Electromagnetic Theory
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Module -2
Electrostatics
Lecture - 11
Potential and Potential Energy

In the last lecture we discussed about the energy of a system of charged particles, and towards the end I made some comments on the difference between the total energy that we calculate for a continuous charge distribution, and the corresponding result that you obtained for discrete charge distribution. Today, we will continue with what is known as the self-energy problem.

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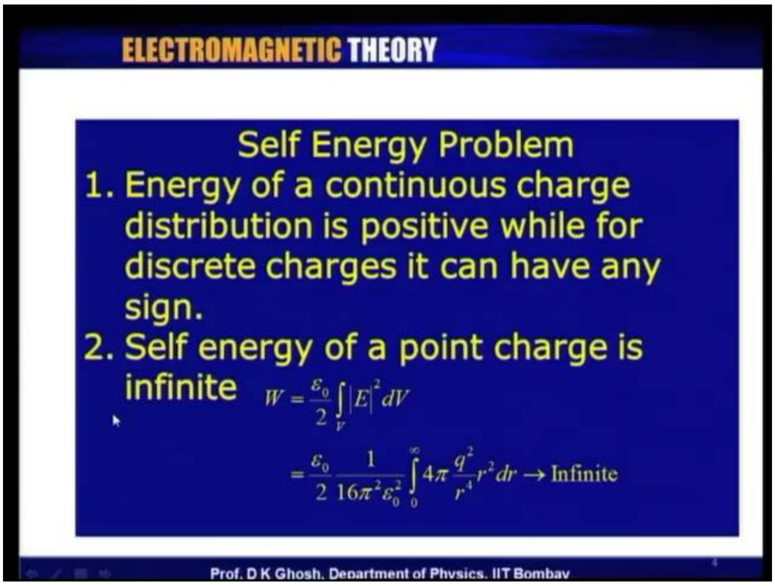

So, let us look at what the problem is in a little more detail. We have seen that the energy of a continuous charge distribution is positive, and that is because we had seen that the energy is given by epsilon 0 by 2 integral over the volume of E square d V.

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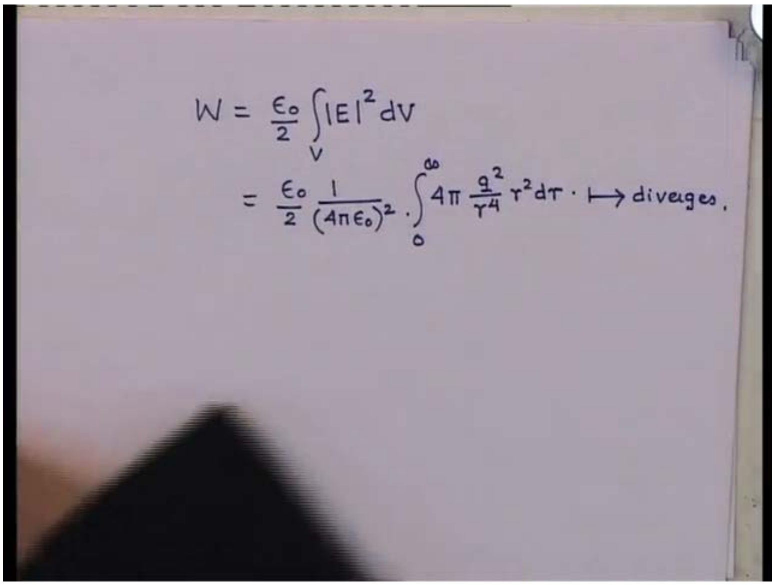

Now, this is an integral over a integral is always positive, and as a result the integral always is positive. However, this contradicts the fact that supposing I had two opposite charges. Since, the interaction is attractive the energy that I would get could be negative. Therefore, when I have a discrete distribution of charges, the total energy could have either sign. So, what went wrong what went wrong in this calculation?

Now, this problem is generally known as the self energy problem, and it would be good to spend a bit of time in understanding what really was the problem. First question is that we assumed that the point charges are given to us, in other words no work was done in assembling the point charge itself. What we did is we assumed that somebody gave us the point charged and they were at infinity initially, so that the interaction energy was 0 then I one by one brought the charged particles and put them wherever they ought to be.

But is there no energy required for creating the charge itself? Now, notice that if I take the discrete charge as charge which is distributed in a sphere of very, very small radius. I will calculate the charge and then take the radius to go to 0. Now, we had seen that this then becomes epsilon 0 by 2 and 1 over 4 pi epsilon 0 whole square and the integral from 0 to infinity I am putting 4 pi from the angle integration then of course, q square over r to the power 4 r square d r.

Notice that in the lower limit this one diverges, in other words the amount of work that I need to make that point particle coming to be is infinite and that is known as the self energy of the charge distribution. Now, this is the issue then.

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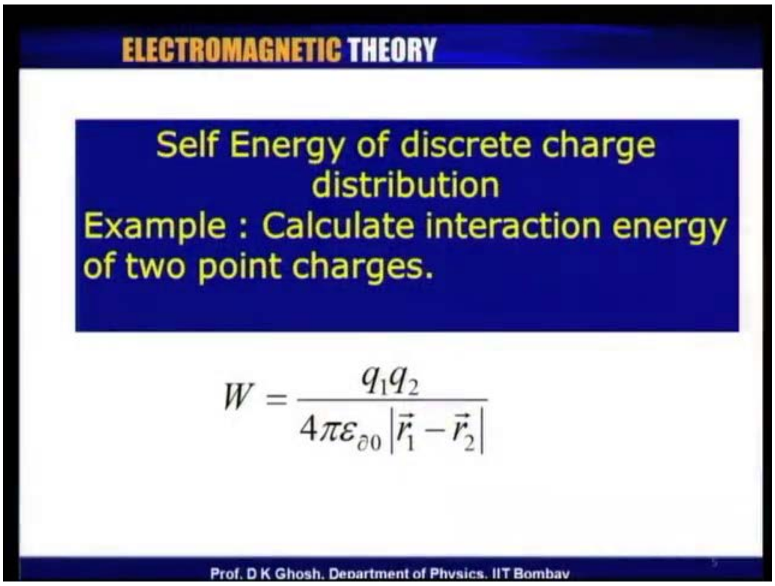

Now, let us then try to calculate using this expression that is epsilon 0 by 2 E square d V the interaction energy of two discrete or point charges.

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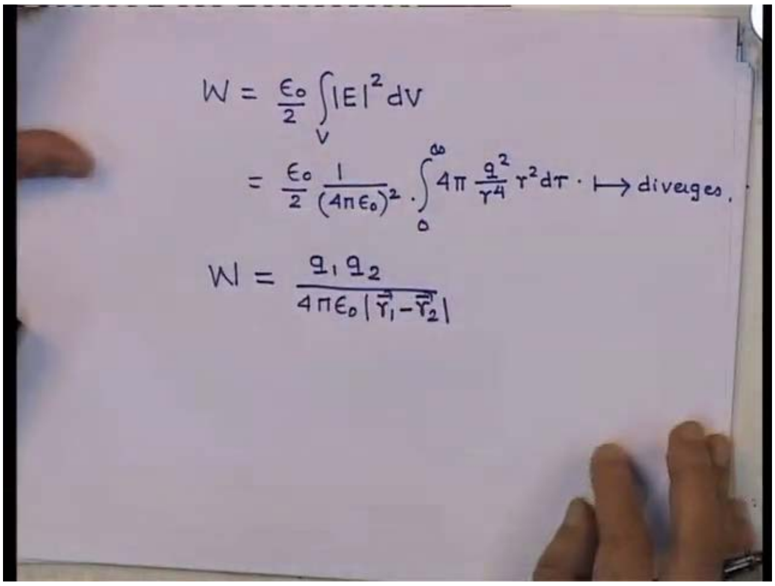

Now, I know that we had shown that for a discrete charge my interaction energy is q 1 q 2 divided by 4 pi epsilon 0 r 1 minus r 2. How does it compare with what we know?

