Electromagnetic Theory
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Module - 2
Electrostatics
Lecture - 13
Poisson & Laplace Equation

In the last lecture, we had talked about the coefficients of potential and capacitance. As we are getting into more formal aspects of electrostatics, we will be spending some time in looking at certain mathematical foundations; in particular, two equations of electrostatics namely the Poisson's and the Laplace's equation.

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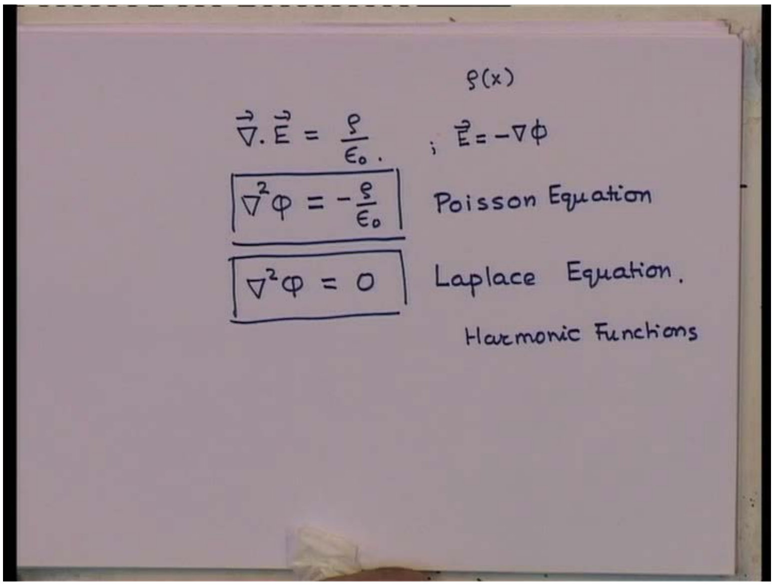

As we are aware that Laplace's and Poisson's equations are rather fundamental to electrostatics. So, if we have a charge distribution, if we have a charge distribution given by the density rho at certain points, then we are aware that according to Maxwell's equations, the divergence of the electric field is given by the rho divided by epsilon 0. Now, as we defined the electric field in terms of potential, that is electric field is given by gradient of phi; where phi is a scalar function. We could immediately obtain an equation for this scalar potential; namely del square phi is minus rho over epsilon 0 del dot del. Now, this equation this equation is known as the Poisson's equation.

Now, supposing I have a region of space in which there are no sources of charge, now in which case of course, this rho becomes equal to 0, and my equation then becomes del square phi is equal to 0, which is known as the Laplace equation.

Now, we have fundamental interest in looking at the solutions of these equations; that is situation where there exists charge density is given by Poisson's equation and region which are source free; where the solutions are given by the Laplace equation. The solutions of the Laplace's equation are known as Harmonic functions.

What we intend to do today is to look at both these equations and look at some formal aspects of the solutions. As well as take some simple example and see how they give us the known results of electrostatics that we have derived from more elementary considerations. Once we have done that, of course we will go over to give you the formal aspects of the solutions of both Poisson's and the Laplace's equation.

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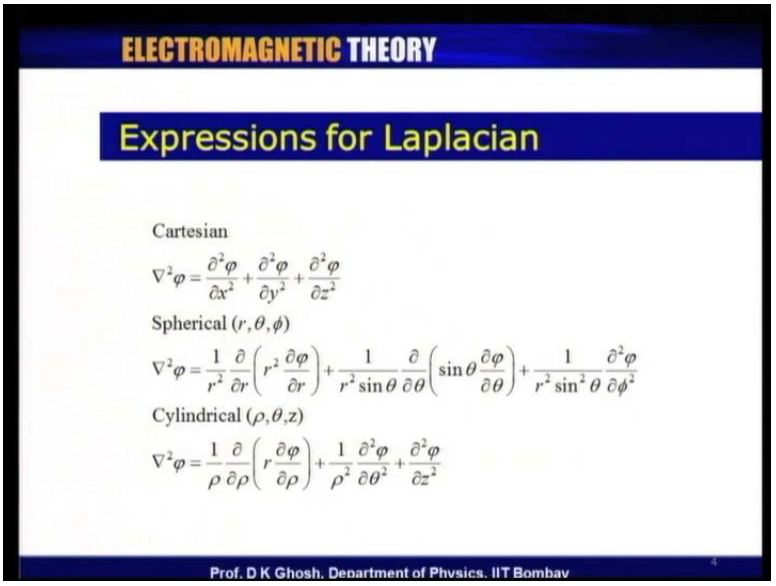

So, let us let us look at first the what are the formal aspects. The since we deal with the del square operator, since we do it deal with the del square operator, We would in general be interested in knowing what is the three dimensional expansion of the Laplacian operator.

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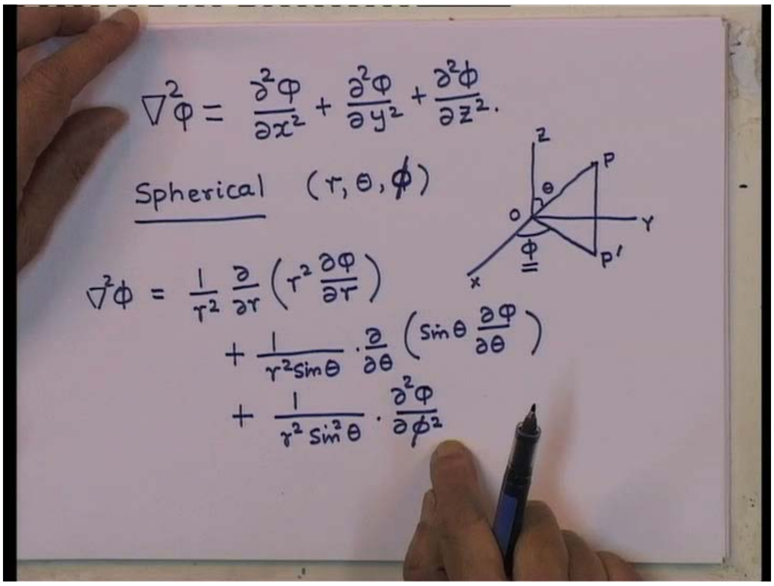

Now, in Cartesian coordinate this is very simple. So, for example, if I want to write down del square of phi; phi is the potential. In Cartesian coordinate, it is simply d square phi by d x square, that is partial, del square phi by d y square and partial del square phi by d Z square. And this is of course fairly simple. Now, very often we need to look at problems which show spherical or cylindrical symmetry. These will be the two most important geometries that will be involved with.

