Electromagnetic Theory
Prof. D. K. Ghosh
Department of Physics
Indian Institute of Technology, Bombay

Module - 2
Electrostatics
Lecture - 10
Potential and Potential Energy

In the last lecture, we had introduced the concept of a potential and talked about potential in a few cases. Today what we want to do is to discuss the relationship of the potential with the potential energy. In general we want to talk about how to calculate the electrostatic potential energy of a system of charges.

So, let us review a few things which we started last time, and one of the things that we talked about last time, but could not complete was determining electrostatic boundary conditions. What happens is that at the interface between two media for instance, supposing one medium is air or vacuum and the other medium is a charged body, you find there are certain conditions to be satisfied at the interface between two media, and these are known as boundary conditions. Till now we have been doing electrostatics, so what I will be concentrating today with respect to the boundary conditions, which we will be elaborated more as we go along, will be the electrostatic boundary conditions.

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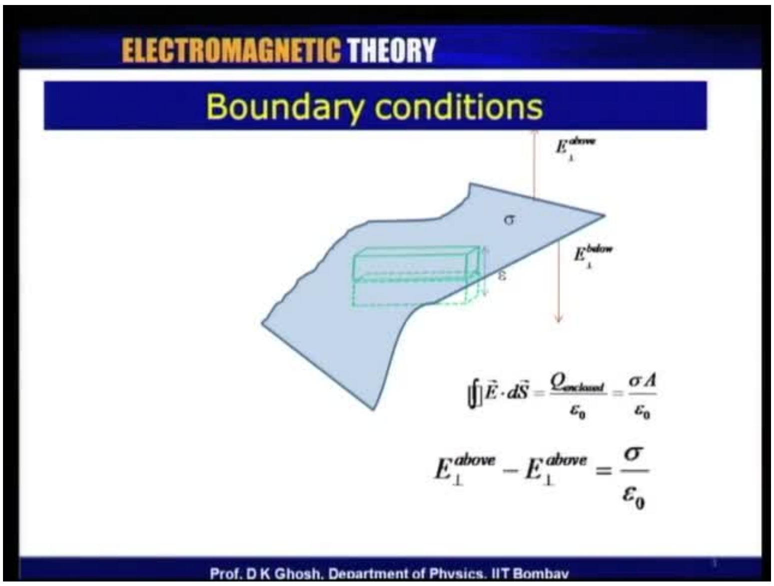

So, let me let me begin with assuming that, I have a charged sheet, an infinite sheet for convenience I take and there is a charge density sigma on this surface. So, it is an ideal surface in the sense that, there is no thickness and the I know that if the charge density is positive, then the electric field above electric field sort of goes away from the direction of the charge. So as a result in this example it is upward above the sheet and it is downward below the sheet.

Now, to look at what boundary conditions we must satisfy, what we do is the following, we take a Gaussian pill box as it is called or just a rectangular parallelepiped of a an infinitesimally small width epsilon, and make it intersect so that half of this lies above the sheet and the other half lies below the sheet. Now, what do we want to do is, we want to, we know already, according to Gauss's law that the flux out of any surface real or imaginary is equal to the amount of charge that is enclosed divided by epsilon 0.

Now, in this case let us assume that the area of the top surface is A, and as a result if I want to calculate the total flux, though this this surfaces of the rectangular parallelepiped. You notice that the, I will take the epsilon going to 0 and as a result what I get is how much is the Q enclosed. The Q enclosed, because I have said that the area of the top surface or bottom surface is A. And this sheet is without any thickness and the charge density is sigma. So, this tells me that the amount of charge that is enclosed by a such an intersecting parallelepiped is just sigma times A. So, integral E dot d S is sigma times A and and let us look at how to calculate this flux. You notice that as epsilon becomes smaller and smaller, the the since the area of all surfaces, other than the top and the bottom becomes infinitesimally small, then there is hardly any contribution to the flux form these surfaces. And by definition, our direction of the normal is perpendicular to the outward perpendicular to the surface, so that they are oppositely directed on the top and the bottom surface.

So, as a result the contribution to the flux form the top surface is the normal component of the electric field multiplied with this area A. Normal component because I have to find E dot n d S. So, as a result what happens is, that from the two surfaces, because the direction of the top surface is opposite to the direction of the bottom surface, the normal's are oppositely directed. So, I get E perpendicular above times the area A, minus E perpendicular below times the area A is equal to sigma A by epsilon 0, which means that E perpendicular above minus E perpendicular below is equal to sigma by epsilon 0.