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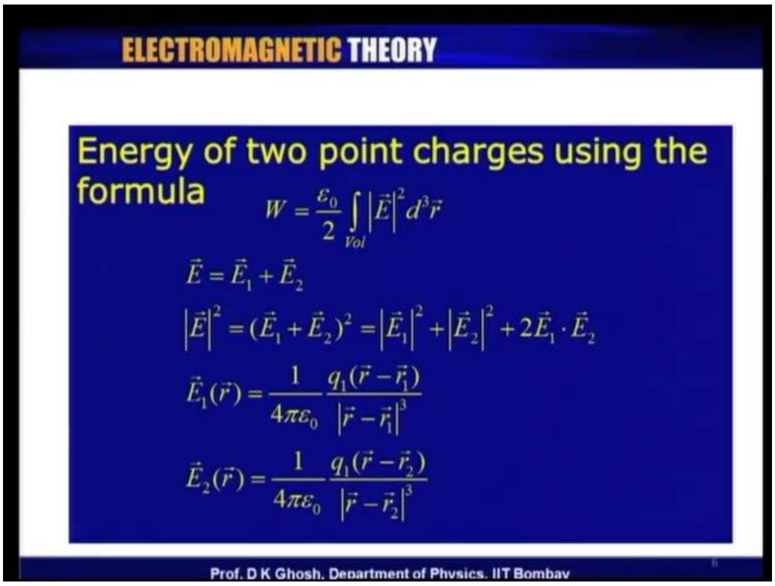

So, notice that the electric field obeys the super position principle. In other words my electric field E is E 1 plus E 2, but when I take a square of that, that is E 1 square plus E 2 square plus 2 E 1 dot E 2. Now, if you take the contribution to this integral epsilon 0 by 2 E square d cube r from these two terms namely E 1 square or E 2 square this will, each one of them will turn out to be infinite and the reason is not very far to see.

The electric field goes as 1 over r square. So, E 1 square or E 2 square go as 1 over r to the power 4 and when I do the integration over space I have only an r square d r. So, as a result I am left with a 1 over r square which is to be integrated out and as a result the lower limit diverges. So, these are the self-energy charge, these infinities are things which we neglect in our calculation, and the reason is that in experimentally experimentally we only measure this difference in energy with respect to self energy. Self energy problem is not totally understood, it it continues to remain a mathematical prescription at this stage, but let us proceed with this. So, what happens?

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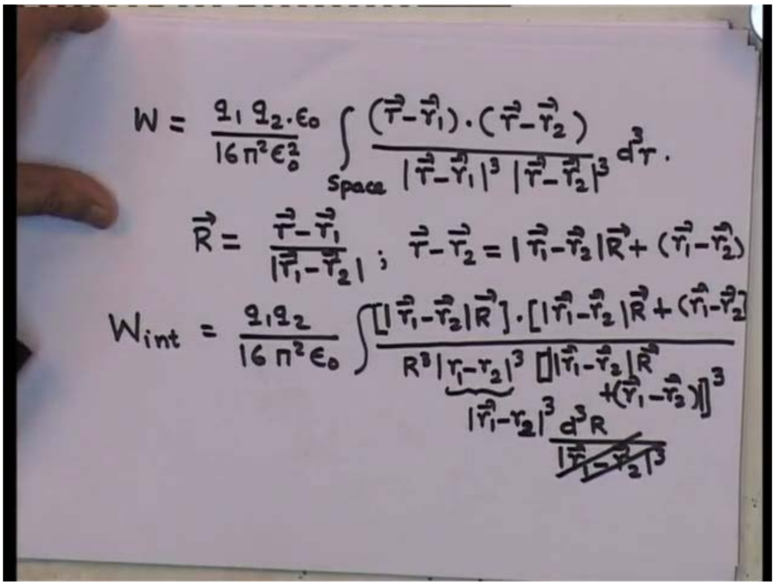

Our interaction energy which is W. Now, I have to take the E 1 and dot E 2 thing and so I have q 1 q 2 divided by 4 pi epsilon 0 whole square which is 16 pi square epsilon 0 square and there is an epsilon 0 on the top because of the expression W. So, there is actually an epsilon 0 by 2, but there is a 2 E 1 dot E 2. So, that is that is there and I have an integration overall space, because its electric field expression r minus r 1 dotted with r minus r 2 divided by r minus r 1 cube times r minus r 2 cube and the integration is over the space d cube r.

This is not a very easy integration to do, but we can do it. I t is instructive and it also gives you some experience of how to handle complicated integration. Let us define a vector capital R, this is actually a dimensionless quantity as r minus r 1 divided by the distance r 1 minus r 2. Now, using this you can write what is r minus r 2. So, you notice r minus r 2 is r 1 minus r 2 multiplied by R then add to this vector r 1 minus r 2, this R is also a vector. So, this is what follows from the definition here. So, the interaction energy W is q 1 q 2 by 16 pi square epsilon 0 integral over all space, let us look at so r 1 minus r 2 dot r r minus r 1 dot r minus r 2.

So, I get r 1 minus r 2 modulus times R that is r minus r 1 dotted with r minus r 2 which is again r 1 minus r 2 times capital R plus r 1 minus r 2 divided by divided by similar thing, but these are cube so it is R cube times r minus r 1 r 1 minus r 2 cube and again we have r 1 minus r 2 modulus R plus r 1 minus r 2 cube. This, all of this is actually modulus cube. Now, there is a d cube r there and you can check immediately d cube capital R is same as d cube small r by r 1 minus r 2 cube. So, d cube R by r 1 minus r 2 cube looks like a horrible expression, but we will be able to simplify it much more. So, let us look at what it is. Firstly let us observe what are going away, there are these r 1 minus r 2 cube. So, here in the denominator and I made a mistake here, so this should have been r 1 minus r 2 cube in the numerator. So, this would cancel with this one and this quantity is if you look at this is written properly in the screen.

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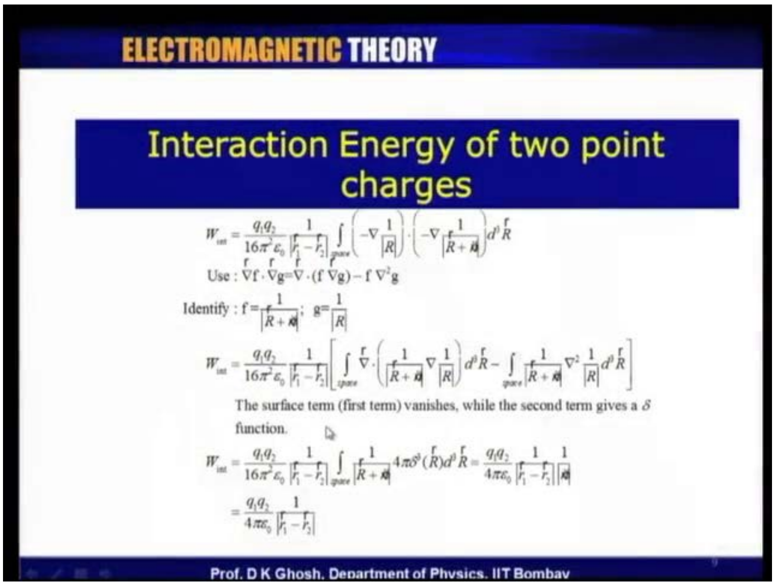

It turns out to be q 1 q 2 by this factor this of course, was there.