So, for example if we look at spherical coordinates, as you were aware that in spherical coordinates we have variables which are r, theta and phi; just to recall for you, r is of course the distance from the origin. So supposing I plot this, my reference plane is X Y Z; r is the distance from the origin of a point P and O P O P makes an angle theta with the z axis. Then from the point p we draw a perpendicular to the X Y plane. And then I have this azimuthal angle phi which is my reference x-axis makes with the foot of the perpendicular that has been drawn from the P point through the point P prime on to the X Y plane.

So r, theta and phi are my variables in spherical coordinates; phi is the known as the azimuthal angle. And so let us look at what happens to the expression for the spherical; for the del square operator in the spherical coordinates. We will not be deriving this because they do not really add to our knowledge, because they are can be found in any standard Mathematics text book. So, del square phi can be written as first the radial part, which is one over r square d by d r of r square d phi by d r. Then I have derivative with respect to theta, and that is r square sin theta d by d theta of sin theta d by d theta of phi, of course. And finally, the azimuthal part which is r square sin square theta d square phi; well, since I am using phi for the potential, let me write this phi in a slightly different way, that is, the azimuthal angle. So, it is d phi square. Well, I guess that is one of the reasons why many text book use v for the potential, but this should not cause much of confusion. So, that is that is our expression in the spherical coordinate.

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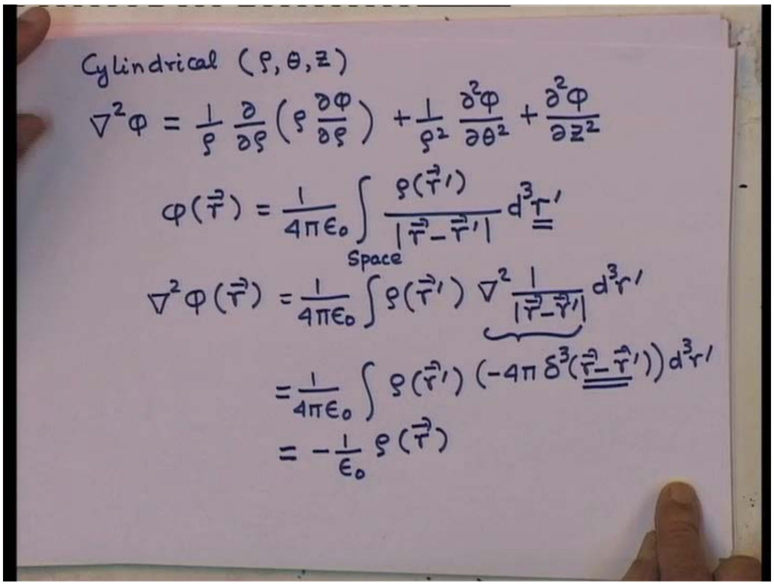

And, the third geometry that we normally talk about is the cylindrical geometry. And in the cylindrical coordinates as you remember that our variables are rho, then of course polar; so, this is basically a polar two dimensional polar things and rho and theta are like polar angles. So, it is cylindrical will be rho, theta and z. So, Z is just the z axis which is same similar to the Cartesian thing. And rho, theta are the polar, the typical polar angles in the X Y line. And the del square operator in polar coordinate is 1 over rho d by d rho of rho d phi by d rho plus one over rho square d square phi over d theta square. And of course the z axis simply remains the way it is in the Cartesian coordinates. So, it is just d square phi over d Z square. Now, we will be using them quite liberally.

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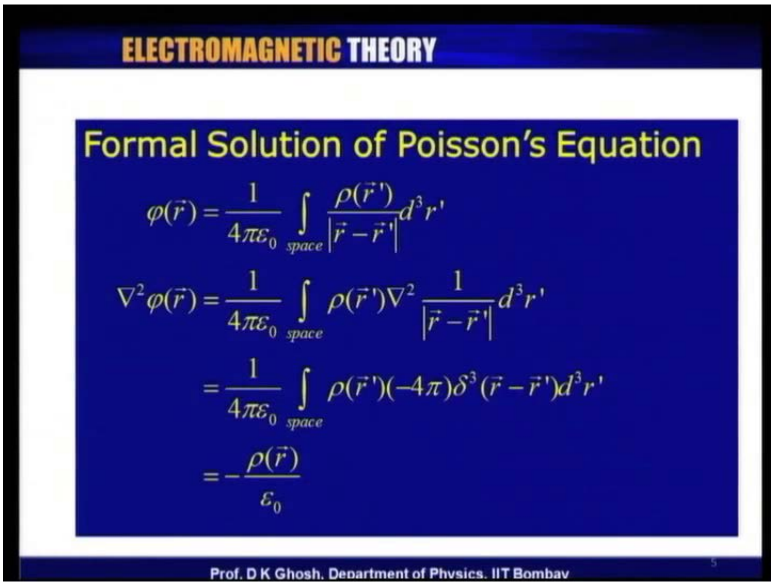

And, so let us look at first what is the formal solution of the Poisson's equation. Now, we actually already know the solutions, but it is a good idea to sort of double check the electrostatic field or the potential that we know, which has been derived from Coulomb's law. And that was simply that, phi at the point r is given by 1 over 4 pi epsilon 0, then the integral is to be taken over all the variables r prime in the entire space and whenever there is a charge density. So, it is rho r prime if the charge density is non-zero. At the point r prime, it comes with a rho r prime divided by r minus r prime modules and the d cube r prime.

So, the integration variable is r prime and we take the integral over all space. At points where there are no charge densities, d rho of r prime will become equal to 0. And so therefore it will not contribute to the integral. Now, let us look at whether this is indeed a solution of the Poisson's equation. So, what we need to do is del square phi r. Now remember that, this del square is being taken with respect to the variable r of the potential, the argument of the potential function. So, this is one over four pi epsilon 0. Now, since the integration is over r prime, I can always plugin, take the del square inside and rho of r prime of course does not depend upon r.