What it implies is this that, if there is a charged surface if there is a charged surface then there is a discontinuity of the electric field, the normal component of the electric field as you cross a charge surface. The difference between the normal component of the electric field, above a surface and that below a surface is given by the charged density of, on that surface divided by epsilon 0.

So, the normal component of the electric field is discontinuous. Now, notice this is all about the normal component. Now, what happens to the tangential component, that is the component which is parallel to the surface?

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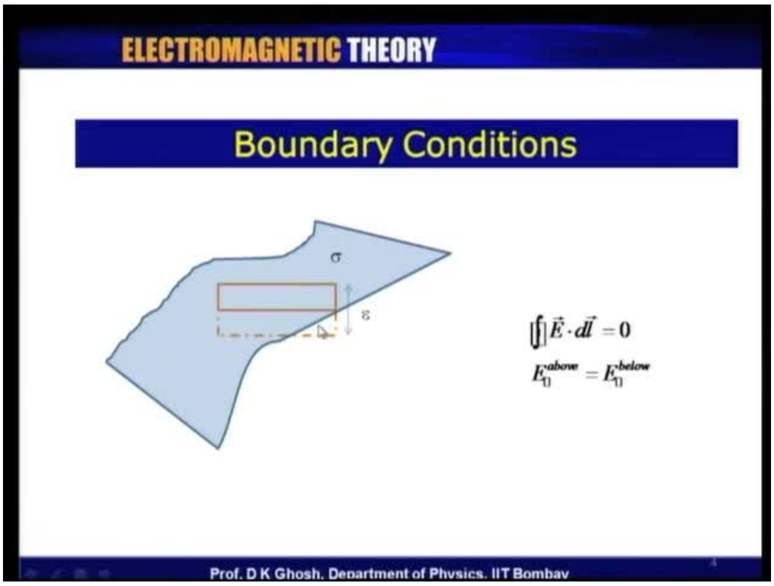

So, once again I start with the same surface, infinite sheet with a charge density sigma, and I am interested in looking at what happens to the normal to the surface. Now, this time what I do is this, I I have a rectangle the previous was a rectangular parallelepiped, but this is just a rectangular and this rectangle is of width epsilon and which is again taken to be small and length l, and I make it half above the charged surface and half below. Now I know that the, because the electric field is conservative, the line integral of the electric field which is E dot d l over any closed path is equal to 0.

Now, if I look at this rectangle and I take a line integral of the closed path, because my definition the direction along which you traverse is tangential to any curve. So, as a result E dot d l contribution from this two surfaces will be along the tangential direction, and there will be no contribution from the two short sides, because epsilon is going to 0.

And here you notice that I am traversing in one direction here and the other direction there… So, as a result E dot d l is e parallel above minus e parallel below times the length l which must be equal to 0. As a result the E parallel above is equal to the E parallel below.

So, what I notice is that, though the normal component of the electric field, has a discontinuity across a charged surface that tangential component does not have a discontinuity. In other words the tangential component is continuous. So, these are two boundary conditions that have to be satisfied at the interface between two media where there is a charge sheet between among the interface.

So, these were about the boundary conditions, so let me now go back to a calculation of the potential energy. You might recall we had said that, there is a connection between the potential and the potential energy. We had seen that if there is a charge, which is located in an electric field, then the eclectic field is of course, is created by another collection of charges. Then this system has the potential energy and the there is a term which is or in that potential energy, there is a contribution because of this charge which is located at let us say the point r.

So, this potential energy associated with our test charge, if our test charge is taken to be a unit charge, is what is the potential at that point. So, this is the relationship between the potential and the potential energy.

