Electromagnetic Theory
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Module - 2
Electrostatics
Lecture - 12
Coefficients of Potentials and Capacitance

Till now, we have discussed some qualitative properties of conductors in an equilibrium situation. Now, what we have said is that in order that the system may be in a an electrostatic equilibrium. It is required that the there cannot be any chargers inside a conductor and correspondingly the electric field would inside a conductor would also be 0. There was also a corollary that the chargers if at all they are there, they must be on the surface and further the normal component of the electric field only can be non-zero on the surface. The tangential component must be equal to 0, because the under electrostatic condition there would be then motion of the charges. What do you want to do now is to look at this problem somewhat quantitatively, let us see how it goes.

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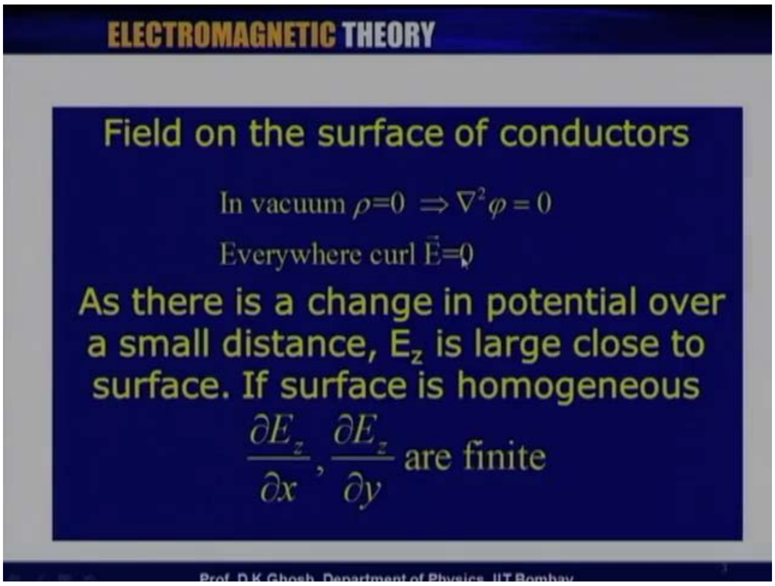

So, in this situation let us look at the electrostatics of a conductor. First thing is that suppose now you look at the space around a conductor then of course I know that the rho is equal to 0. And so therefore the Laplacian of the potential function would be equal to 0 in vacuum. Further I know under electrostatic conditions the curl of the electric field must be equal to 0. So, let us look at what does it actually mean. Now, what we are saying is there is a change in the potential over a short distance. This would mean, let us say that the surface of the conductor is in the x y plane, then the normal component namely the E z - the z component of the electric field must be large close to a surface. Further if I assume that the surface is homogeneous.

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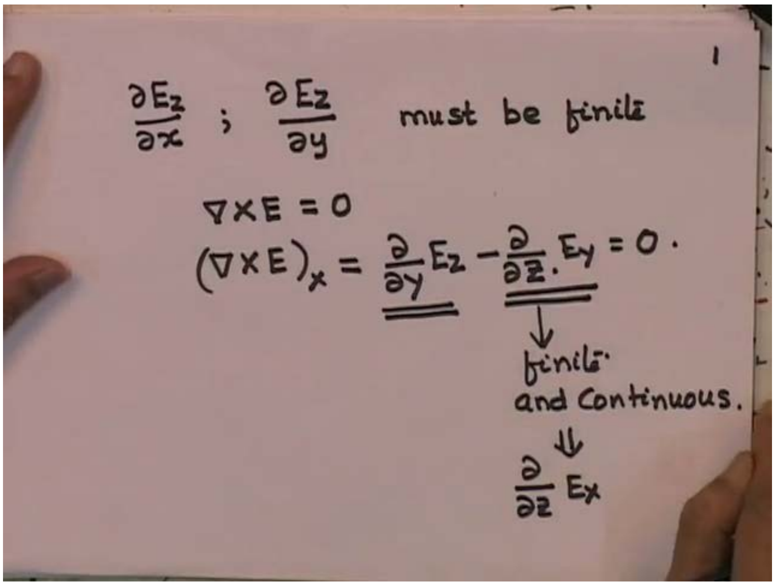

Then we can write that partial E z by e del x or partial E z by d y x and y being on the surface, they must be finite now let us look at what does it mean with respect to the curl equation. So, for instance I would know that the curl of the electric field is equal to 0. Now, if the curl of the electric field is equal to 0 let us take some component for example, let us let us take the x component of this equation. So, del cross E is x component. So, this is for instance d by d y of E z minus d by d z of E y must be equal to 0. So, notice what we have said is that d by d y of E z is finite and continuous.

Now so that tells me that d by d z of E y must also be finite and continuous on the surface. And by symmetry this will also, this is also finite and continuous. But notice the tangential component of the electric field E y or a similar equation is valid for d by d z of E x. So, that is also finite and continuous, but I know just inside the surface just inside the surface the electric field is equal to 0. So, as a result what we are trying to say is this that the E y the since the tangential component of electric field is 0 just inside the surface and we have said this is finite and continuous. This tells me that the tangential component of the electric field must be 0 on the surface, so this is of course we had obtained this by sort of purely qualitative argument that under electrostatic equilibrium this is what we expect.

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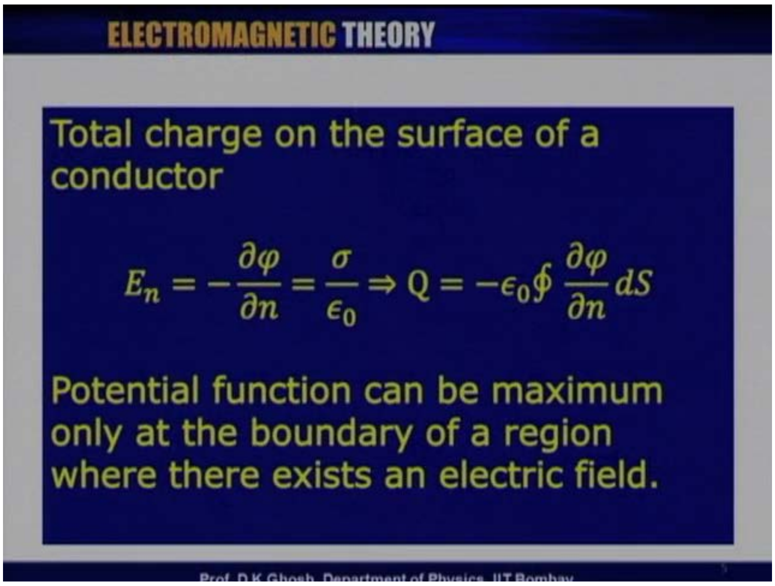

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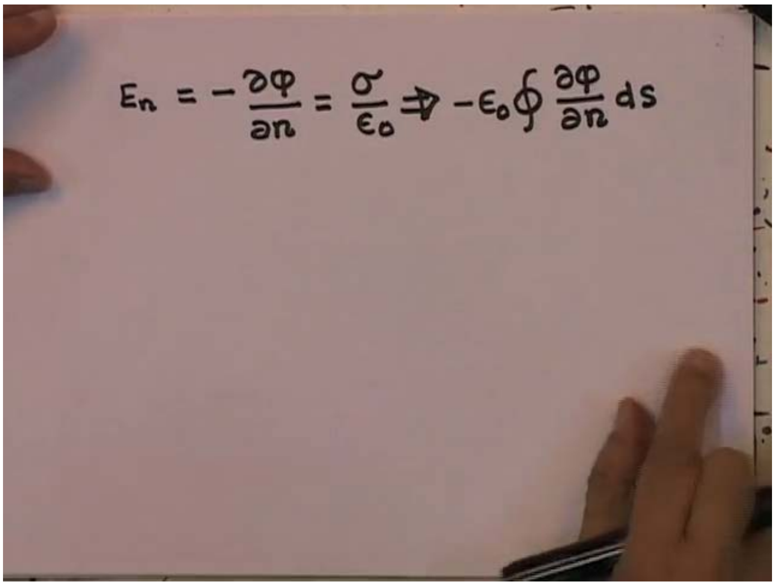

Now, let us look at the total charge on the surface. Now, notice that the normal component of the electric field which is given by if i is the potential, then I can write this as minus d phi by d n and that is equal to sigma by epsilon 0. So, this if you integrate this this tells me because there are charges only on the surface. This tells me that minus epsilon 0 the surface integral of the normal derivative of the potential, this quantity. That is the total charge that is there on the surface. Now, I want to prove a theorem which says that the potential function can be maximum only at the boundary of a region, where there exists an electric field. Now, let us see why.