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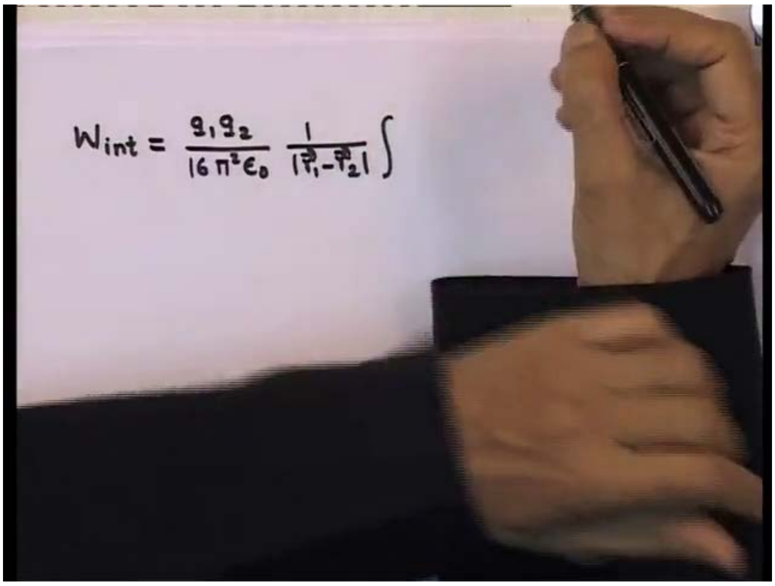

So, let me let me copy it here so that we can understand it better. So, this is q 1 q 2 by 16 pi square epsilon 0 1 over r 1 minus r 2 integral over space.

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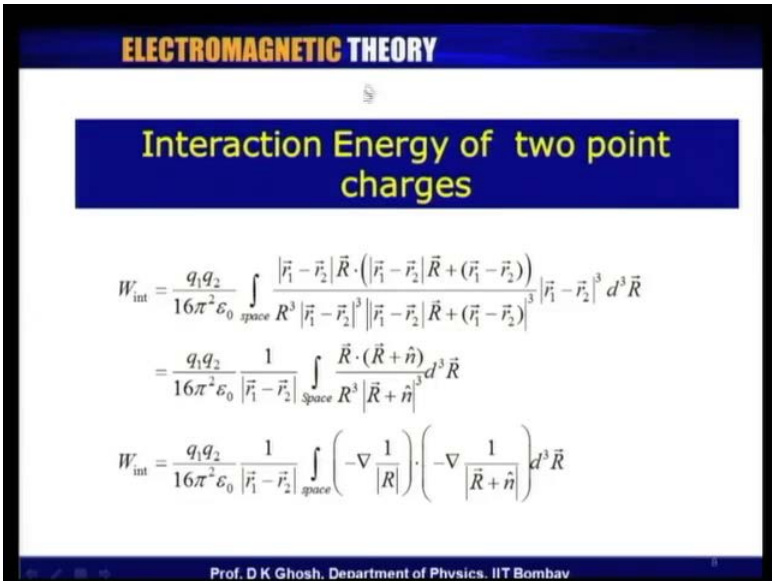

Now, notice one thing that in the previous expression I had these quantities that is I have vector R by R cube or here this is a vector. Now, what I will do is this, what if this vector is along r 1 minus r 2 direction. Now, r 1 minus r 2 direction if I define a unit vector to be n then vector r 1 minus r 2 is modulus of r 1 minus r 2 times the unit vector n so that the modulus of r 1 and r 2 will come out. Now, you have vector R by R cube or vector R plus n by vector R plus n cube. Now, vector R by R cube is nothing but gradient or minus the gradient of 1 over modulus of R.

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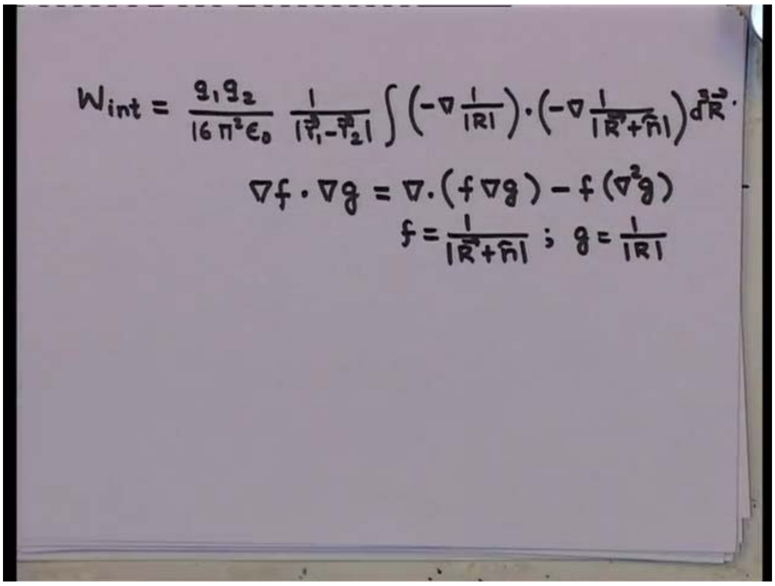

So, with this I get minus the gradient of 1 over modulus of R dotted with another minus the gradient of 1 over modulus of R plus unit vector n which I told you is at the direction of r 1 minus r 2 and d cube r. Now, this expression I will try to simplify by using some vector algebra and in doing so what I am going to do is this, I am going to use an algebraic identity which is gradient of f dotted with gradient of g is given by del dot of f times of gradient of g minus f times del square of g, this can be trivially obtained by using chain rule differentiation on del dot of f times gradient of g. And I choose f is equal to 1 over modulus of R plus n and g to be simply 1 over modulus of R.

Now, look at what happens. This is del dot del so I get del dot of f gradient of g minus f times del square g. So, coming back to the screen again I find that W int is q 1 q 2 by 16 pi square epsilon. Now, this term now which is let me write it down because there are some confusion there.