So, therefore del square operator which is a operator operation with respect to r does not take care of that. So, del square of one over r minus r prime d cube r prime. If you recall, we had shown that del square of one over r is given by minus 4 pi times three dimensional delta function of the argument. So, therefore, this is one over four pi epsilon 0 integral rho r prime and so it is minus four pi. And since it is a three dimensional function, I write down delta cube of r minus r prime and d cube r prime. Recall the property of the delta function that, delta function if it is inside an integral, then it only contributes at one point; that is, the point at which its argument becomes 0.

And, so therefore all that I need to do is to, wherever there is a variable r prime there, I need to put it as equal to r because r minus r prime becomes 0 there. There is a minus four pi, plus four pi; that cancels out. I am left with minus 1 over epsilon 0 and rho at the point r because of the delta function. As you can see that this is indeed the Poisson's equation for the potential function phi of r. And so therefore I know that this is indeed the solution of the electrostatic problem.

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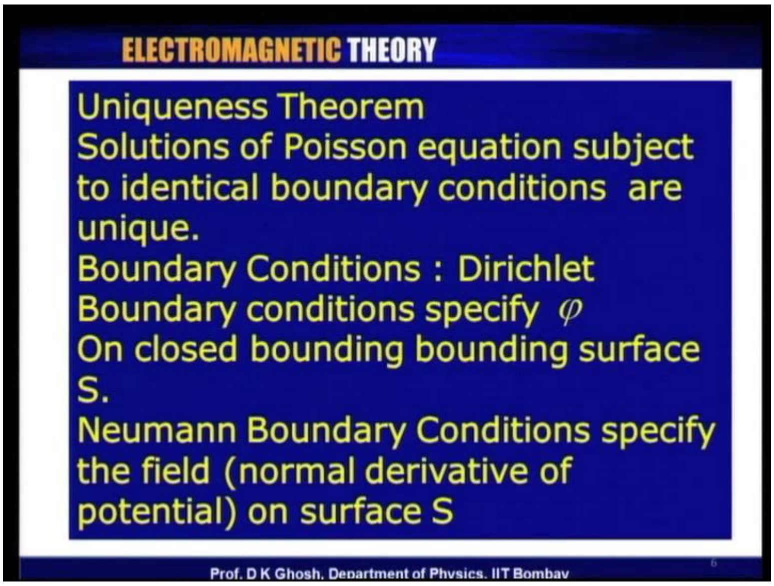

Now before we go for some illustration, let us look at the general structure of these solutions. What are the general properties of these solutions? The one of the most important point is this. So, you know we are solving basically differential equations. Now, while solving differential equations we also need what we know as or what we call as boundary conditions.

So, differential equations are always solved subject to certain boundary conditions. And so we are going to state a theorem which says that, that if you take Poisson's equation and you have to solve that subject to a given boundary condition, then the solution of those equations will be unique. Now, the uniqueness has a great advantage. If you know that the solution is unique and if by intuition or by some other way, you can show that a solution or a given function satisfies the differential equations and the boundary conditions specified, then you know that those are or that solution is the only solution.

So, now what is meant by boundary condition? Now, typically what happens is this. That supposing you want to find out the solution of the Poisson's equation in everywhere in space, but in that space I have, let us say a set of conductors; we are aware that the conductors are equipotential. So, therefore if I have a set of conductors, I could specify I could specify the values of the potentials on those conductors.

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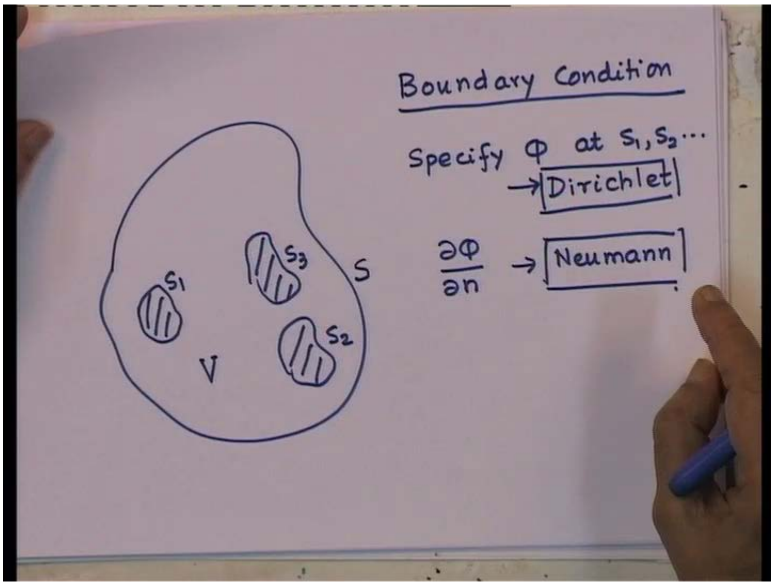

And, of course I supposing in addition, there is an overall boundary to that volume, so for instance, I have a big bound volume like this; so, this is my outside boundary S. And let us suppose that these are some surfaces which are conducting surfaces. So, I specify that I know the potentials at S 1, S 2, S 3 and of course S. Now, I need to solve the problem potential problem; that is, the Poisson's or the Laplace's equation inside this volume, which is bounded by, which is exterior to the surfaces S 1, S 2 and S 3. Now, the uniqueness theorem states that the solution that you obtain will be unique, but before we go to a proof of that, it is not always necessary that the values of the potentials are given on S 1, S 2, S 3 and S. Instead, what we could have is we could specify the electric field, which is of course normal to the conducting surfaces.

So, the boundary conditions which are there with us are of two types; so, the boundary conditions on which are applicable on these surfaces. So first is, we specify the values of the potential. The one you specify values of the potentials on the conducting surfaces and the boundary, these are normally known as the Dirichlet boundary condition. So, these are known as the Dirichlet problems. Now, if on the other hand you specify essentially d phi by d n normal derivative of the potential, which is nothing but the electric field subject to a sin, these are known as the Neumann boundary condition.

So, what we are going to do is to prove that, solutions of Laplace's or the Poisson's equations subject to Dirichlet or Neumann boundary conditions will be unique. Now I must point out that, the type of boundary conditions that we have must be a one type; where you cannot have on certain surfaces Dirichlet and on certain other surfaces Neumann boundary conditions; because I mean, such a problem would be known by some other name like quasi boundary condition. And one can show that the solutions then need not be unique. So, let us look at what is it.