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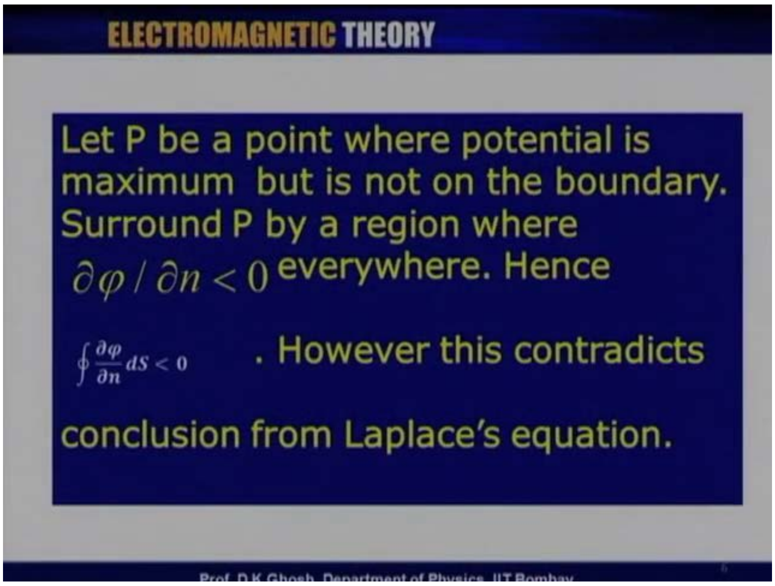

Now, let us suppose this is naught 2, supposing P is a point where the potential is maximum, but contrary our statement that a function, the potential function can be maximum only on the boundary.

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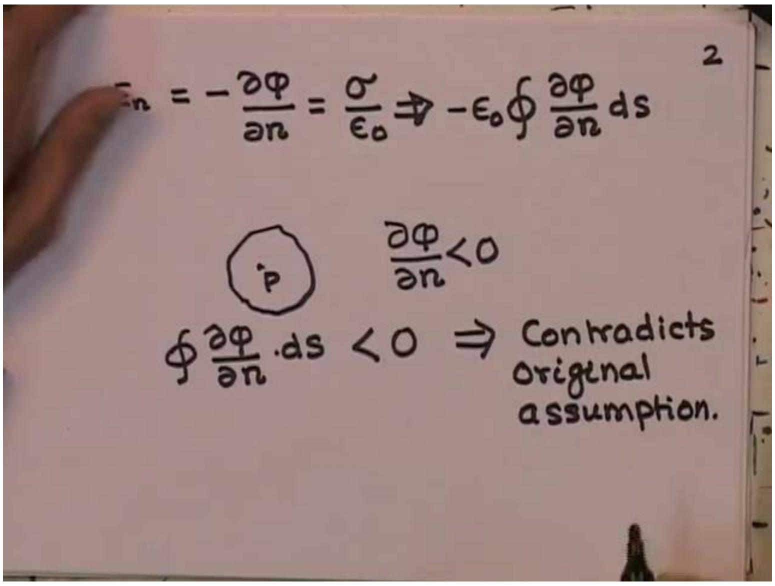

Let us suppose this point P is not on the boundary, but let us so just inside. Now, what we can do and then do since P is a point just inside I can always surround this point P by a small region, where since the function is maximum at the point P then at every point on this surface I must have d phi by d n less than 0 that is because a point P is a maximum Now, since d phi by d n is less than 0 everywhere, this implies that I must have d phi by d n, if I integrate over the surface this must also be less than 0. But remember that this is contrary to what we obtained from Laplace's equation because this quantity is nothing but the total amount of charge that is enclosed. But I know that since this region is within the surface, with in the material of the conductor, then my total charge must be equal to 0. So, as a result this contradicts my original assumption and as a result we state that the potential can be maximum only in a region which is on the boundary.

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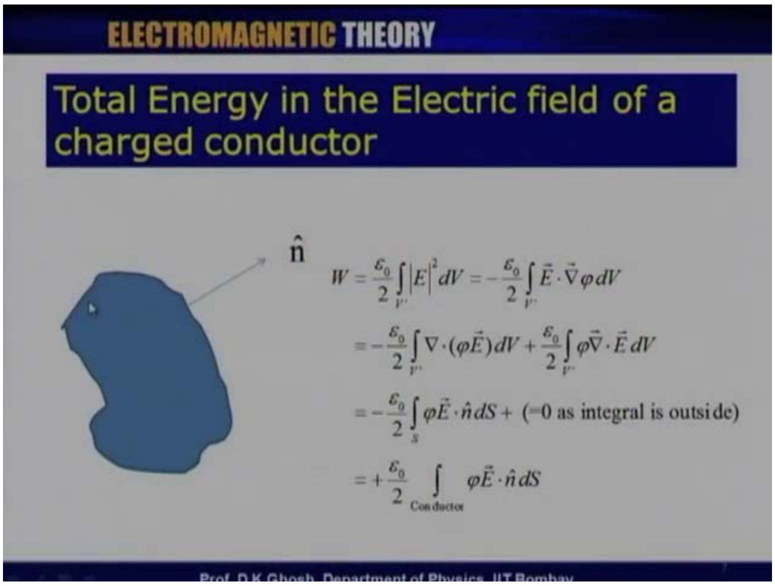

So, this is what it is. Let let us now look at the total energy of a conductor or a system of conductors in an electric field. So, let me this is a picture. The outward normal as we have seen is given by n and what I want to do is this. That let me write down an expression for the total energy.

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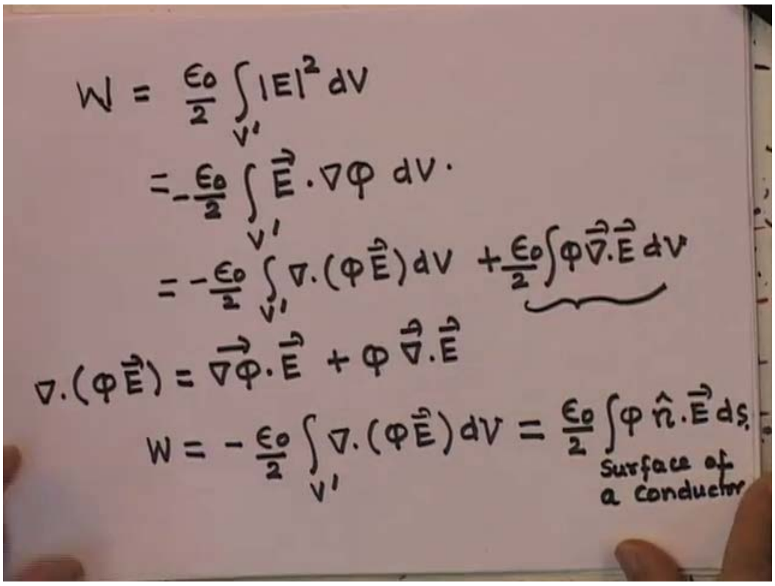

Now, this total energy is as you remember is epsilon 0 by 2 integral over the volume. Now, I will tell you why I have put in a prime there, so E square dv. Now, what I am calculating here is the amount of energy that is stored, in the electric field that is created by the system of conductors. In other words the volume of the conductors itself I am subtracting it and and this region all space minus the region volume of the conductor is what I have indicated by v prime. Now, since E square is E dot E. What I will do is I will re write this expression in a slightly different way. I will write this E dot and one of the E's I will write as the minus gradient of v, minus gradient because so as not to confuse with this volume v I have called it minus gradient of phi the potential.

So, this integrated over volume over this v prime. Now, what I will now do is, I will use what is del dot. Let me write down. what is del dot of a scalar times a vector? So, this I know is given by grad phi dot E plus phi del dot E. So what I do is that, this grad phi dot E which is the same as the E dot grad phi, I write it as del dot phi E minus phi del dot E. So, there is a minus sigh there. So, I will write it as, so minus epsilon 0 by 2 integral v prime del dot of phi E d v plus epsilon 0 by 2 integral phi times del dot of E d v. Now, notice that this integral, the second integral epsilon 0 by E phi del dot of E must be equal to 0 because I am looking at region outside the conductor and there are no charges there so del dot of E must be 0. So, as a result my total energy W becomes minus epsilon 0 by 2 v prime del dot phi E d v.

