi x over L, where m prime is a is another index. Now, if you integrate this you will find only when m is equal to m prime it gives you L by 2. Otherwise when you take the integral from 0 to L it gives me 0. So, that is why it is written as L by 2 times. This is called a chronicle delta. Delta m m prime, that is equal to 0. If m is not equal to m prime and is equal to 1, if m is equal to m prime. Now, if you recall that integration, that you notice that I come to this equation here.

I integrate I I multiply this both sides by sin m pi x over L 1 and n pi y over L 2 and integrator. Actually speaking what I do is to multiply with, I should have said m prime n prime, but I know that unless m prime equal to m and n prime equal to n these C m n's will go way. So, this is what I get there, this is what is get. So, that tells me that if I know the function f x y and I will be given everything else that I require there and in principles if I can do those integration. Then in principle C m n, these coefficient are known to be.

(Refer Slide Time: 31:01)

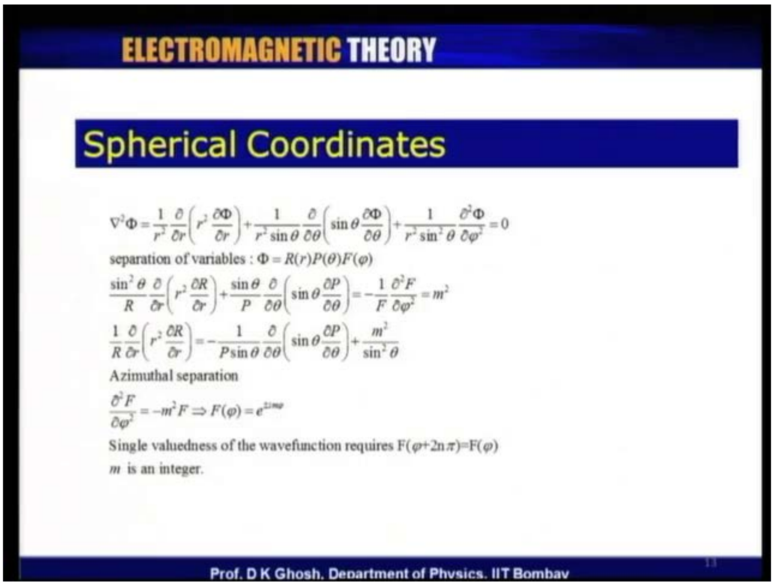


So, let us let take a very special case. If I have take a special case, where I take this function f x y let say it is constant. Now, to go back notice this, that in C m n I have two integrals, a double integral over x and y, but if this f x y is constant then of course, these integrals become two product of two different independent and differs. Sin integrals are very trivial integrals to be done. So, I use the fact that integral of d x sin m pi x over L is L by m pi cosine n pi x over L and the limits of integrations are from to 0 to L 1.

So, in for d x integration this is L 1 and that is L 1, but this is a generic relation. Now, in the upper limit I have cosine of n pi. I know that is nothing but minus 1 to the power m, lower limit is of course, 1. So, this integral is L by m pi minus 1 to the power m minus 1. This tells me that this has a value which is 0, if m is even because then it becomes 1 minus 1. If m is a odd, I get minus 2 there because minus 1 to the power odd number is minus 1 and I also have another minus 1 there. So, I get. So, the result of this equation is, it is equal to minus 2 L by m pi if m is odd and is equal to 0 if m is even.