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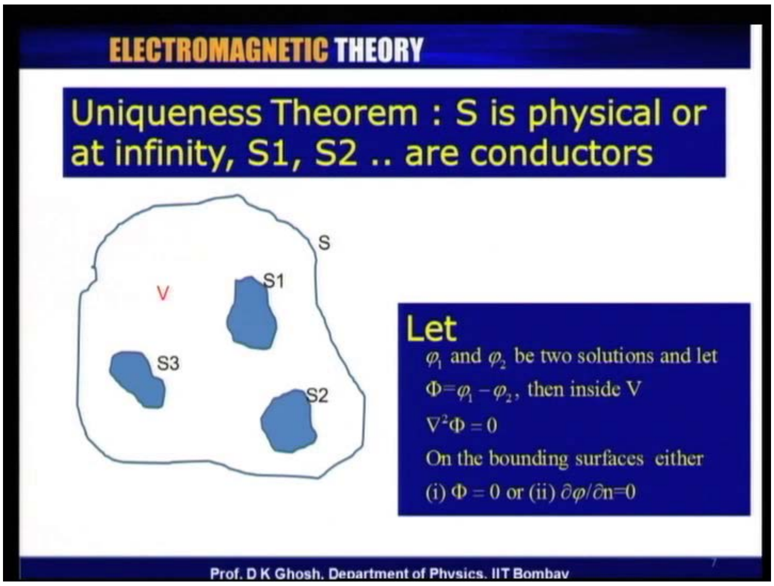

Now, so as the same picture is being shown on the screen, so let us suppose that the statement that we have made is not true.

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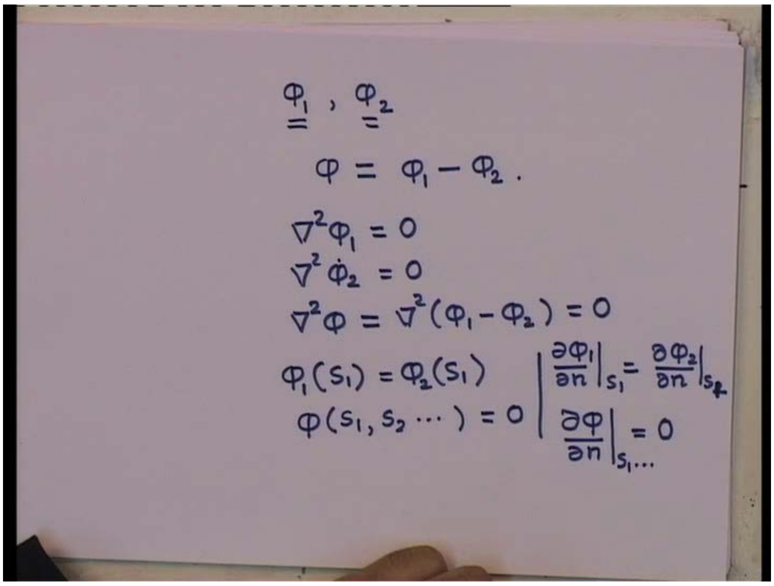

That is, suppose phi one and phi 2 are two solutions of the same differential equation namely Laplace’s or Poisson’s. And let us also further state that they satisfy the same boundary conditions, either Neumann or Dirichlet. And so now, we are interested in finding out the whether this is possible. Now, let me define let me define a solution; general potential given by the difference between phi one and phi two. Now, since phi one satisfies the Laplace’s equation or Poisson’s equation, del square phi one equal to 0 let us say. And Del square phi 2 is also equal to 0. This tells me that del square phi which is equal to del square of phi 1 minus phi 2; that must also be equal to 0. Now, this is true, whether phi 1 and phi 2 satisfy Poisson’s equation or Laplace’s equation. In all cases, phi one minus phi two, which is phi satisfies a Laplace’s equation. Now, since we have made...so, this is what we want to solve for phi my volume V. Now, since I know that on the surfaces S 1, S 2, S 3, etcetera, either Dirichlet or Neumann boundary conditions are valid.

So, I have phi one. For example, if I am taking about Dirichlet boundary condition, then phi one on S 1, for instance, is same as phi two on S 1. And similarly for S 2, S 3, etcetera. Now, that means phi at S 1, S 2, etcetera must be equal to 0. Now the same situation is true, if instead of Dirichlet and Neumann, supposing I say d phi one by d n one on S 1 is equal to d phi two over d n at S 2, then again by subtracting we find the d phi by d n on S 1 S 2; whatever, sorry, this is same surface S 1. This must be equal to zero.

So, you notice that once the boundary conditions are the same, the potential phi on each one of the bounding surfaces or d phi by d n on each one of the bounding surfaces must become equal to 0.

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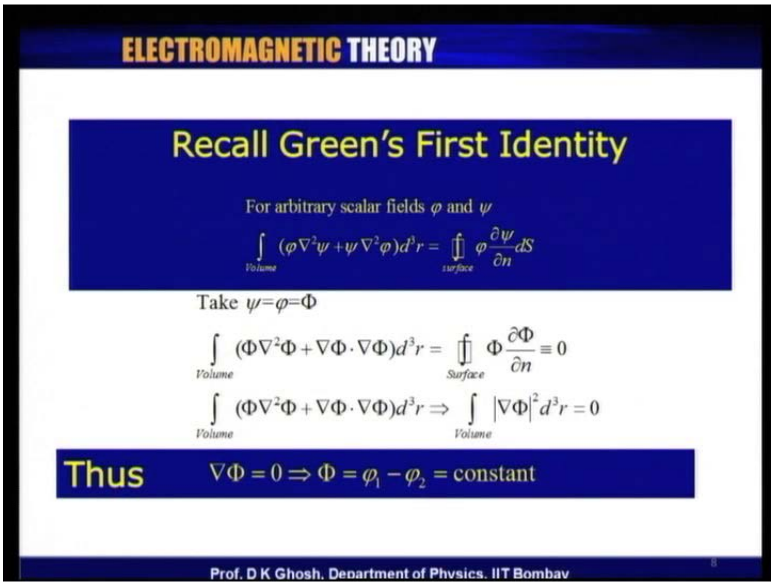

Now in order to prove this, let me take you back to some mathematical exercises that we have done in our first few lectures. One of the things that we proved, we called it Green's first identity.