Now, what I now do is this, that I convert this integral into a surface integral. That is because this is a divergence thing here. Therefore, I will convert this into phi E and dot d s. So, this quantity then I will get as equal to epsilon 0 by 2 integral phi n dot E d s. So, this is E dot del phi and E dot del phi and that is equal to this quantity minus this quantity. So, this is what I get as the total energy of the electrostatic a system of conductors and this is a. what is the difference? The difference that I have note down that this v prime, if you recall the volume v prime was the volume of the space around the conductor.

Now, when I converted this into a surface integral, what I need is the surface integral will be along the direction which is outward normal to such a region. So, if you look at this picture again, you notice that this is the direction of n on the surface of the conductor, but if you look at the region surrounding this area, then direction of the normal to that area would have been directed into the conductor. And so since this when it is converted into a surface integral the direction of the normal is inside the conductor and this n is directed outside the conductor. So, hence this minus sign is adjusted and I will write this as over the surface of a conductor.

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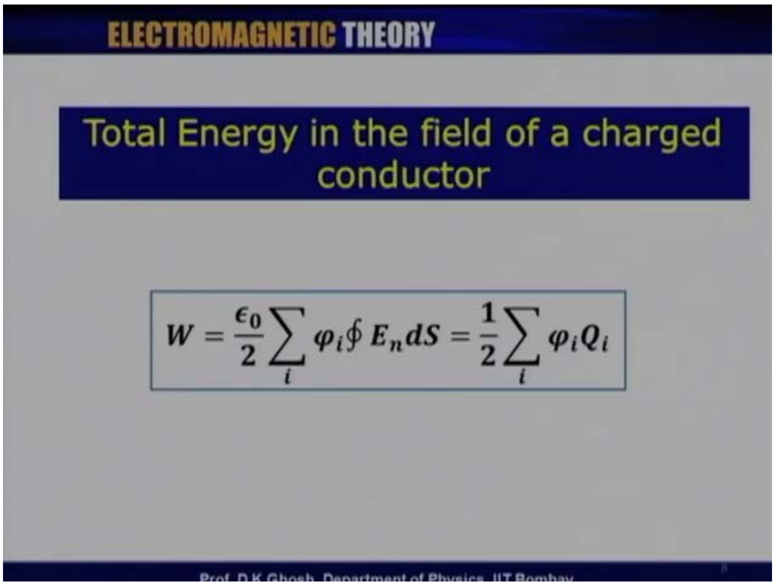

So, this is the expression for the total energy. So, supposing i have a set of conductors then I know that the potential on the surface of the conductor is fixed. So, if phi i is the potential on the surface of the i th conductor then my total work done is obtained by simply summing over and this is the result and since this quantity epsilon 0 E and d s is nothing but the charge on the i th conductor. So, the an equivalent expression for the total energy is also half sum over i phi i Q i.

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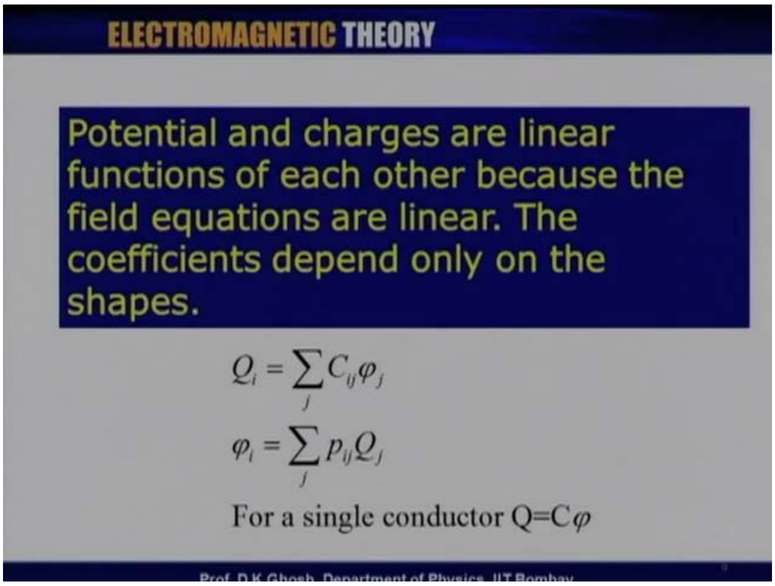

Now, let us proceed and try to look at this in a slightly different way. I know that the electrostatic equations that we have talked about. The Maxwell's equations they are all linier questions. Now, as a result the charge and the potentials, they are expected to be linear functions of each other. Now so these coefficients the coefficients of linearity the constants they would then depend only on the geometry of the conductors on the shapes.

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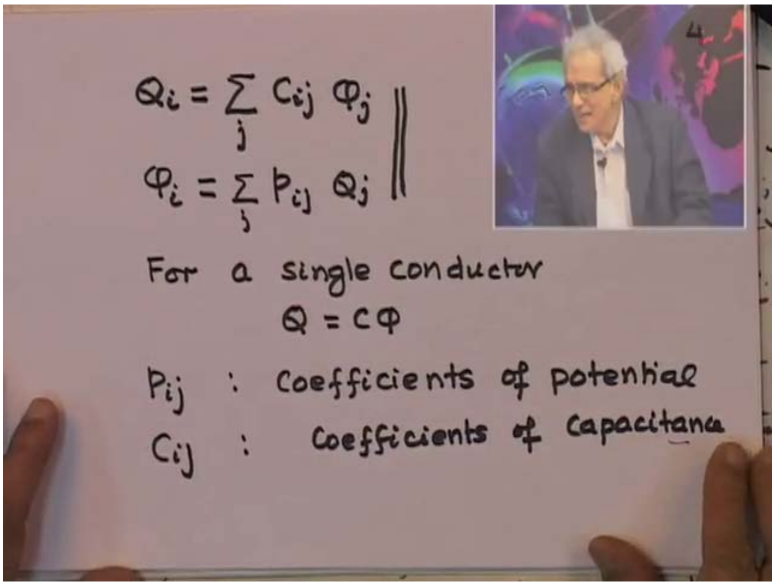

So, let me write down the charge on the i th conductor in terms of the potentials on the various conductors that is there I have talking about a system of conductors. So, I can write in general this as sum over j C i j some coefficients which depends upon i and j. j is the summation index i is that particular conductor I am talking about. Times Q phi j and similarly, I can write phi of i the potential on the i th one as sum over j, some coefficients again p i j Q j. Now, if i had a single conductor then of course, is a very familiar expression for a single conductor.

I expect Q to be equal to C phi and this is very well known to you as the capacitance of the conductor, but for a system of conductors this is the set of equations. This coefficients phi p i j they are known as coefficients of potentials. So, p i j's are coefficients of potential and C i j are called coefficients of capacitance. So, let us let us express my total energy expression, in terms of these coefficients that we have talked about. So, for instance let me take an expression which I wrote down.

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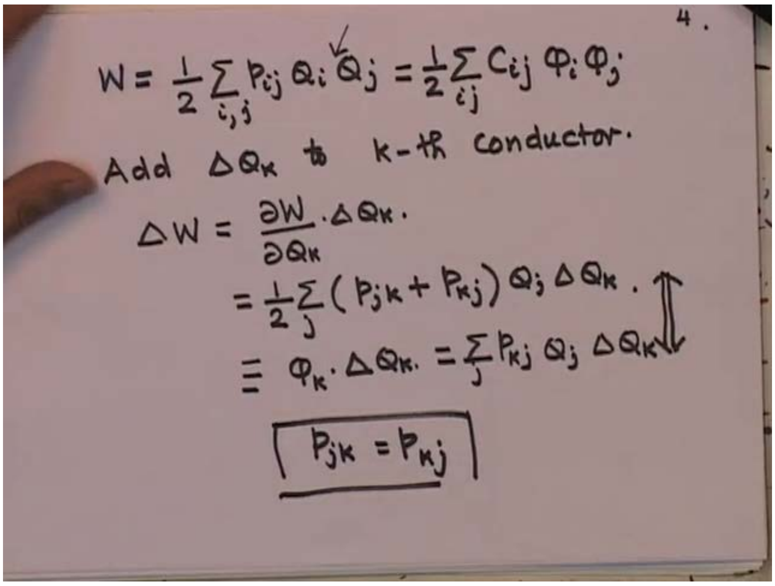

So, some time back so the total energy is half if you remember this is phi i Q i or phi j Q j if you like. So, expressing the potential in terms of the charges I have sum over i j then p i j Q i Q j. So, this is this sum over j p i j Q i is my phi j and the sum over j phi j Q j with the factor of half is my total energy, which is also can be written as equal to the other around when you express the charge in terms of the potentials. You can also express this as sum over i j c i j the coefficients of the capacitance and phi i phi j. Either of these expressions is good enough and now I am interested in finding out something about the properties of this potentials. Suppose, we add a small charge delta Q to let us say Q th conductor so we add delta Q k to k th conductor.