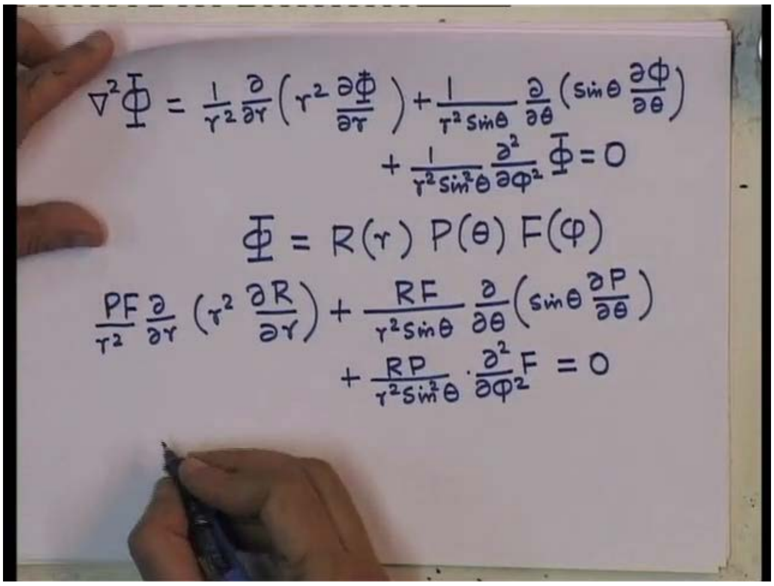
So, if you look at this expression for C m n. C m n is L by L 1 L 2 etc. I have this f x y which is now constant and product of two integrations. Each one of the integral, I know how to do it. So, if you plug it in you find this to be equal to 16 over m n pi square 1 over sin hyperbolic k m and L 3. This will be non zero only if only if neither m nor n is even, because if any one of them is even because it is being multiplied they inter corresponding integral is 0. So, when both m and n are odd C m n is non zero, in this case. So, that essentially completes our solutions in the rectangular coordinates. With this we will go over to the solution of Laplace's equation in spherical coordinates.

(Refer Slide Time: 34:18)



The spherical coordinates, Laplacian form is little complicated. But it turns out that the principle of separation of variable, that we talked about still works here the with a slight bit of a modification. So, let us look at what does it get me.

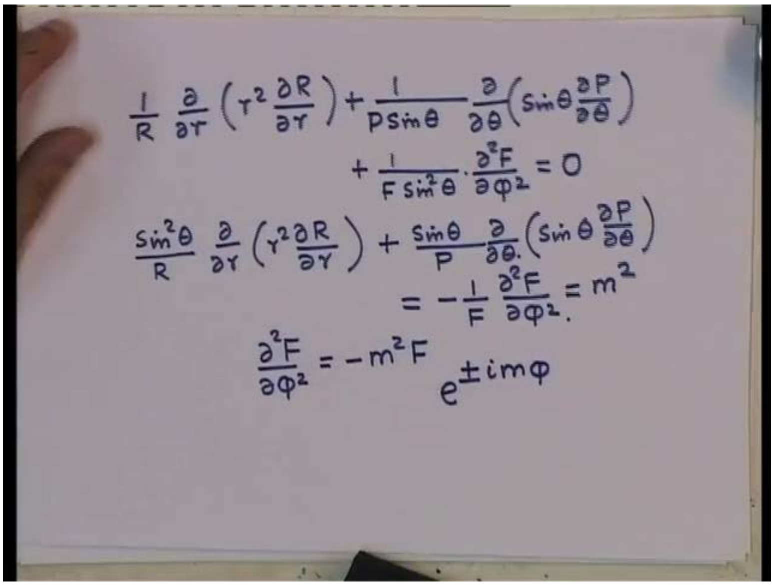
(Refer Slide Time: 34:45)



First my del square of phi here, I will be a little careful in writing the potential function phi because as the azimuthal angle is also noted by phi. So, this is given by 1over r square d by d r of r square d phi by d r, is the radial part Polar the the angle part is r square sine theta, d by d theta of sine theta d phi by d theta. Finally, I have got 1 over r square sine square theta d square over d phi square, this is my azimuthal angle of phi. That is equal to 0. Now, what we will do is we will have the same trick, that is look at the possibility of separation of variable in this problem.

So, we write capital phi as product of three functions capital R which is a function of r, capital P which is a function of theta and let us write capital F which is a function of phi. By if you plug it in, let see what you get. So, first is of course, I have got 1 over r square d by d r of r square d R by r and P times F will come out, because that would not depend upon r. This one is similarly, 1 over r square sin theta, d by d theta of sin theta and this time it is d P d theta and R and F will come out and 1 over r square sin square theta d square of d phi square of F and this time R and P will come out, this equal to 0. Now, you do the same thing again that is divide this equation by R P F and the other thing that we notice is there is a 1 over r square all over. So, therefore I can take that r square to the right hand it becomes 0. So, I am left with the equation which looks like the following.