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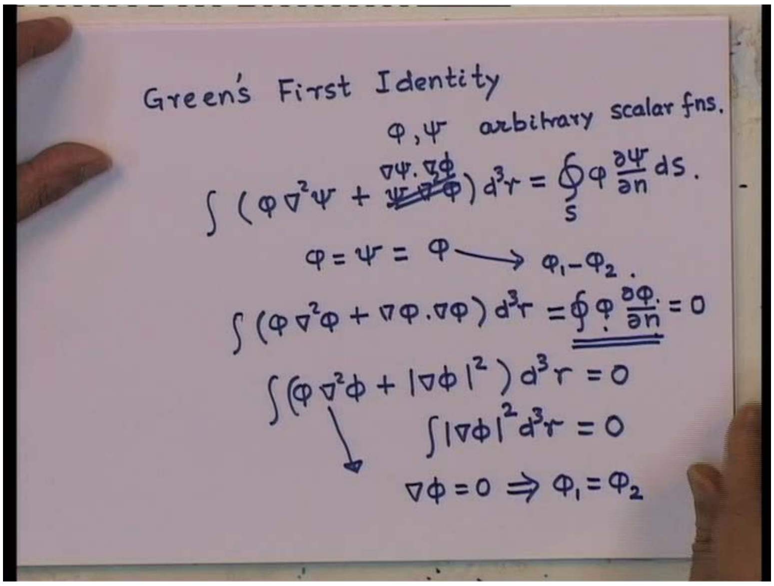

What we said there is that, supposing we have got two scalar fields given by let us say phi and psi, then If you take an integral over a given volume bounded by a some surface S of phi del square psi plus psi del square phi integrated over the volume, this is given by the integral over the surface which is bounding surfaces of phi d psi by d n d s.

Now, recall that this is true this is true for phi, psi being arbitrary scalar functions. Now since they are true for arbitrary scalar function, I am going to do the following. I am going to take phi to be equal to psi as a special case, and equal to the phi that I have chosen, that is, phi of our problem.

So, this phi is of course the arbitrary, but this is my potential; this is the phi 1 minus phi 2. Now, so let us substitute it here and see what we actually get. So what do we get is, if you take this, this will be phi del square phi. So, I have phi del square phi plus del phi dot del phi d cube r; that quantity is equal to integral phi d phi by d n. I made a small error here. This term should have been del psi dotted with del phi for the first identity.

Now, you notice. This right hand side is on the surface. And we have seen that on the surface, either phi is equal to 0 or d phi by d n equal to 0 depending upon the type of boundary condition you have. So whatever be the boundary condition, whether it is Dirichlet or Neumann, this quantity is equal to 0. Now, so that tells me that phi del square phi plus, well, del phi dot del phi is absolute del phi square d cube r; that is equal to zero. But remember that phi satisfies the Laplace's equation; phi one and phi two could satisfy Poisson's equation, but phi satisfies always Laplace's equation. So, del square of phi 0.

Now if del square of phi 0, this tells me del phi absolute square is equal to 0. And we all know that I cannot have an integration which is some of positive quantities only and get a value 0, unless each one of the term is equal to 0. So, which tells me that gradient of phi is equal to 0 which implies that phi one is equal to phi two, which is nothing but my statement of uniqueness.

Now, what we will do is this. Before I go to formal solutions of this problem, I am going to be looking at some simple applications which you are all familiar with, and the results also we know. So, this is simply reconfirm or reassure us that, the solutions that we obtain by the mathematical process that we are going to elaborate now. Also, gives us the solutions which we are aware of from more elementary considerations.

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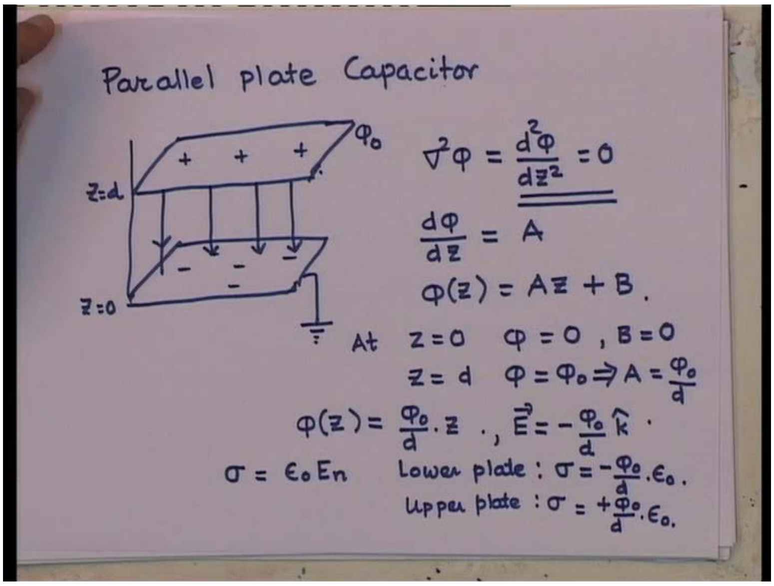

So, let me first pick up or take the example of a parallel plate capacitor. So, they I have this parallel plate capacitor here. I have got two plates; one at…So, let me take the z axis as the vertical axis. So, I have a plate which is, let say at Z equal to 0. And this plate, I will be grounding it. And there is another plate there is a another plate at let us say Z equal to d, this is the separation between the plate.

So, you notice that and let us make a statement that Dirichlet boundary condition is true, namely the upper plate is maintained at potential V 0 phi 0. Now, in the region in the region the region between the two plates they have no sources of charge. So, therefore in the region between the two plates, I have Laplace's equation for the potential; which is del square phi. But remember that since I take the plates to be infinite in dimensions, the there is no x and y variation of any physical quantity that we care to calculate. So in other words, any variation that is there has to be only with respect to the z axis. So, this is nothing but this tells me this is simply d square phi by d Z square and that is equal to 0, which is basically reduced to a one dimensional problem.