Now, that will be mean and change in the energy delta W which can be written as d W by d Q k delta Q k. Now, take this expression for the total energy, then you can see that I need to differentiate this expression with respect to Q k. So, there are two terms there I do a chain rule differentiation. So, delta Q i by delta Q k gives me a delta function delta i k and delta Q j by delta Q k will give me another delta function delta j k and by using the delta chronicle delta to remove one of those summations, I get this sum over j, p j k plus p k j Q j delta q k.

And but this must be identical to because I have added an amount of charge delta Q k to the k th conductor. If the potential of the k th conductor is phi k, then the change in the energy must be phi k delta Q k, and so this quantity by writing this phi k in terms of the corresponding potentials I can rewrite it as sum over j p k j Q j delta Q k. Now, what you can do is you can compare these two expressions there. If you compare these two expressions, you immediately realise that p k j must be equal to p j k that is the coefficients of potentials are symmetric.

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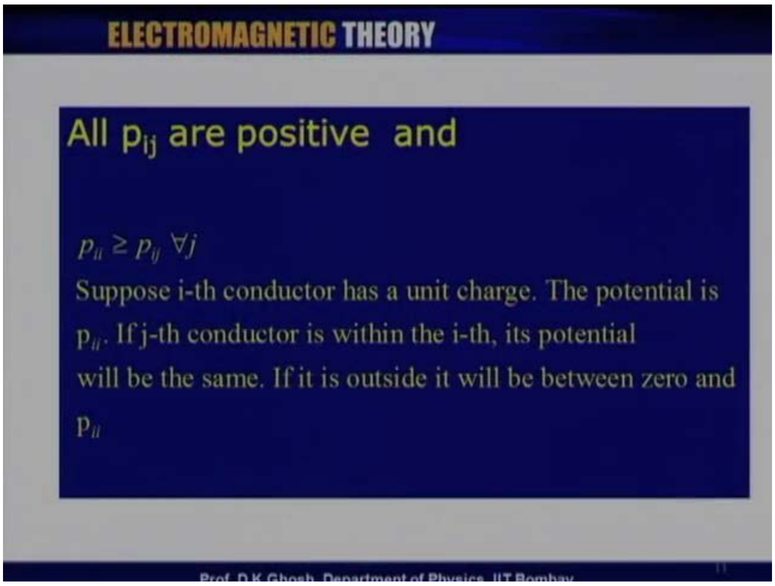

So, this is one of the properties of the coefficients of the potentials. Now, the second thing that I need to show is that these potentials these potentials are positive and while the p i i is greater than all p i j for i not equal to j. So, in order to do that look at the following. Let us define this 0 of the potential to be infinite at infinite distances and let us suppose that I have a unit charge or let us say the i th conductor. This is the i th conductor and i have a plus 1 unit of charge.

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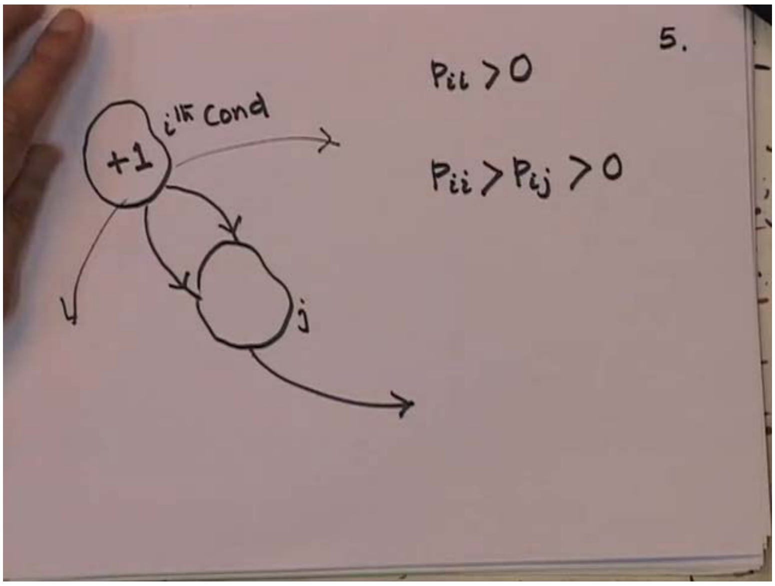

This is the only charge that is there. Now, since the reference of the potential is at infinite. So, this tells me that the potential of this is necessarily positive. In other words p i i is greater than 0. So, that is what I start with. Now, let me now think about a conductor j which is hear but, it has no charge in that case the lines of forces which emanate from the i th conductor can go to infinity in one of the two ways. It can directly go to infinite distances. Alternatively what it can do is these lines can terminate on the j th conductor and from the j th conductor they can go to infinite distance. So, this tells me that the potential of the i th conductor must be greater than the potential of the j th. So, in other words my p i i must be greater than p i j which my definition is greater than 0. So, this is a one of the properties of the coefficients of potential.

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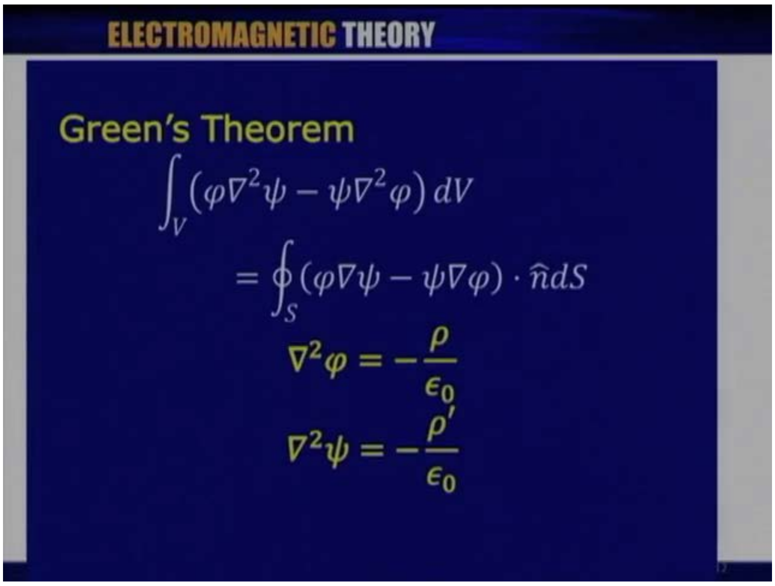

Before we proceed further, let me recall for you when we were doing vector calculus, we had discussed something called a Green's theorem. Just to a recall for you what was Green's theorem, we said supposing have a region v and in that I have given two potentials, scalar potentials - phi and psi.

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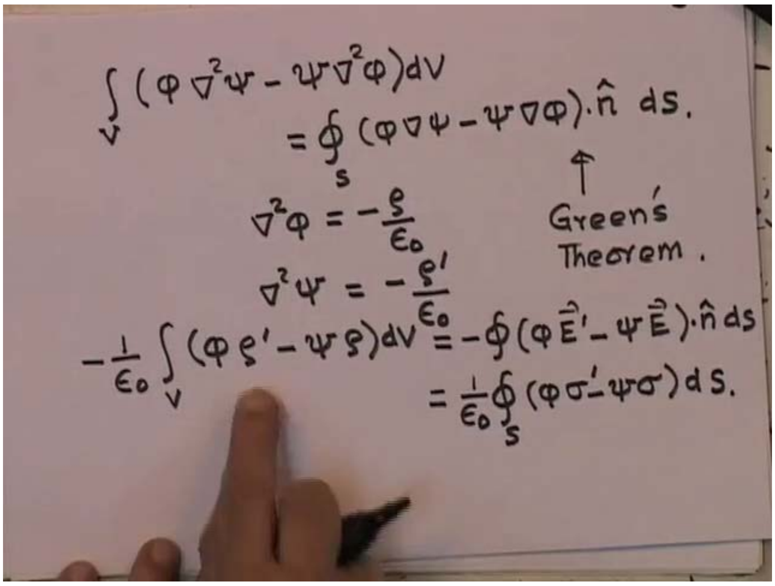

Then what we proved there by Green's theorem was phi del square psi minus psi del square phi, volume integral. This is equal to the surface integral of phi grad psi minus psi grad phi dot n d s. Of course, since we said phi and psi are potentials defined within volume. So, they will tell me that del square phi is minus rho over epsilon 0 that is the potential phi corresponds to a charge distribution rho and potential psi corresponds to a charge distribution rho prime. This is Green's theorem and these are the two Laplace's equation with which we are very familiar.