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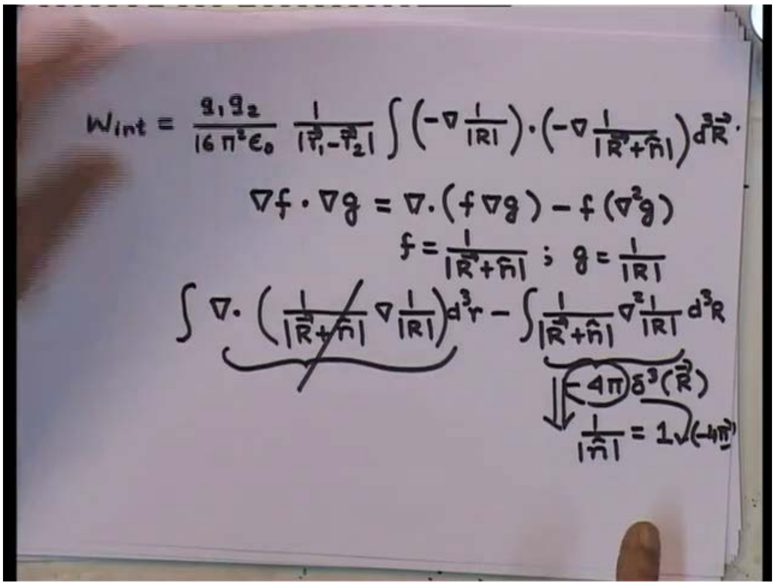

So, let us look at what do I get? I get del dot of 1 over R plus m which is my f times gradient of 1 over R d cube r. Then minus integral of 1 over R plus n del square of 1 over R d cube R. Now, this is something which we have been coming across regularly. Using divergence theorem I can convert this integral to an integral over the surface and this surface since it is over all space is at infinite distance. Therefore, I need the values of the functions there and therefore, these will be will go to 0 and it will minus.

So, what I am left is with this term and I know that del square of 1 over r is minus 4 pi times a delta function of R. This will enable me to do this integration and I will be simply left with 1 over modulus of n which is of course, equal to 1 and this factor minus 4 pi which will come there. So, using this you notice that the interaction energy turns out to be correct, there is a minus here, there is a minus there. This term goes to 0. So, I am left with q 1 q 2 by 4 pi epsilon 0 r 1 minus r 2 as it ought to be in case of point charges.

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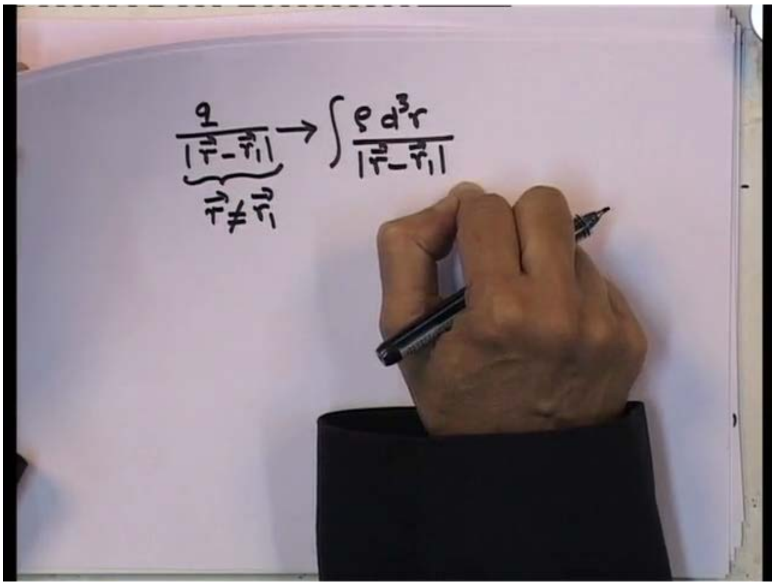

So, when we did discrete charges, we went over to continuous charge distribution by making a prescription that 1 over, q over r minus r 1 for example, we change this to rho d cube r over r minus r 1. Now, this prescription did not take account of the fact that in this term the effect of a charge at its own position is to be excluded. In other words this term had to be for r not equal to r 1, this restriction was removed there.

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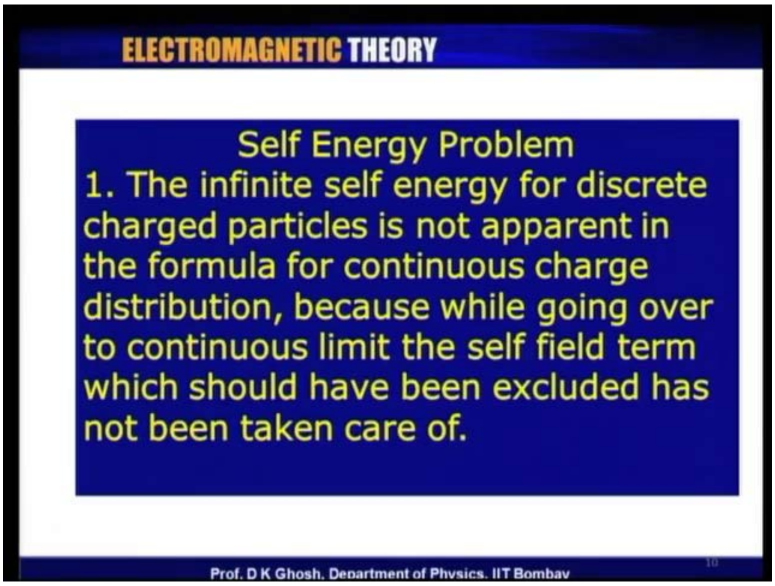

In other words that when we went to the continuous limit the self field term which should have been taken in to account has not been taken care of explicitly. So, why is it that for the continuous charge distribution for instance we worked out the energy of a uniformly charged sphere and we found this was a definite positive quantity. Why, what happened to the infinity there? The point actually is this there is an essential difference in physics. A point charge which can be regarded as a delta function density that is no matter how small you take the extension of the charge is there is the charge still resides there.

In other words the charge is contained in literally a 0 volume. Now, in a continuous charge distribution even though the density like in that example we have taken density to be constant, but I can take the limit of the volume element as small as I like and the amount of charge that will be contained there in spite of the fact that the density is non zero, but the amount of charge can be made to go to 0. So, as a result there is no self energy term there. Therefore, whenever the charge density has a delta function like behaviour, I will get the self energy problem.

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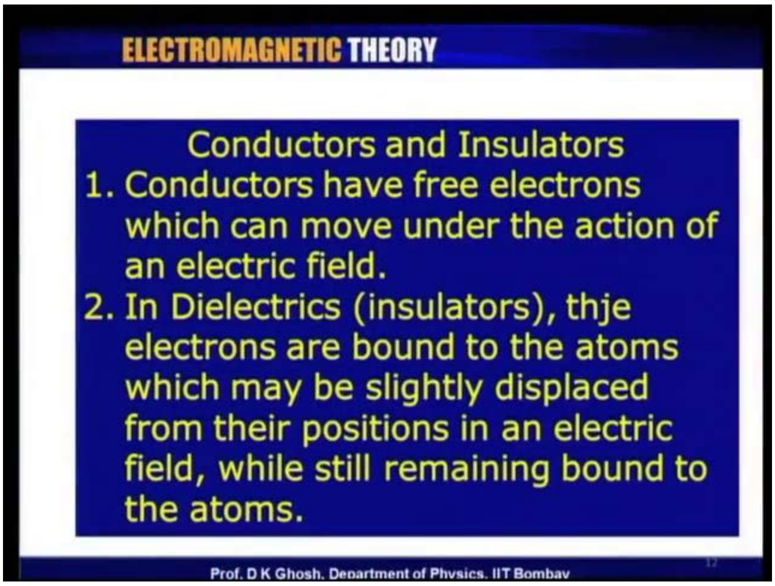

Now, as I made a statement earlier that this self energy problem is not a completely understood problem, but this is the best that we can do at this stage. Having done, talked about, having talked about the energy of charge distribution let me now go over to a discussion of the electro static field due to a conductor. Now, you are all familiar with what is a conductor, but let us formally define it. Conductors as the name suggests are materials which conduct electricity. These are characterised by having free electrons which when subjected to a an electric field these, they move inside the material. This when we say free electron what you mean is this electron electrons belong to the crystal as a whole and are not tied down to an atom.