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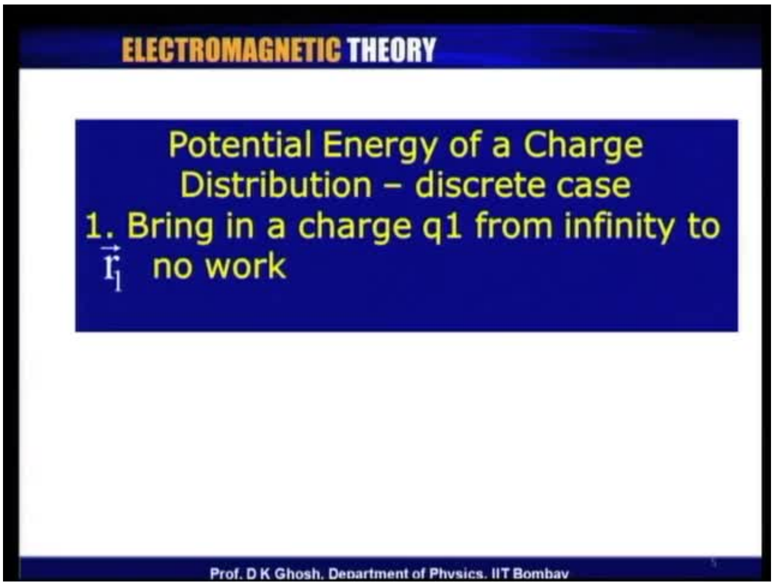

Let us look at it a little more quantitative. To do that, I would take a collection of a charges. To begin with I will take the collection of discrete charges and suppose I bring in charges, such that my final configuration is that the charge q 1 is at the position r 1, a charge q 2 is at the position r 2. How do I calculate? How do I calculate the potential energy of such a configuration?

Now, so I will give you a prescription of what to do. We assumed that, initially all the charges are at infinity. We have seen that it is good to take the reference point of potential energy to be 0 at infinite distances, because forces the coulomb forces vanish at infinite distances only being a long range force, so initially all may charges will be assumed to be at infinity and infinitely separated. Let me now do the following, supposing now because all charges are at infinity, there is no electric field in the finite region of space. So what I do now is, bring in a charge Q 1 from infinity to and put it in its own location which is r 1.

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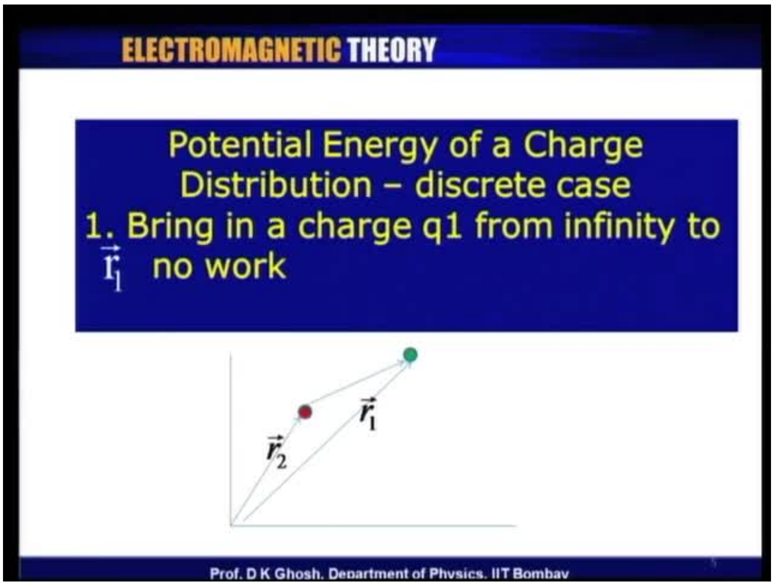

Now, when I do that, when I do that no work is done, because this charge is not subject to any electric field, because it is the first charge being brought in from infinity and there is no field excising there at all. The situation changes when I bring in a second charge, so suppose I have brought in a first charge, which is at the position r 1 and now I want to bring in a second charge. Now this second charge this second charge will be placed here, that is with respect to an arbitrary origin, the first charge is at r 1, this will be placed there,

So, let us look at put that here, now notice that this second charge that we have bought in, experiences a field because already the first charge is in its place. So what I need is, I have to do some work, because there in an electric field due to the first charge, and this will exert a force on the charge the second charge that I am bringing in as it comes from infinity to its location. And in order to take care of or work, in order to make sure that the charge comes and is located at the point r 2. We meaning the external agency have to do work to overcome this force.