Now, let me let me now look at in the following way. That using these expressions del square phi is minus rho etc I can rewrite these expressions there as minus 1 over epsilon 0 integral over the volume of phi del square psi is rho prime by epsilon 0 minus psi rho d V and that is equal to once again I have got del psi and del phi. Let me write it in terms of the corresponding electric field. So, these will be minus integral of phi e prime minus psi e dot n d s and since the potential on the surfaces for the conductors are constants, so I can take them out of the integral if you like. So, I have got 1 over epsilon 0 integral over the surface phi sigma prime minus psi sigma ds. So, if you compare these two expressions, what you find is what is known as Green's reciprocity' theorem and that is written as integral phi rho prime d V plus surface integral, phi sigma prime d s that is equal to psi rho d v plus psi sigma d s.

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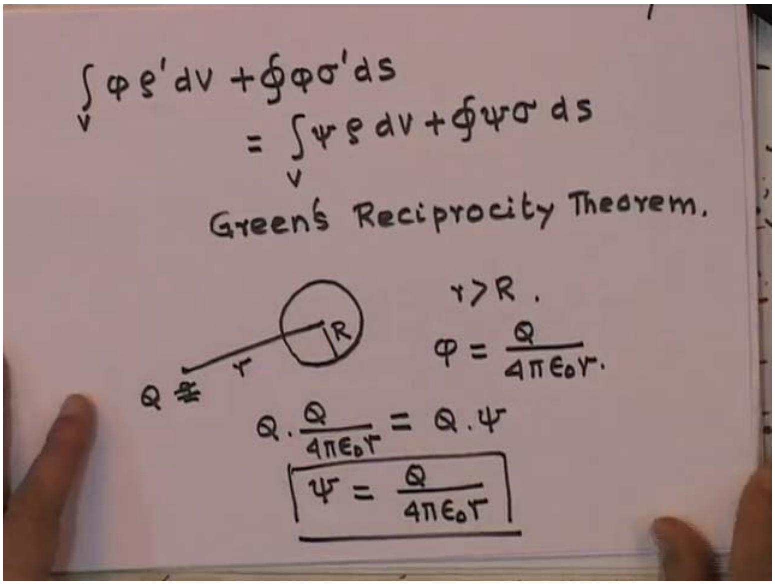

So, this is known as Green's reciprocity theorem. There are many interesting consequences of such a theorem. So, let me let me give you some simple examples. Let us talk about a point charge at a distance r from the surface of the sphere, where the radius of the sphere is r so r greater than R. Now, so the point charges here is q. Now, I am interested in finding out what is the potential charge that is generated on the surface of the sphere because of this charge q is being located at r. Now so what I do is I look at this is a system of two conductors the point charge is being considered as an infinitesimally small conductor with volume charge density equal to 0 and this is just a charge q there.

Now, what I can do is this that I can use reciprocity theorem now, to instead of putting the charge q here and finding the potential there I can instead put the charge on the surface because I want to use this relations. I can put the charge on the surface of this theorem and then of course, I know that the potential that it generates at a distance r is simply given by phi equal to q divided four phi epsilon zero r. So, according to the Green's reciprocity theorem here I can write then q times q by 4 pi epsilon 0 r. This is the same as if I put the charge q there.

Well I have written it a small q but, make it a capital q. The times the potential on the surface of the sphere now that immediately tells you the potential on the surface of the sphere is given by q divided by 4 pi epsilon 0 r a result which is very familiar to you. So, it is a neat application of Green's reciprocity theorem. So, for a general discussion let us suppose my point charge is called one and my sphere is called two.

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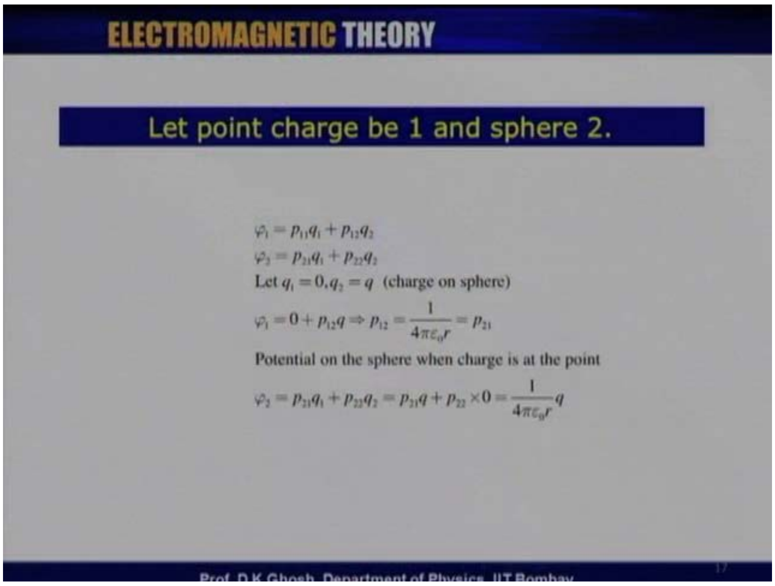

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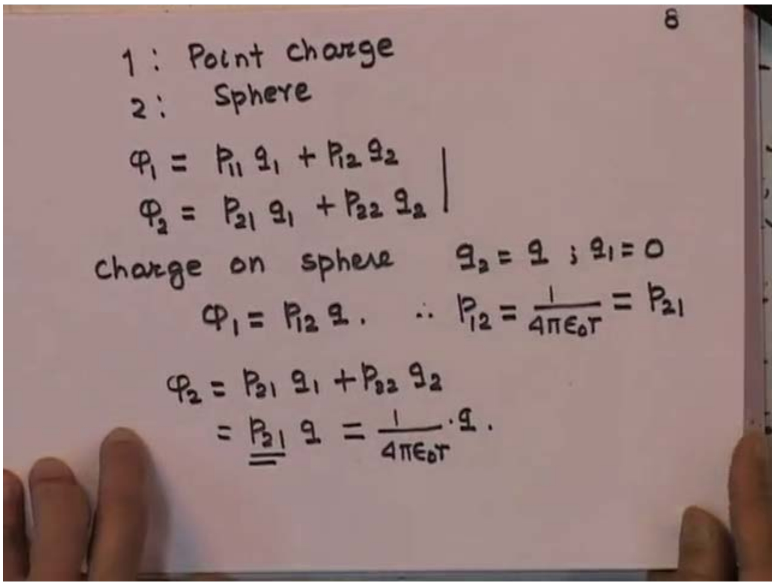

So, one is my point charge and two, I represent my sphere. Then my phi 1 that is the potential on the, at the point charge is given by by the definition of the coefficients of potential p 1 1 q 1 plus p 1 2 q 2 and likewise phi 2 that is for the sphere it is p 2 1 q 1 plus p 2 2 q 2. Now, if I now say my charge is on the sphere if charge, is on the sphere this implies that q 2 is equal to q, but q 1 is equal to 0. So, that tells me phi 1 is simply given by p 1 2 q now, that tells me that p 1 2 is phi 1 by q which is 1 over 4 phi epsilon 0 r but by symmetry of the coefficient. This is also equal to p t 1 p 2 1. So, I have determine p 1 2 and p 2 1. Now, let us look at this expression and this time I am looking at phi 2 equal to p 2 1 q 1 plus p 2 2 q 2 and let me now put the charge on the at the point and therefore, q 1 is equal to q

So, this is equal to p 2 1 into q and this is equal to 0 so therefore this plus 0 and that I have just now found out what is p 2 1, which is equal to p 1 2 therefore, it is 1 over 4 pi epsilon 0 r times q. So, that immediately tells you what is phi 2. So, these are techniques in which the coefficients of potential and capacitance are used to determine things.