Now, you can easily solve this equation. As we know that, this means d phi by d Z is constant. Let us call it A. And phi as a function of Z is then given by A times Z plus a second constant B. Now, we need to now supplement to the boundary conditions. So, at Z equal to 0, phi is equal to 0; which tells me that the constant B is equal to 0. Now, at Z equal to d, the value of the potential is phi 0. And you notice my equation is now because

B is 0; that my equation is phi Z equal to A z, so put Z is equal to d. and on the left hand side put phi 0, which tells me that A must be equal to phi 0 by d.

So, therefore the solution of this equation for an arbitrary Z is: in place of A, I must put phi 0. So, that is simply equal to phi 0 by d times z. So, this is the potential phi of Z as a function Z. And this satisfies the two boundary conditions that we have talked about. Now, I can calculate the electric field in the region between the plates because I know that minus the gradient; of course the gradient in this case is nothing but the derivative with respect to z. So, therefore the electric field is directed electric field is directed along the minus z direction and it is given by minus phi 0 by d times unit vector K. Now, which is consistent with the fact that my upper plate will have positive charges, and the lower plates will have negative charges; because this is connected to the earth. And the electric field, of course is constant in the region between the plates.

Now once I know the electric field once I know the electric field, I can calculate the charge density on the plates. Now, while calculating the charge density you have to be little careful. You have to notice that the normal component of the electric field is the charge; gives me the charge density which is basically sigma, is epsilon 0 times E n. So, if you take the lower plate; since on the lower plate, the direction of the normal is along the positive K direction, so my sigma is nothing but minus phi 0 divided by d. And of course, there is an epsilon 0 there. And on the upper plate, however the charge density sigma is just the opposite because the direction of the normal is along the minus z direction. So, that will be plus phi 0 by d into epsilon 0. Having obtained the charge density, I can simply multiply to the area of the plate. Of course, strictly speaking we have taken the area of the plate to be infinite, but we can multiply it with the area of the plate and get the total amount of charge to be given by phi 0 d epsilon 0 by A.

So, to get the charge and you know that, so the charge Q on either plate plus or minus, of course is area phi 0 d by d times epsilon 0. And you know that the potential difference between the plates is phi 0. And then this immediately gives me that, if I equate these two capacitance times phi 0, it gives you the your well known expression for the capacitance; which is A epsilon 0 divided by d. So, this is the simplest one dimensional parallel plate capacitor problem. Now, let us continue with different example.

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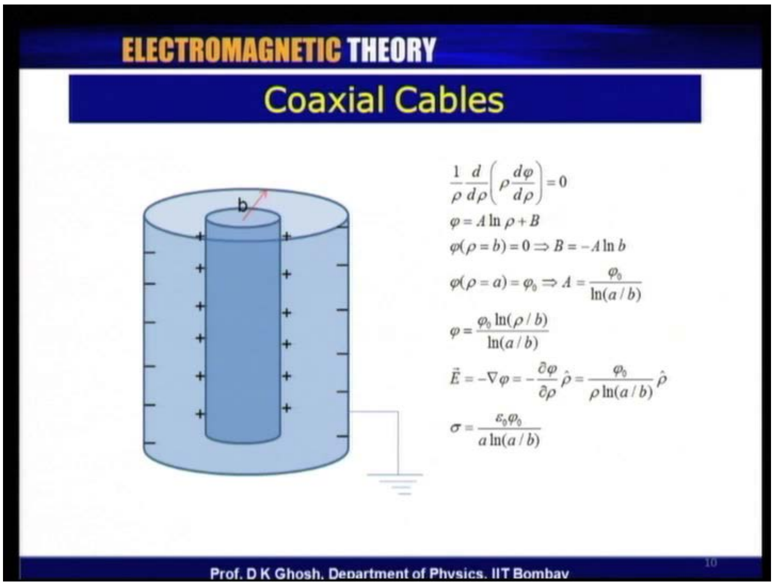

Now, let me take an example of a cylindrical symmetry. Now, in a cylindrical symmetry I take a coaxial cable. I have shown on the screen. So, I take a coaxial cable of an inner radius a, and an outer radius b. And once again if you recall the expression that I had for the cylindrical geometry, this was given by here. Just recall back. So, this was 1 over rho d by d rho d phi by d rho, etcetera, etcetera. Now if my coaxial cable is infinite, then I have what is known as the cylindrical symmetry of the problem. And the only dependence of various physical quantities can then be on rho, which is the distance from the axis of the cylinders to whichever point we are talking about, along the X Y plane or in the X Y plane.

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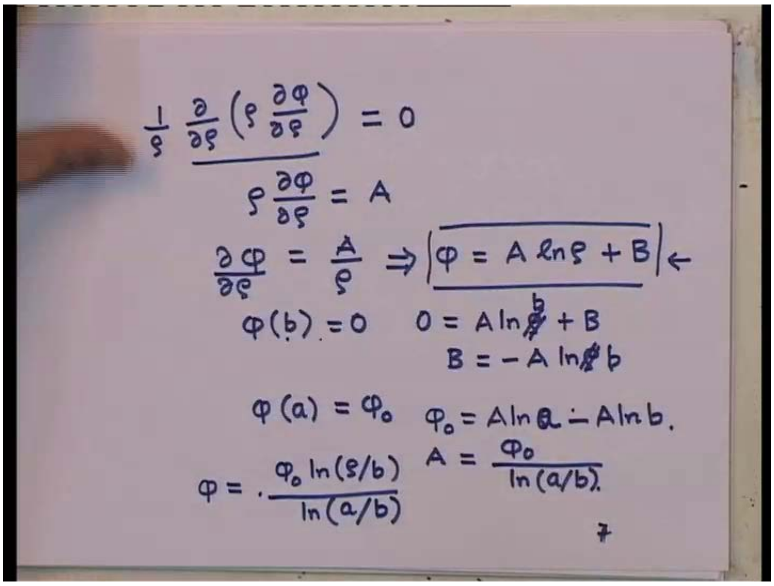

So, the equation that I need to solve, then would be one over rho d by d rho of rho d phi by d rho. So, this is my del square and this is equal to, well, let us say I take the Laplace’s equation in this case; suppose, it is equal to 0. I am solving Laplace’s equation because my inner conductor and the outer conductors are of course charged, but I am looking at the solution in the space in between where there are of course, no charges. So, density rho is equal to 0. So, this tells me d by d rho of this quantity. This is equal to 0, which tells me again that rho d phi by d rho is a constant. And let us write that constant as equal to some A. So, this gives me phi is equal to d phi by d rho is A by rho, which have the solution that phi is equal to A log rho plus another constant of integration which is B. Now, once again what I do is I earth the out... ground the outer conductor which has a radius b.