On the other hand there is another class of material known as insulators and we will be using the word dielectrics and in these the electrons are bound to the atoms and when you apply an electric field from outside in, though these electrons can be slightly displaced from their mean position they still remain bound to the atom and as a result do not move around within the material. In other words they do not conduct electricity, in spite of the fact that all conductors offer some resistance to flow of electrons and all insulators to an extent may be a small extent conduct electricity. For our purposes I will assume that when I use the word conductor the conductivity is infinite. In other words it does not offer any resistance at all and likewise an insulator has 0 conductivity or infinite resistance, but we will continue to talk about it. Now, let us look at properties of a conductor.

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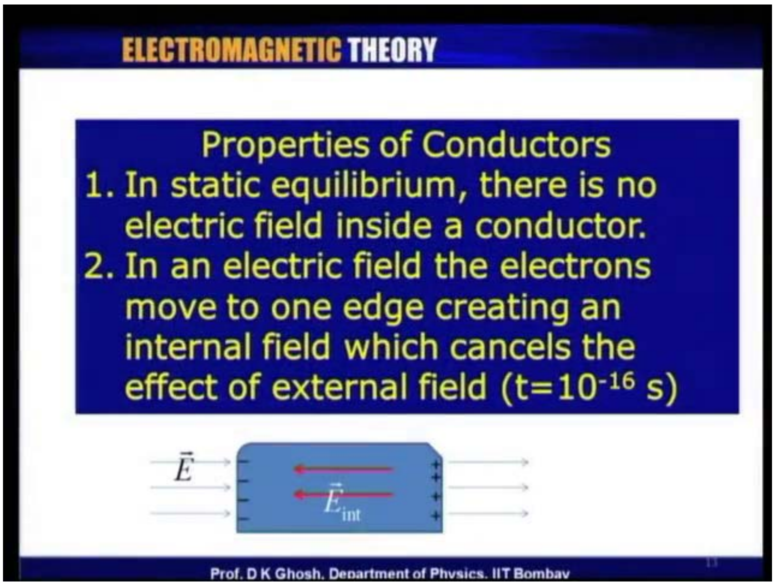

Now, I am talking about electro statics, when we talk about electro statics it means I am looking at a static phenomenon. Now, if I am looking at an equilibrium situation there cannot be any electric field inside a metal. Why is it so? The reason is this that if you apply an electric field, the electrons in the conductor being free would move around and this statement itself negates the fact that the system is in equilibrium. So, there cannot be an electric field inside a metal in situation of equilibrium, but what happens? The, there are electrons which are free, but we are saying on one hand if I have an electric field the electrons should be able to move around but we are saying electric field inside a conductor should be 0.

Now, conductors they respond to the situation in a very smart way. Almost as soon as you apply an external electric field the charges, the free charges they move to an edge of the conductor. So, for example, suppose I take the carriers or the free charges to be electrons which are negative charge. Now, if the negative charges move to one side the other edge becomes positively charged, that, this implies that inside the material there is an electric field created which is opposite to the direction of the external electric field which is here and this happens practically instantaneously. Typical time during which such adjustment takes place is of the order of 10 to the power of minus 16 seconds and this is the way the electric field inside the conductor become 0.

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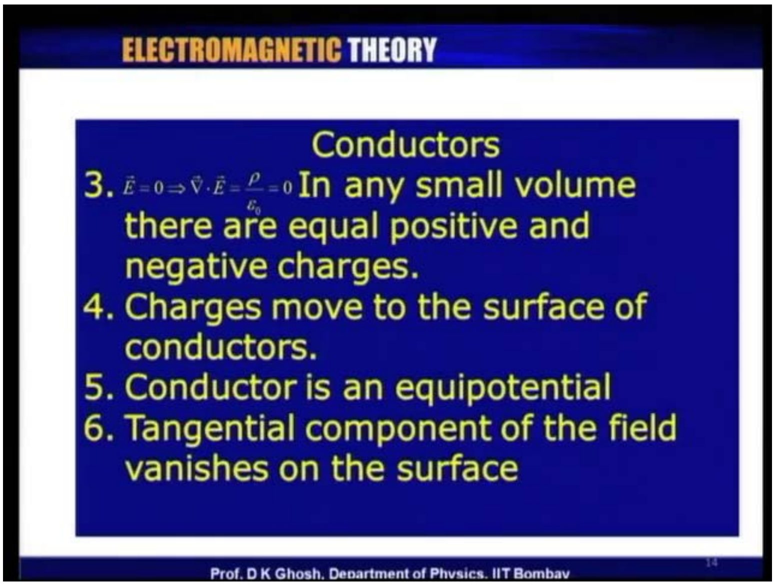

Now, let us look at some consequences there.

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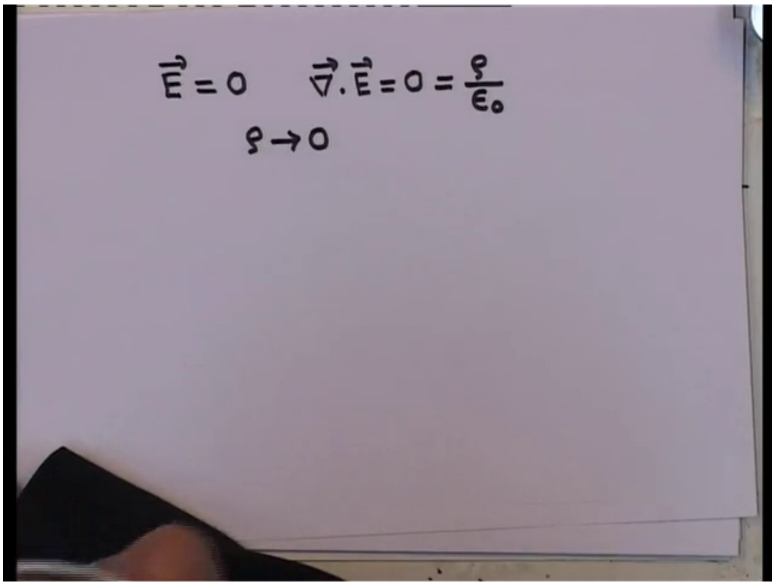

Now, if the electric field is 0 inside, then you know that the divergence of the electric field will also be 0, but according to Gauss's law divergence of the electric field is charge density divided by the epsilon 0 which means charge density is 0. Physically, this means that if you take any small volume inside the material there will be equal amount of positive and negative charges therein and these charges, the free charges they would move to the surface of the substance.

Now, we have said that the inside of the electric field, inside of the material has 0 electric field. Another equivalent way of making that statement is that the conductor is an equipotential, because if the electric field is 0 the potential whose gradient is the electric field must be constant. Now, what happens on the surface? On the surface the charges are moved so obviously there can be electric field, but there are some restrictions. There cannot be a tangential component of the electric field on the surface. Tangential means along the surface because if they did then equilibrium will be disturbed because the charges will be subject to such an electric field and move about. However, there is no restriction on a normal component of the electric field to be there on the surface.