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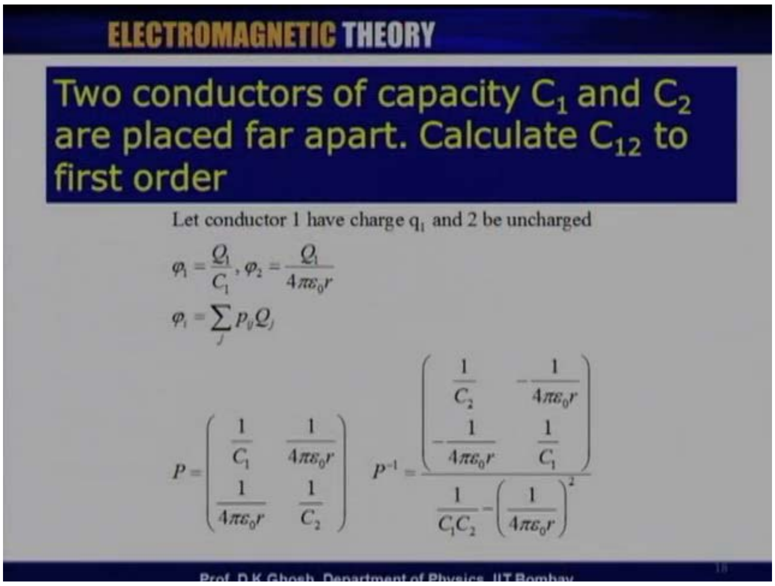

As a second example consider two conductors of capacity C 1 and C 2 and let us suppose, they are placed far apart and I need to calculate the coefficient of the capacitance C 1 2 to first order. Now, once again let us let us talk about the conductors one.

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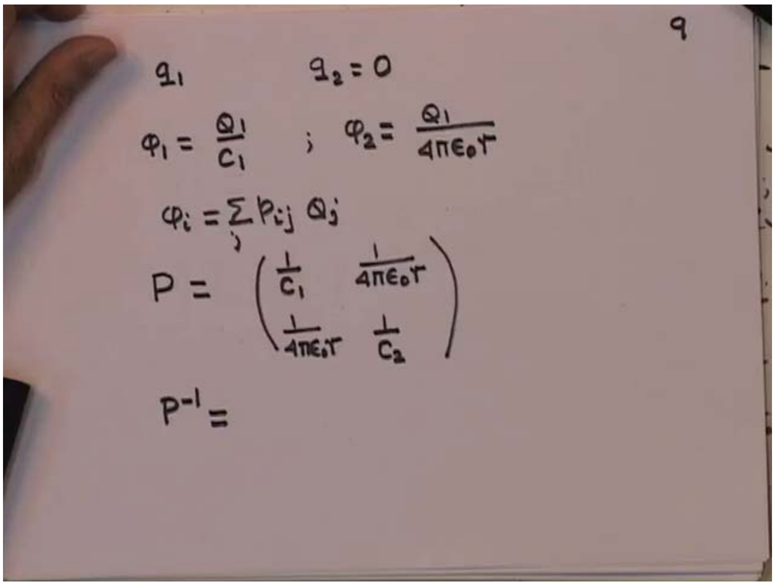

Having charge q 1 and conductor q 2 q 2 is equal to 0, it is a uncharged conductor. Then by definition of the capacitance my phi 1 is equal to q 1 by C 1 and pi 2 is the potential on the other sphere, which we just now worked out is q 1 divided by 4 phi epsilon 0 r because at a large distance it is the same as the field at a point charge. Now so my phi i which by definition is sum over j p i j Q j. I can rewrite, I can write my potential coefficients in form of a matrix. So, I have already calculated that p 1 1 is one over C 1 by symmetry it turns out that p 1 2 must be p 2 two must be 1 over C 2. This is 1 over 4 pi epsilon 0 r. This we have just now obtained and by symmetry 1 over 4 pi epsilon 0 r must be p 2 1 as well.

Now, once we have got this matrix, you can invert this matrix. This is the 2 by 2 inversion. It is easy, I will not write it down but, you can see it on the screen this is p inverse matrix and once you have got the p inverse matrix notice that I could now use this to obtain the charges from the potential. So, basically we have a method of determining the coefficients of potential and capacitance by smart application of green's reciprocity theorem.

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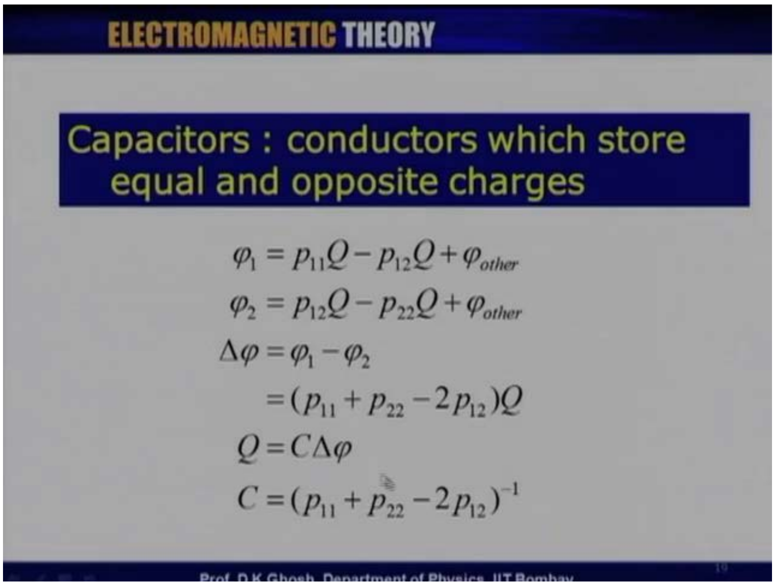

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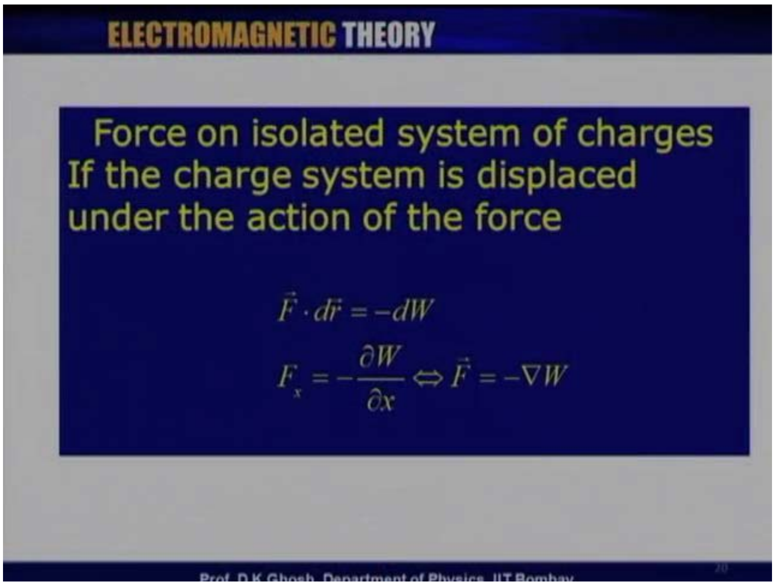

Now, let us look at a collection of isolated system of charges. I am looking at force on conductors. Now, there are two issues there. The suppose, I consider an isolated system of charges. Now, in this charge suppose there is a force F acting on it and under the action of this force supposing the system is displaced by an amount d r, in which case the work done is F dot dr. Now, if the work done is F dot d r. Now, this work must have been done at the expenses of the potential energy or the electro static energy of the system of charges.