(Refer Slide Time: 37:44)

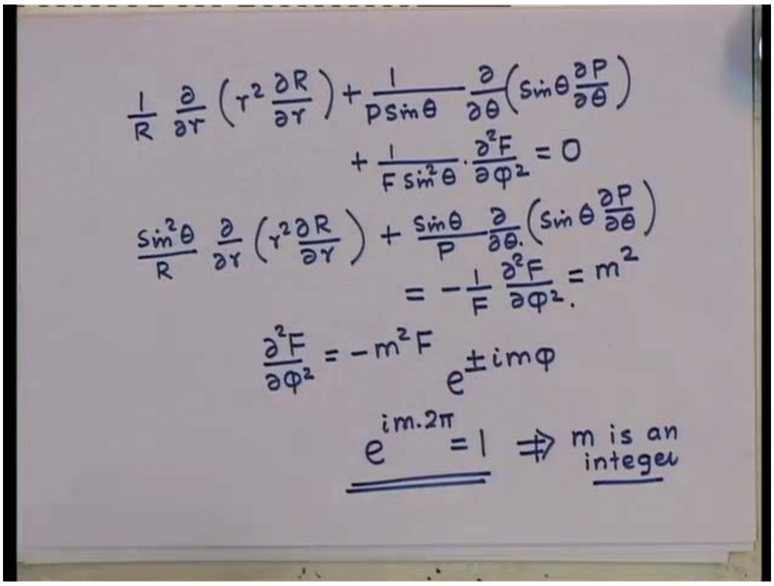


I get 1 over R d by d r of r square d R by d r plus 1 over sin theta times P d by d theta of sine theta d P by d theta plus 1 over F sine square theta d square F by d phi square, that is equal to 0. The the first thing that we need to do, is to realize that in this equation in this equation I can take the sin square theta multiply the entire equation by sine square theta and realize then these two equation, these two terms the radial term and the polar term, they are both theta appears there. On the other hand this will give me just 1 over F d square F by d F square.

So, let me let me do that. So, let me multiply this with sin square theta by R d by d r r square d R by d r plus sin theta by P d by d theta sin theta d P by d theta. This is equal to minus 1 over F d square F over d phi square. Now, I give the same argument again, the left hand side is a function of r n theta and small r and small theta where is on the left hand side, the right hand side however, depends only on phi. We are excepting this should be equal to that. That is not possible unless each one of these terms happens to be a constant. So, let us let us write that constant to be equal to some quantity like m square.

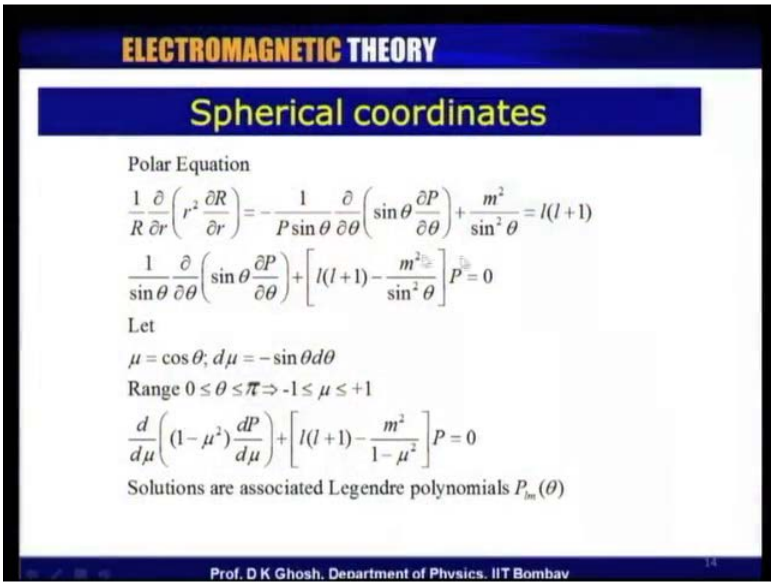
Now, I can I am now in a position to solve what is known as the azimuthal equation. the azimuthal equation then becomes d square F over d phi square is equal to minus m square F. We know that the solution of this equation is, e to the power plus or minus i m phi. What is m? The value of m can be fixed by saying that if phi increases by 2 phi. Let say I am I am back to the same point, if the azimuthal angle changes by an amount 2 pi I am back to the same point. So, that value of F must be cyclic in the sense, whatever is the value of F at some value of the azimuthal angle phi that must be having the same value, when the azimuth increases by an amount 2 pi. Which is possible if e to the power plus or minus i m into 2 pi is equal to 1.