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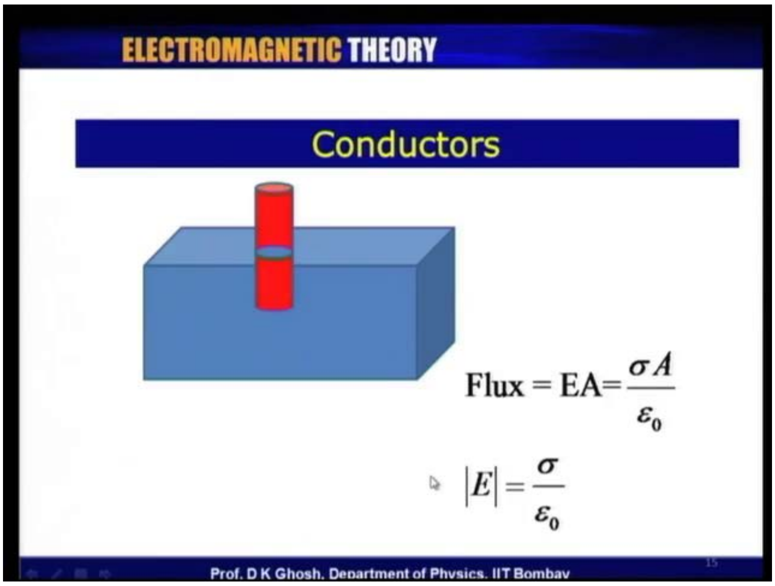

So, let us look at some properties of, more properties of the conductor. Supposing, I have a conductor which is shown here and we have seen that there are charges on the surface which is here. Now, I will take a Gaussian cylinder and as we did earlier half of that cylinder will be outside and half inside. I know there are electric field normal to the surface of the conductor assuming that the charges are positive they are directed outward from the surface of the conductor.

So, the flux through this cylinder by Gauss's theorem is equal to 1 over epsilon 0 times the amount of charge enclosed and where is this charge enclosed? Charge is just enclosed in this cross section where this cylinder intersects the top surface, taking the area of that patch to be A, if the charge density is sigma then sigma times A is the amount of charge contained there. Now, notice one thing so the flux is the electric field times the area and that is equal to the charge which is sigma A divided by epsilon 0 and that tells me that the electric field is sigma by epsilon 0.

Now, remember there was no electric field in the bottom half and the electric field then has a magnitude sigma over epsilon 0 just outside the charge surface. I would like you to recall that earlier we had talked about an infinite charge plane and we had said that the electric field is sigma by 2 epsilon 0 going outward on either direction and the reason was the following. That unlike in a conductor when I talked about a charged plane then when I had the Gaussian surface from both the edges I had flux so that the total flux was

E times 2 A and not E times A which gave me that half the sigma by epsilon 0, but there is no contribution to the flux from inside a metal because inside a metal or a conductor the electric field is 0. So, flux contribution is 0. So, this is the essential difference that is there. The next statement that I want to make is the surface of a conductor is also equipotential.

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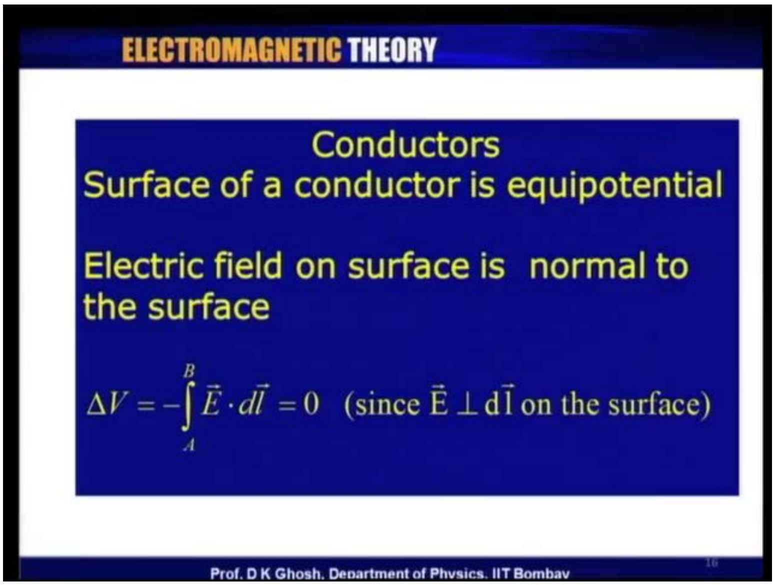

Now, we have seen that the electric field can exist on the surface, but then it has to be normal to the surface. So, if you take two points on the surface A and B the potential difference between those two points delta V is minus the integral from A to B of E dot d l.

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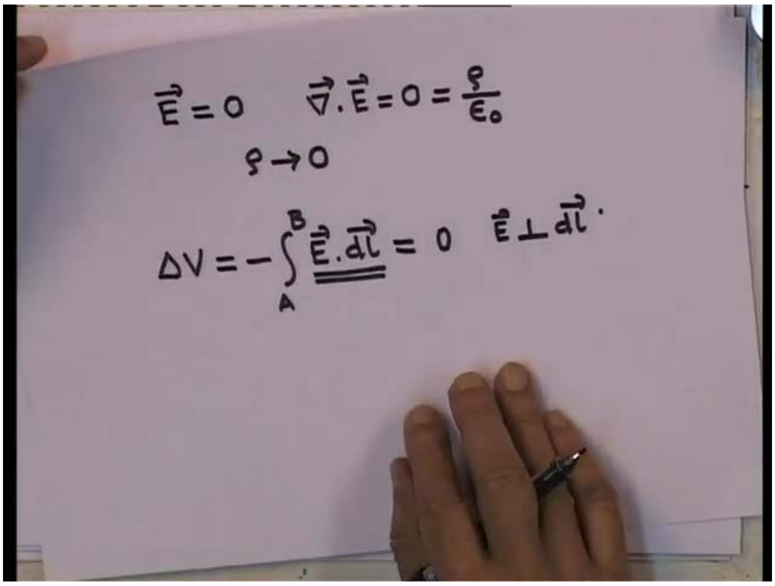

Now, if the potential difference delta V is A to B E dot d l and I know that E is perpendicular to the direction of d l because E is normal to the surface and d l is on the surface. So, this is equal to 0, because electric vector is perpendicular to d l. So, for any two points A and B the potential difference is 0 which implies that the surface is an equipotential.

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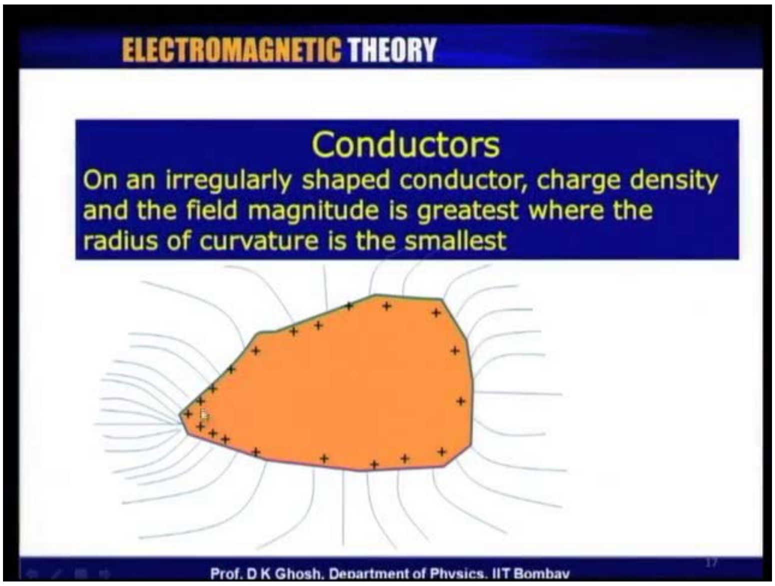

Supposing, you take an irregularly shaped conductor, then it turns out that the charge density and the field magnitude is is the most that is the electric field is the strongest and charge density is maximum where the radius of curvature of this irregularly shaped body is the smallest. Now, I will not be able to give a rigorous proof of this statement, but it can be understood in the following way.