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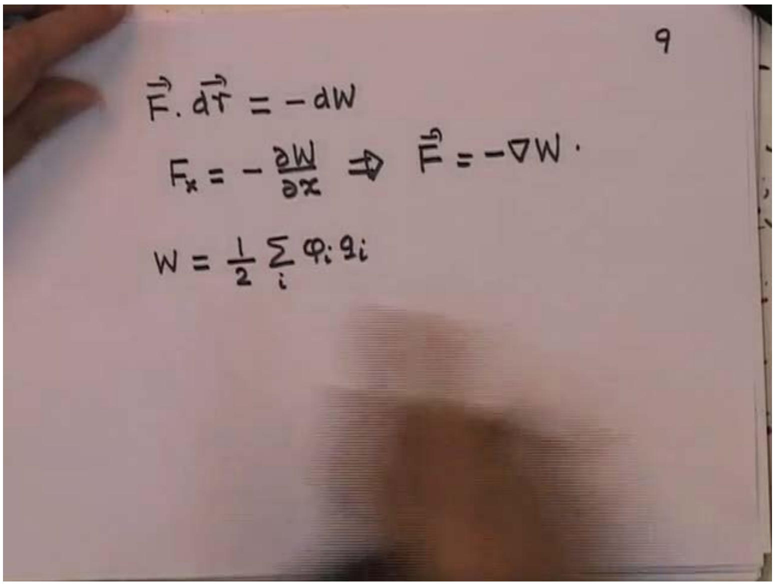

Now, since the system is isolated I have assumed that no charge can flow out of it. A similar situation could exist when for example, I have a charge capacitor from which the battery is disconnected. So, now the for instance the x component who offers them can be obtained from a consideration of the change in the energy. For instance x component is then given by minus d W over d x and equivalently this means that I can write the force F as equal to minus gradient of W. The other situation is when I have a system of conductors at a fixed potential. Now, if the potential is fixed for example, a system of conductors which could be connected to a battery. This tells me that my total energy W, if the potentials are fixed is given by half sum over i phi i q i. Now, I can write down what is q i as you can see it here.

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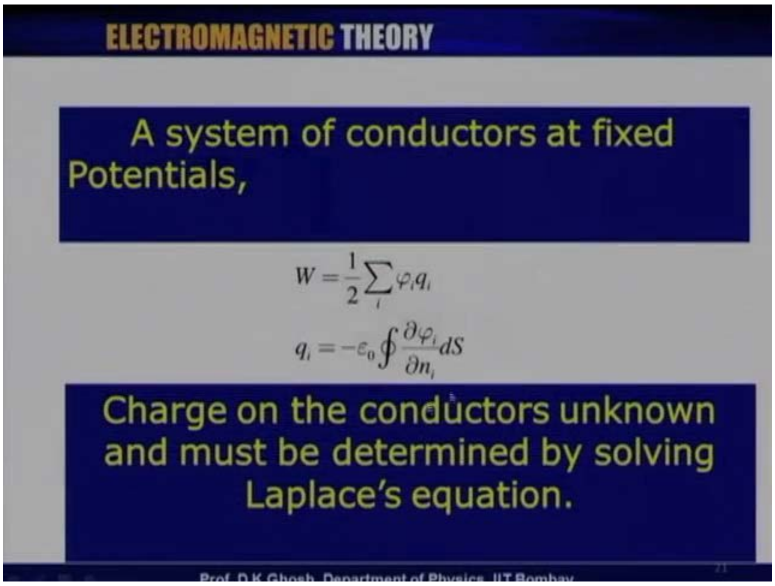

In terms of the potentials of the individuals now, notice in this case my charge on the capacitor conductors is unknown and must be determined by solving Laplace’s equation.

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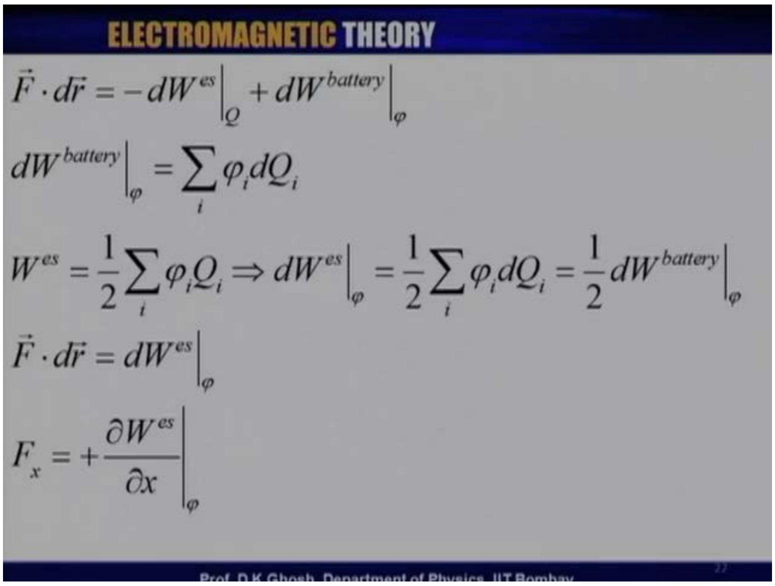

Now, let us look at a general situation. Now, if I have a system which can exchange both charge and change its potential for example, F dot dr. Now, this can happen in two ways. One of the ways is to reduce the potential electrostatic energy of the system, keeping the charge constant. The second one is it can draw charges from the battery. In other words that it can it increase its potential energy at the expense of the chemical energy, that is there in the battery. So, I know that in that case I can write the change in the energy of the battery in order to keep the potential fixed, that must then provide you certain amount of charge and that must be sum over I phi i d q i. The electrostatic energy we have seen is half sum over i phi i q i. So, if you combine these expressions so for instance this one tells me d W e s by phi is given by this and that then becomes equal to half the change in due to the energy supplied by the battery.

So, the force expression now becomes for the situation what the battery is able to supply energy is d W es by d x at a constant potential. Let me complete or or close this discussion with giving you couple of examples.

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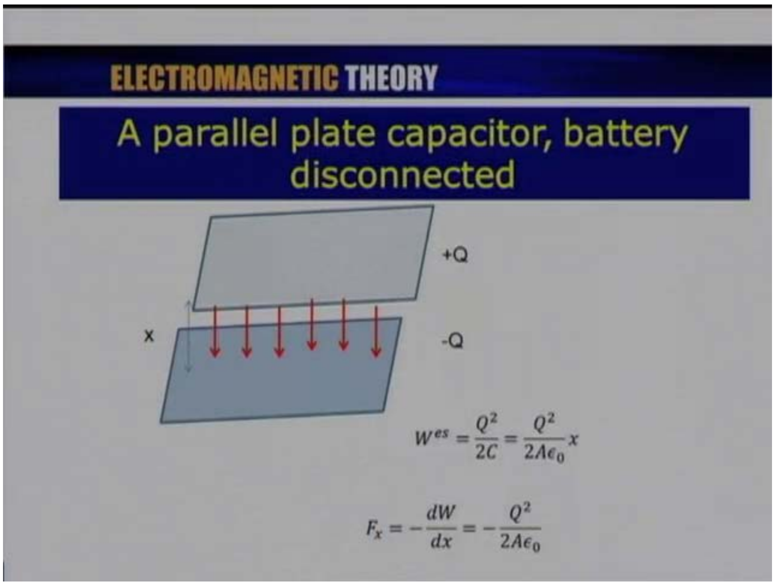

So, for example, let us consider a parallel plate capacitor from which the battery has been disconnected. So, what I will assume is that the initially the two capacitors are charged to plus Q and minus Q and let me assume that one of the plates is fixed. Let us say by bolt it is not allowed to move the other one is allowed to move.

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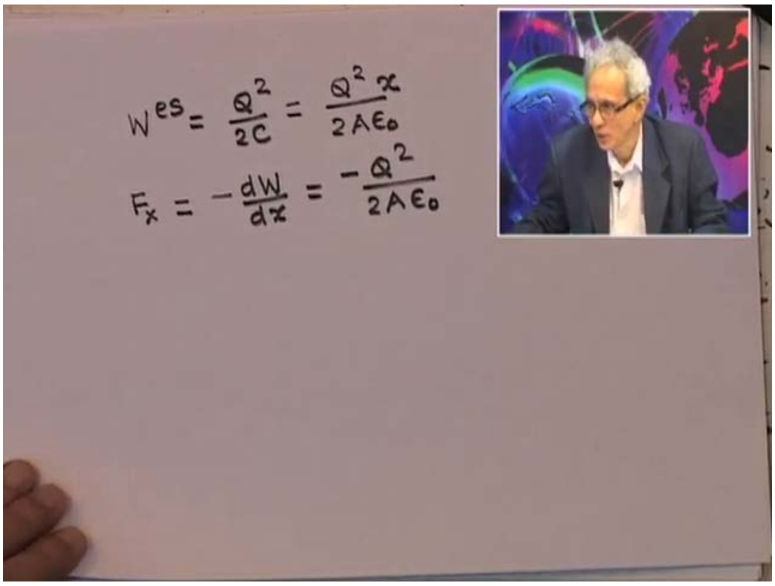

Now, I know that the electrostatic energy of the problem is W e s and that is equal to Q square by 2 C. Now, the capacitance expression is known to be so it is Q square divided by 2 a epsilon by 0 times x, where x is the instantaneous distance between the two plates.