(Refer Slide Time: 41:57)



So, if e to the power i m into 2 pi that is equal to 1, then it is possible. First which tells me that m is an integer, positive or a negative integer. So, in this equation by m is an integer that has been worked out. Let us look at the next equation. Now, having done that identified m.

(Refer Slide Time: 42:32)



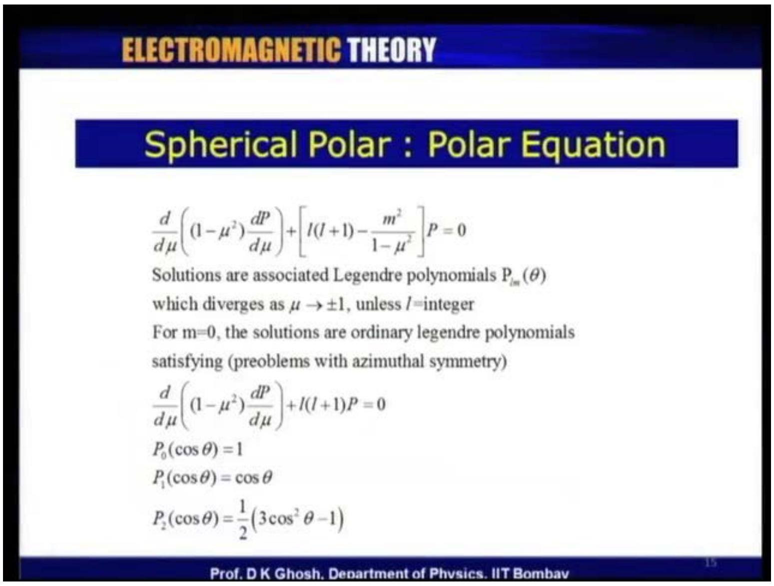
So, I will now write the equation in this way. Remember I have divided by sin square theta but now, I have brought the sin square theta to wherever it is. I have written 1 over R d by d r of r square d R by d r is equal to two terms. One is this term which was already there and the other one is m square by sin square theta. This as made it possible for me to separate r and theta, the polar separation. Therefore, once again with the type of argument that we have been given, each one of these terms must be constant.

I do not know what that constant is, but anticipating some result. Let me put that constant to be equal to L into L plus 1 of course, I have not yet said what is L therefore, I can write my constant anywhere I like. So, I have got now two equations. So, the main equation is the polar equation is 1 over sin theta d by d theta sin theta d P by d theta plus this L into L plus minus m square by sin square theta into P equal to 0. You can simplify this equation by making a substitution of variable mu is equal to cos theta. If you do that mu is equal to cos theta, d mu is equal to minus sin theta d theta.

Now, since I know that the polar angle goes from 0 to pi the mu which is taken to be equal to cos theta goes from minus 1 to plus 1. Straight forward to check that this equation can now, be written in terms of the new variable mu like this.

We will not actually be trying to solve this equation, but it turns out that this equation is well known in the field of differential equation. The solutions are known to be associated Legendre polynomials. So, this is an equation whose solutions people know, we will be talking about the nature of solution little latter.

(Refer Slide Time: 45:08)



Let us look at that. So, this is called associated Legendre polynomials. So, my angle part of the different of the Laplace’s equation. Now, has an e to the power i m phi times the associated Legendre whose, which are characterized by L in addition to m. Now, incidentally there are two types of solution. There one is written as P L m another is written as q l m. Now, q l m goes to infinity as mu goes to plus or minus 1 P L m does not, but P L m also goes to infinity that is it diverges as mu goes to plus or minus 1 unless L happens be an integer.