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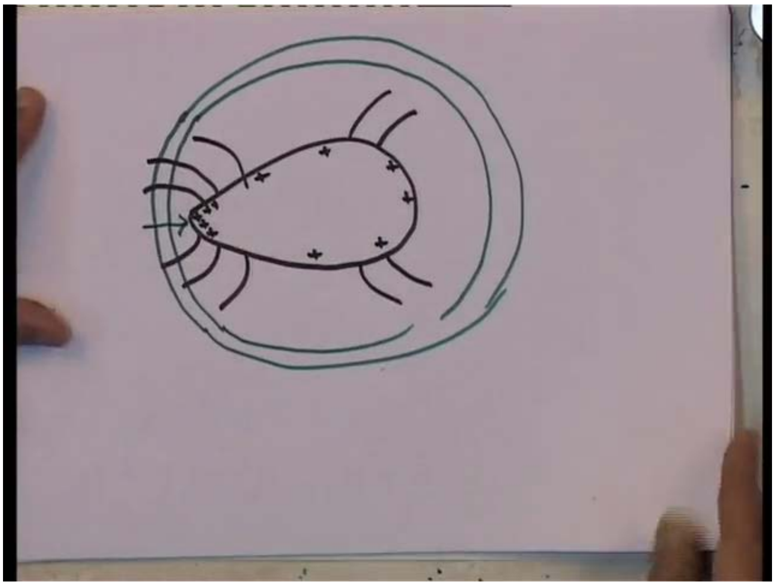

Suppose, I take an irregularly shaped body, I have, so let me take them to positive charges on the surface. Now, what happens is that the electrical lines of forces are like this, as has been shown in that picture. Now, notice that this body if you look at large distances it would appear like a point charge. Now, since it appears like a point charge the equipotentials are spheres.

So, therefore, these spheres they are equipotentials and there will be collections of sphere system of spheres and supposing I have to draw this spheres around this and let me assume that this spheres will be drawn such that the corresponding potential is separated by an amount delta V. Now, as these spheres are drawn as I come close to this place where the radius of curvature is smallest then I expect I expect these spheres to be much closer than they would be here.

So, the equipotentials will be such that they would be more concentrated near this left end edge which means the electric field will be the strongest in this area. A sort of qualitative argument, but for an irregularly shaped body this is correct.

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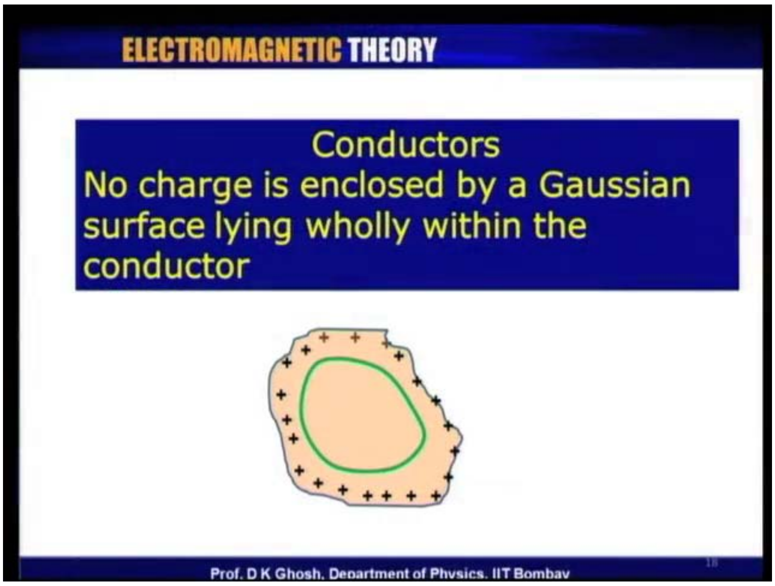

So, let us look at the few things about the conductors. First is we have seen that if there are charges they must reside on the surface. You can see it even by from Gauss's theorem. Imagine this is a metal and take a Gaussian surface which is shown in green completely inside the metal. Now, since the there is no electric field inside a metal the amount of charge that must be enclosed by this Gaussian surface must be equal to 0. Therefore, the extra charges that are there they must move to the surface.

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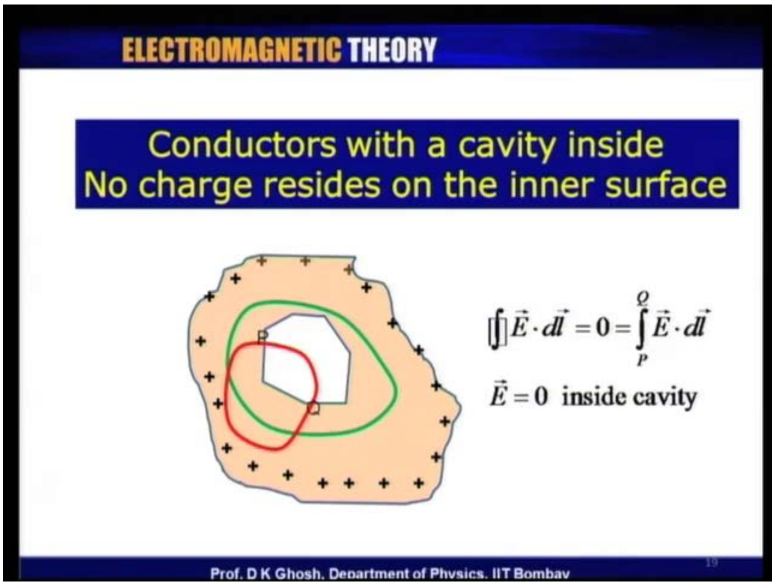

Now, some interesting problems consider a metal a conductor irregularly shaped, but with a cavity inside and let us also also assume that the cavity does not have any charge. Now, it follows that that in such a situation the inside surface of the cavity cannot contain any charge that is all free charges must go only to the outside surface. The proof of this is exactly the same as before. Take once again a Gaussian surface which is totally inside the metal which tells me once more that the amount of charge enclosed must be equal to 0 because the flux through such a Gaussian surface is 0 since the electric filed is 0, which means no charge can reside on the inner surface.

Look at this red contour which I have drawn. So, this red contour intersects the cavity at points P and Q and I know that it is a property of the electrostatic field that integral E dot d l over any close contour is 0. So, as a result since the for the part of the contour which lies wholly inside the conductor there is no electric field, the contour integral of the electric field is simply the integral from P to Q of E dot d l and this is true for any arbitrary P Q which is possible only if electric field inside the cavity is 0.

Now, this is the principle which is adopted for making what is known as a Faraday's cage. This tells us that electric field cannot penetrate a metal. So, as a result if you want to insulate some particularly sensitive apparatus from external electric disturbances you should encapsulate them inside a cavity within a metal.