So, therefore we say that phi at b is 0. Now, phi at b is 0; that tells me that 0 is equal to A log rho plus B. So, which is B is equal to minus A log rho. So, I need to of course obtain the constant A now. And that is done by… phi in the inner conductor has a value phi 0. So this, if I plug it into this equation, I get phi 0 is equal to A log rho plus B. So, phi 0 is A log rho plus B which is, well, we have a… what I should have done is, since b is equal to 0 I should have written it is A log b. So, it is A log b and here my phi 0 is A log a plus B, which is minus A log b.

So, that tells me that A is equal to phi 0 divided by log a minus log b, which is log of a by b. So, that tells me that phi is given by… well, if you combine this you got A log rho minus A log b; which is nothing but a log rho by b. And of course the constant a is phi 0 log a by b. So, what we get is phi is given by phi 0 log of rho by b divided by log of a by b. So, this is the potential. Now, once again I do exactly the same thing as I did earlier. Namely, I calculate the electric field which is nothing but the negative derivative of the potential.

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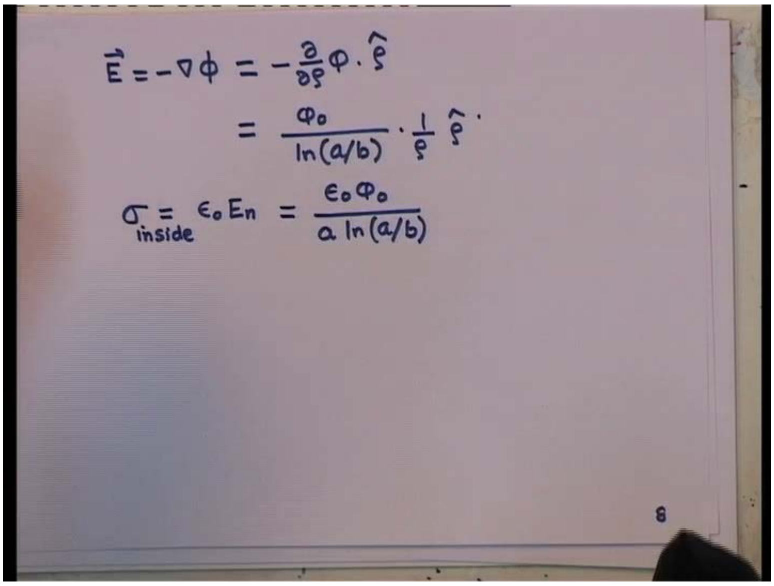

So, the electric field E is minus gradient phi. But you recall that there is really no dependence in the theta and the z. So, this is nothing but minus d by d rho of phi times the unit vector. And this is of course fairly straight forward to differentiate.

And so, you get phi 0 divided by the constant log a by b. And the log rho is to be differentiated. So, which is one over rho, and of course along the rho direction, well, the normal component of that now gives me the charge density, which is epsilon 0 E n. Now, once again you have to realize that the normal component for the inside conductor is outward, which is along the positive rho direction and the normal component for the outside conductor is along the negative rho direction. And so therefore, this is epsilon 0 phi 0 divided… Now, I need to calculate the charge density on one of the surfaces. For example, supposing I calculate charge density on the inside surface, I get rho is equal to a.

So, it is a log a by b. Now, these are charge densities. And you can essentially now calculate the charge per unit length. Remember that the coaxial cables are of infinite length. So, I need to calculate charge per unit length, which is simply obtained by multiplying the rho with the area and which is take a unit length and of course, whatever is the area of that. And that is the reason why the charge density on the inner and the outer surfaces are not symmetric. And that is because the outer one has a radius b. And so therefore, if you take the same unit length it has a much bigger area and the inner one has a radius a, which is smaller.

So, you could multiply that. And then of course you know the two conductors are being maintained at a potential difference phi 0. And once again you can find out the capacitance of this cylindrical conductor. I leave the algebra to, you know to be done at home.

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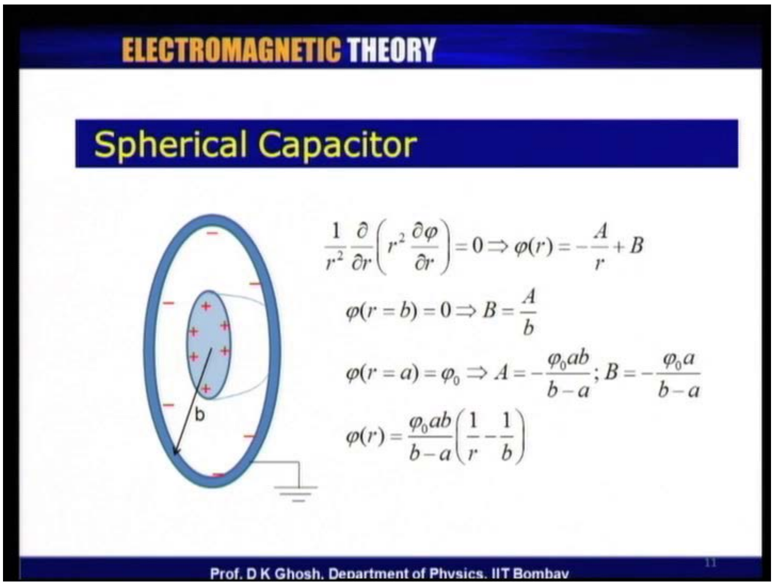

As a third example, let me take a spherically symmetric situation. They are all they all boiled down to… all these examples I am giving they all boiled down to one dimensional problem. So, once again I have a spherical capacitor as soon here, I have shown a cross section here, actually. The outer one has a radius b; the inner one has a radius a. And once again I have complete spherical symmetry. Now, spherical symmetry implies that there is no dependence on theta and phi.