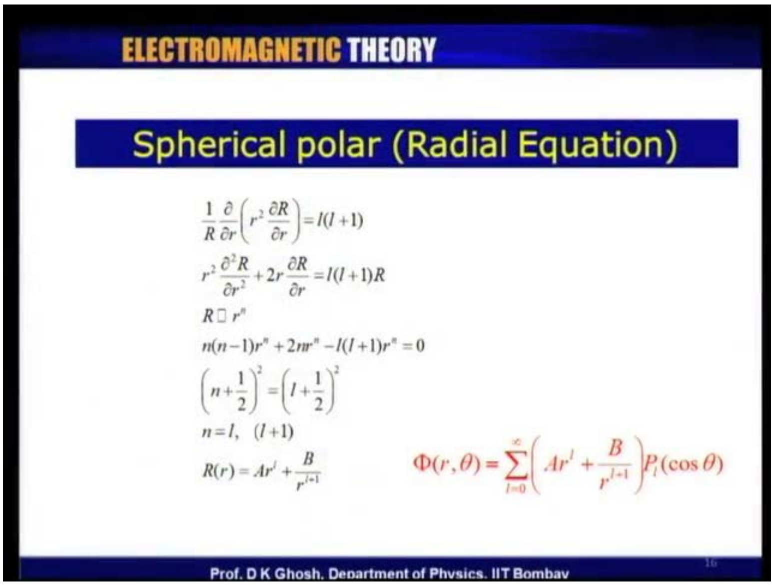
So, therefore I take the integer. In that the integer L the, L to be an integer and then the solutions are given by P L m which are of course, functions of cos theta and of course, you have to multiply it with e to the power plus or minus i m phi. Let us take a simple specific case. Suppose in my solutions do not have azimuthal dependence. What it means is, my problem does not have a dependence on the azimuthal angle in other words there is azimuthal symmetry in the problem. The entire in azimuthal symmetry in the problem, the solutions cannot depend on phi. The only way to do it is to take the m to be equal to 0.

Now, if you put m to be equal to 0 in the equation of associated Legendre polynomials, the equation that I get is d by d mu 1 minus mu square. This time I have removed the L m index d P by d mu plus L into L plus 1 into P equal to 0. This is known as the equation for Legendre polynomial. The previous one is called associated Legendre polynomial. So, the Legendre polynomials are essentially special cases of associated Legendre polynomial having m is equal to 0.

Now, the solutions of these Legendre polynomial are also well known. You can actually check very easily the solutions it happens to be a polynomial in mu which is a polynomial in cosine of theta. For instance you will find that the zero th order polynomial is 1. It is very straight forward to check that a P is equal to 1 the left hand side is identically equal to 0 because d P by d mu. If my, if L is equal to 1 L is equal to 0 P 0 cos theta is 1. If L is equal to 1 then of course, this becomes 1 into 1 plus 1, which is 2 and you can check that P 1 of mu is nothing but mu itself. Now, you can, that is why fairly straight forward because if P is mu d P by d mu is 1.

I have got a 1 minus mu square there. So, when you differentiate this with respective to mu I get a minus 2 mu there. Therefore, I get a minus 2 mu there. Here I have got L is equal 1. So, it is 1 into 1 plus 1 which is 2 and P is mu. Therefore, this equation is satisfied. Likewise you can find that P 2 cos theta is half into 3 cos square theta minus 1. So, this is the spherical polar equation. So, what are we seen we have seen that when I have azimuthal symmetry, my solutions are given by the angular part of the solutions are given by Legendre polynomial. I have not yet touched the radial equation. But in the remaining time let us look at what the radial equation looks like.

(Refer Slide Time: 50:23)



If you recall that the radial part was 1 over R d by d r r square d R by d r that was equal to a big expression with angular part and we put both of them as L into L plus 1. So, this is the equation. Split up this equation you get a equation like this. This is actually a equation which is very easy to solve. You can check, if you take capital R to be going as r to the power n, then the equation is satisfied and the value that n takes is L or L plus 1. So, as a result the complete solution of this equation, which whose properties will be talking about in the next lecture is given by A which is a constant, R to the power L plus B by R to the power L plus 1 into P L cos theta. Next lecture we will look at some applications of these solutions in three dimensions.