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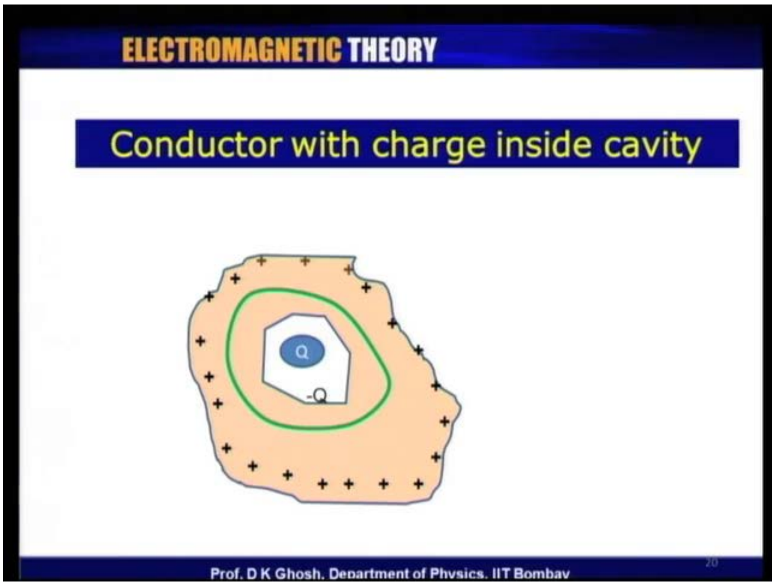

Now, situation changes if by some means we have been able to put some charges inside the cavity. Well it does not look like it can stay there like this. What it means is that there is an insulating handle by which it is kept inside the cavity so that it does not touch the sides. Now, supposing I have to take once again a Gaussian surface shown in green then this time the charges are enclosed within in spite of the fact that the flux is equal to 0. Now, if the flux is 0 the net charge enclosed has to be 0, but I have only a charge Q there. This implies that a charge minus Q must come to the inside surface of the cavity. The last thing that I want to do is to look at a very interesting problem that if I have a charged surface in a metal it turns out that the surface experiences a force or a pressure, there is an electrostatic pressure. Now, how does it work?

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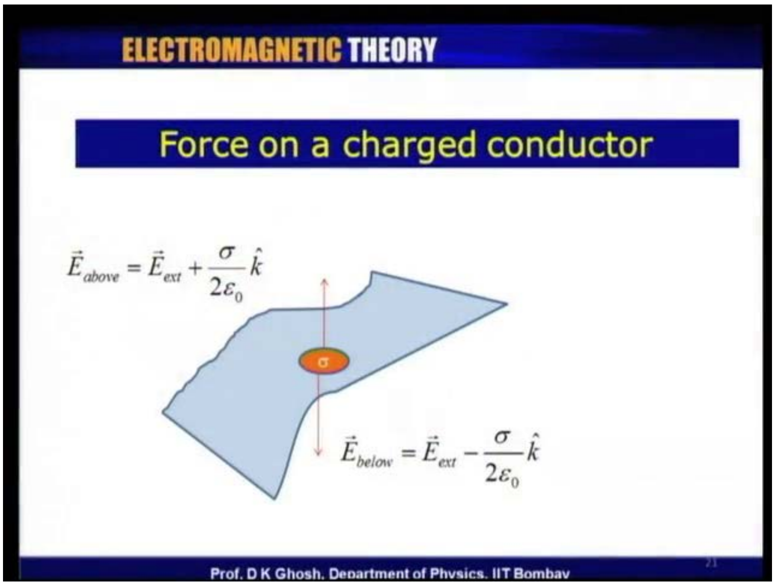

The, if you look at this picture now, notice this that there is charge density everywhere. Now, let me mentally separate it in to two parts. There is a small infinitesimal patch which I have taken here, the charge density is same everywhere, but this is a very small patch of area A let us say and I want to find out what is the force exerted on this small area. Now, where does the force come from? I know a body cannot exert a force on itself. So, what I do mentally is to think of the entire charge surface as consisting of two parts. One my little area here and the other what I call as the balance.

Now, what about what about the electric field, what about the electric field here? Now, notice that the electric field above the charged surface has two parts. One due to local this localized charge, small area and we had seen that now this is like, we are trying to remove this part by creating hole. So, as a result there is no discontinuity across this. So, this would mean that the electric field above would be the electric field due to the balance of this charges which I call as E external plus the electric field to due to this charge distribution which is sigma by 2 epsilon 0 k and below it is the same expression excepting now it is a E external minus sigma by 2 epsilon 0 k. The E external is the electric field due to the rest of the charges there.

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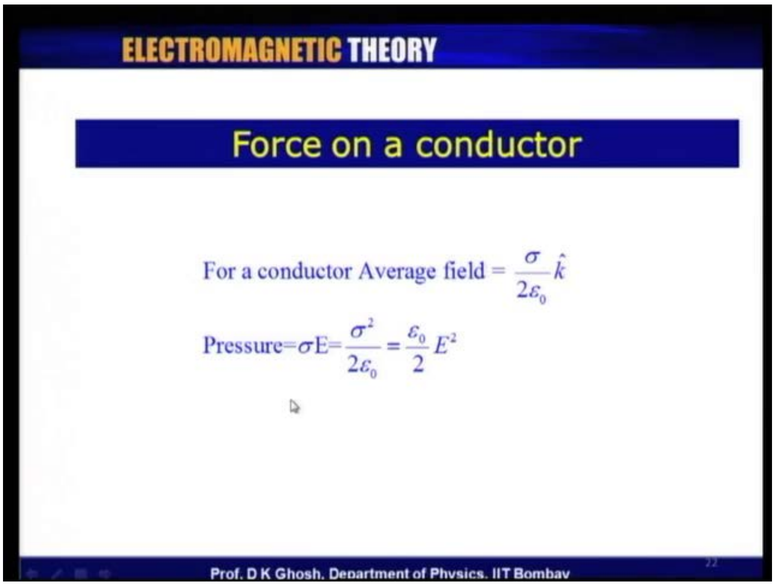

So, this means that the force that is exerted on that small piece because it cannot it cannot exert a force on itself is nothing but the force due to what I have been calling as the external things, but what is this external things? If we just add up this two you find that the external field is nothing but the average of the field above and the field below. Now, this is something which I know how much it is. This average field for a conductor surface is sigma by 2 epsilon 0, because I know that the field is sigma by epsilon 0 and 0 inside. So, it is sigma by 2 epsilon 0. Therefore, the pressure on the charged surface is sigma times the electric field which is sigma square by 2 epsilon 0 equal to epsilon 0 by 2 E square.

So, let us look at what we have done today. We first talked about the self energy problem, we realised that the discrete charge distribution differs from the continuous charge distribution because of self energy. This is I repeat a rather difficult problem to understand, but it is good to realise that one can at least mathematically understand it.

We defined conductors and insulators and looked at the properties of the conductors. As a special case we talked about the force that a charged surface of a conductor experiences. We found that the inside of a conductor is a equipotential, the charges reside only on the surface, the electric field can only exist on the surface and can be normal normal and because there is a charge sheet there is a discontinuity of the electric field between above the charge surface and below the charge surface. In the next lecture we will continue with the properties of the conductor by talking bringing in the concept of a capacitor.