Now, then we have seen that the x component of the force is given by minus d W by d x and that is equal to minus Q square over 2 a epsilon 0. Minus sign shows that the x would be decreasing and as a result the fixed plate is trying to attract the other plate this is of course, once again a situation that is well known to us. Let me then give you another example of a situation where I have a system of a parallel plate capacitor again and what I have done is this time this parallel plate capacitor both of them are maintained at the same potential.

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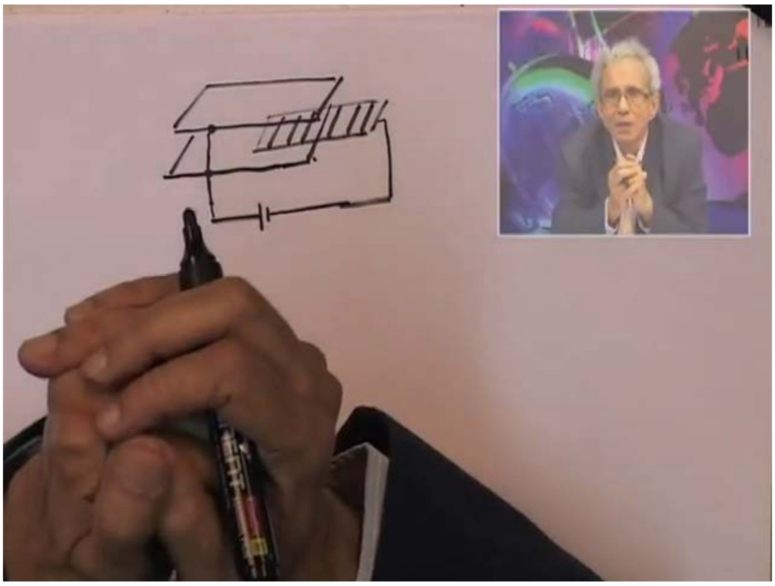

So, that is very easily done I can sort of connect both these plates and then of course, connect them to a battery and the other hand I can ground. But what I am now doing is I am connecting this 0 potential the grounding to a conducting plate and I am trying to introduce this inside the space between the two capacitor. Now, remember before I introduce the conductor the grounded conductor since, both the plates were at 0 potential were at the same potential there was no electric field between the plates but once I have decided to so let us just look at this situation. Once I have decided to put in this conducting sheet. Let us suppose it has entered by an amount x now how much is the force that is exerted notice that these two plates have been kept at both same potential and this is my grounded sheet. This is grounded.

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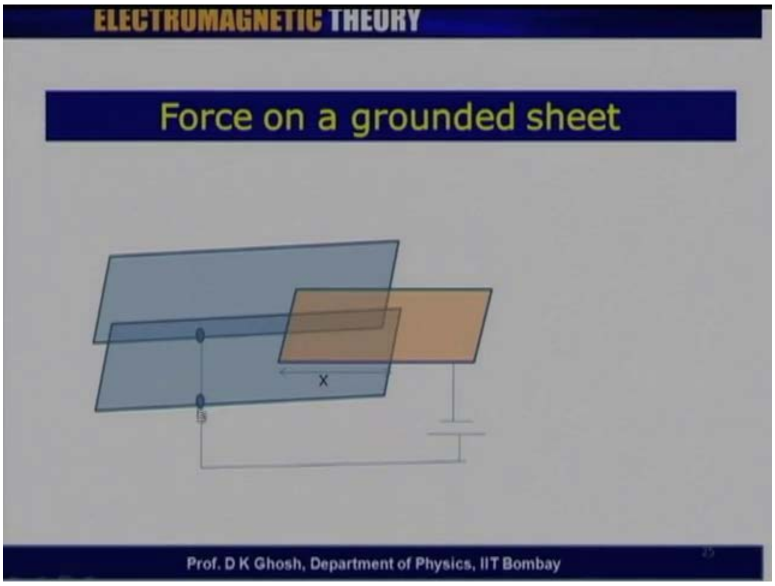

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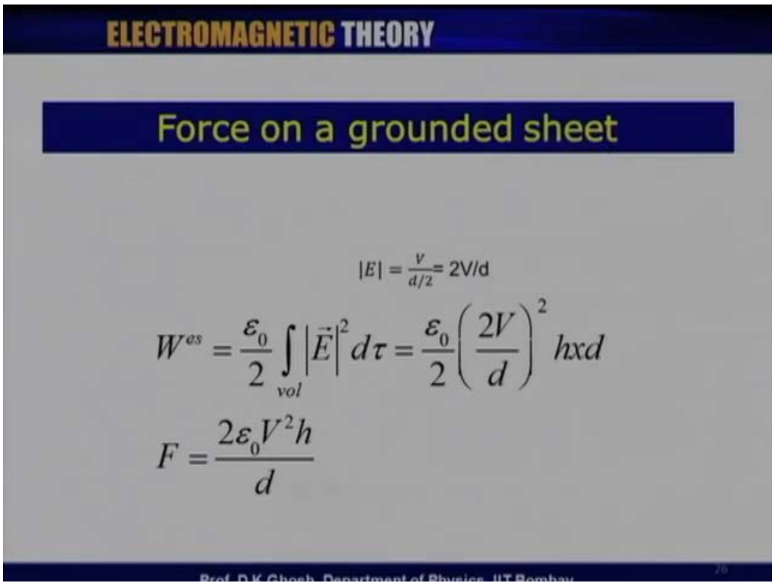

Now, if I look at that then in the region where this grounded sheet has entered and I have put it right at the centre let us say. Now, an electric field has been established and this electric field has a Magnitude equal to V divided by d by 2 because d by 2 is the distance between the grounded sheet and either of the plates which is equal to 2 V by d. Now, the electrostatic energy of this situation is epsilon 0 by 2.

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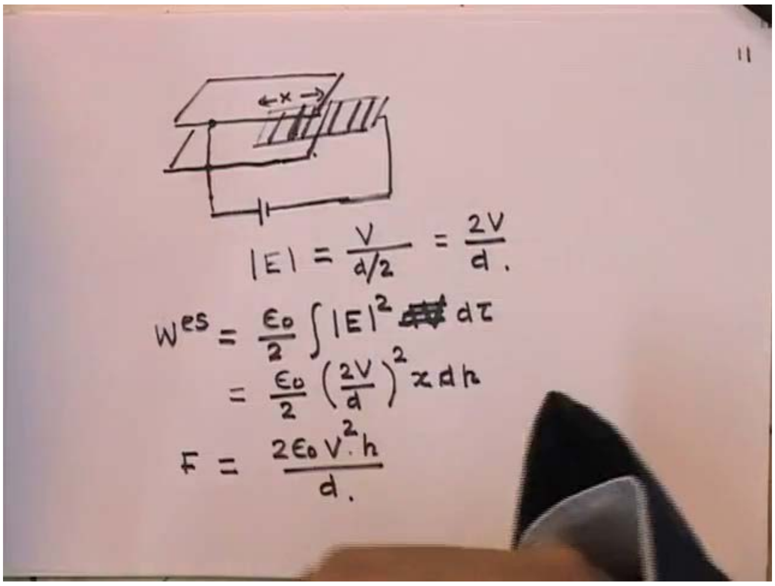

Now, I need to integrate it over all space. So, E square d V or d tau well let me not confuse I have said potential is difference is V so let me call it d tau and notice that if this distance is x then the eclectic field exists only in that region. So, therefore, this and its constant so the integration is easily done. So, the integral will be epsilon 0 by 2 2 V by d whole square times x times d. x is the this distance and d is there and I must add an h because I am also looking at the volume of the whole thing. That is this width that is there so that tells me that the force is given by 2 epsilon 0 V square, just differentiate it with respect to x times h divided by d.

The sign of the force tells me that the system is trying to pull the conductor in. So, what we have done today is to look at some electro statics situation in a quantitative fashion. We have seen that from basic electro static equations. We can define coefficients of potential and the coefficients of capacitance and in terms of which I can determine. For example, by use of Green's reciprocity theorem I can convert a problem for which the solution is not known, to a problem for which the solution is known to be. So Green's reciprocity theorem is useful in doing a mapping. The for system of arbitrary system of conductors I do not know the details but, the problem can be stated in terms of the coefficients of potential and the coefficients of capacitance and one can obtain the total energy.