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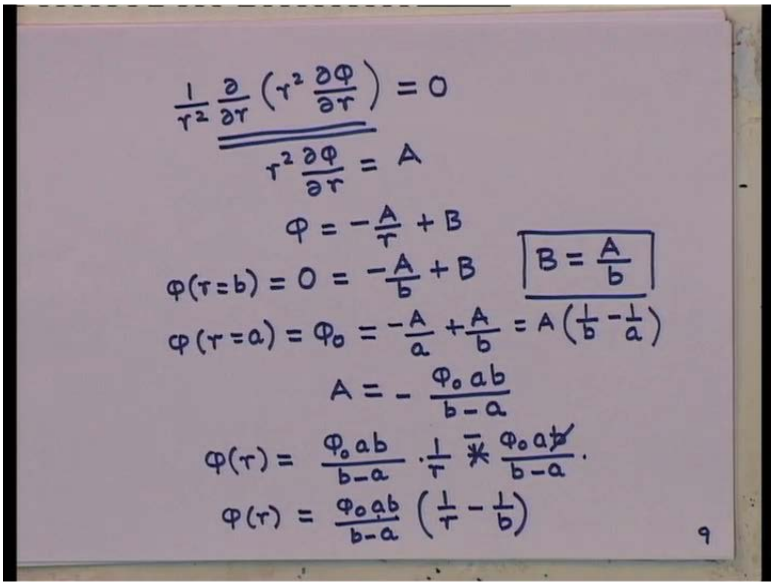

So, I get my del square operator as we had given an expression earlier; as one over r square d by d r of r square d phi by d r. This is equal to 0. Now, which immediately means that d by d r of r square d phi by d r is 0. So, r square d phi by d r is a constant; which, let us call it as A. So, d phi by d r is A by r square, which gives me that phi is minus A by r plus a constant B. Now, let us let us try to evaluate these constants A and B. On the outer conductor, the potential value is 0. So, phi at r is equal to b is 0, which is equal to minus A by small b plus capital B, which is the constant; which means B is equal to A by small b.

The second condition is, the inner conductor which is phi at r is equal to a; that is the constant phi 0. So, let us plug that in. So, this is equal to minus A by r is equal to A and plus B, which is instead of writing B, I will write A by b. So, which is equal to A times one over b minus one over a; which tells me that the constant a is phi 0 a b, well, by a minus b. But since b is greater than a, let us write it as a minus of b minus a. So, this is my constant. And as a result, my phi at any point r is given by, there is a minus sign already there, so phi 0 a b by b minus a times one over r plus the constant B. And the constant B we have already seen is a by b. So, which is phi 0 a b by b minus a, but with the minus sign here and divided by b; which means this b cancels out.

So, this tells me that phi of r phi of r is given by this expression. But actually speaking you can sort of simplify some of these expressions a little bit. And get, for example, you will get phi 0 a b by b minus a. You have one over r there. And since there is a there and there is a a b there, so I get minus one over b there. So, this is the expression for the potential, in case of a spherical conductor. Let us complete this job.

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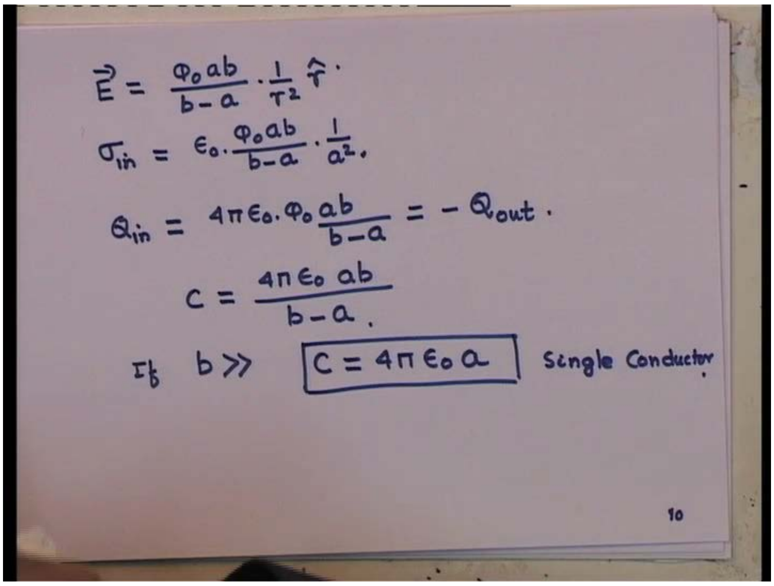

So, the electric field which is minus gradient of the potential is phi 0 a b by b minus a. Derivative of one over r is minus one over r square, but there is a minus sign there; so, this into one over r square into unit vector r. Once again because of the way we have chosen our axis, the normal direction is the radial direction. The inner capacitor, the inner conductor has outward normal along the radial direction and the outer conductor has in the opposite direction. So, the charge density of the inner conductor is epsilon 0 phi 0 a b by b minus a into one over a square.

How much is the charge contained there? So, Q inside which will actually turn out to be opposite of the charge that is contained outside is simply obtained by multiplying the charge density with the area 4 pi a square. So, it is 4 pi; a square will cancel with this a. I have an epsilon 0 phi 0 a b by b minus a. Well, you can show that this is also negative of Q outside. So, the capacitance of a spherical conductor, which is simply obtained by dividing the charge in either conductor, magnitude of the charge in either conductor by the potential difference between them which is phi 0 is nothing but 4 pi epsilon 0 a b divided by b minus a.

Now, suppose I have just a single conductor. Now, this I can do by taking the outside conductor to infinite distances; that is, if b is very large, then you notice I can neglect this a in the denominator and then this b will cancel out with that b and I will be left with the capacitance equal to 4 pi epsilon 0 times a. This is the capacitance of a single conductor. Actually speaking, this should not come as a surprise to us because if I have a single conductor, I know how much is the potential due to that, and the capacitance can be very trivially obtained from the expression for the potential of a single conductor.

So, let us summarize what we did today. We spent some time today in talking about the formal methods of Poisson's and Laplace's equation. We proved that given for given boundary conditions, either of Dirichlet type or the Neumann type; what the Dirichlet means, give the potential values on the conductor and Neumann means give the normal component of the electric fields on the conductor. Give for a given set of boundary conditions, this solution is unique. And so what we did is to take simple exercises of problems which we had done in the past of a parallel plate capacitor, a cylindrical coaxial cable and spherical capacitors and solved them by this high power methods; which are of course not really necessary. But this sort of gives us a confidence by looking at the fact that the results are familiar to us. This gives us a confidence that this system works. In the next lecture, we will be talking about the formal solutions of Laplace's equations.