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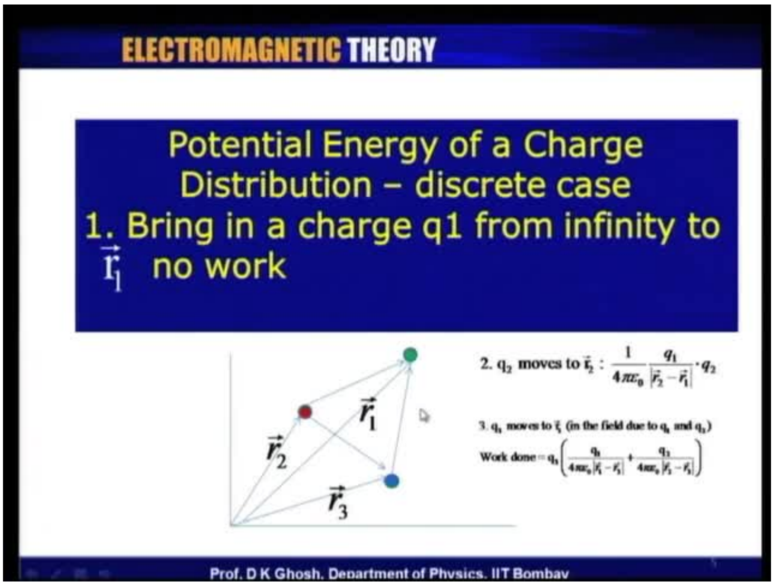

I already know the charge, I already know the electric field, because the electric field is due to the charge q 1 therefore, the electric field at this position which is r 2, is 1 over 4 pi epsilon 0, q 1 divided by the r 2 minus r 1 and directed of course, along the line joining them. So, the work that needs to done is to 1 over 4 pi epsilon 0, q 1 by r 2 minus r 1 times the charge q 2.

Now, this work that you have done obviously will now become the potential energy of the system of the two charges. Now, let us bring in a third charge, let us bring in a third charge now this third charge, is brought in to a position r 3, this being brought into a position r 3. But now this charge experiences an electric field due to the presence of both the first charge and the second charge. And since its at r 3, so when q 3 moves to r 3, the work done is q 3 times electric field due to q 1 as well as the q 2, this what is written here.

Now, I can go on like this a fourth charge or fifth charge and things like that. So, what we actually see? What we have seen is that, the amount of work that is done is equal to the amount of work that is done. Suppose we are looking at we are looking at the work done in assembling n number of charges.

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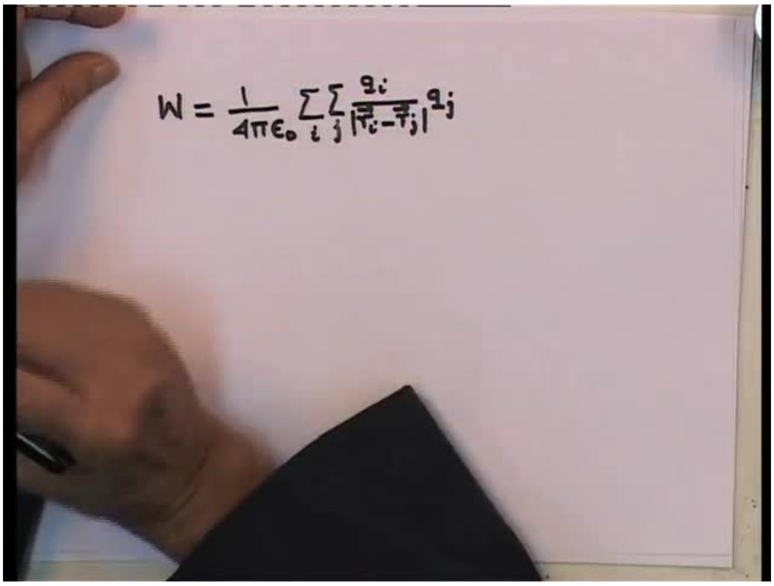

So, this work done is 1 over 4 pi epsilon 0. Now, let us look at how much work is done in bringing, let us say z charge. When all other charges are already there, so let me talk about the presence of charge as sum over I, so I have q i by r i minus r j vector times q j. Now, this is the amount of work that I will be doing, if I bring in that j th charge and locate it at the position r j, when a system of charges are in their respective place. Now, so it follows that that in order to get the total work done, I must sum over j so that each charge j is in the field of each every other charge.

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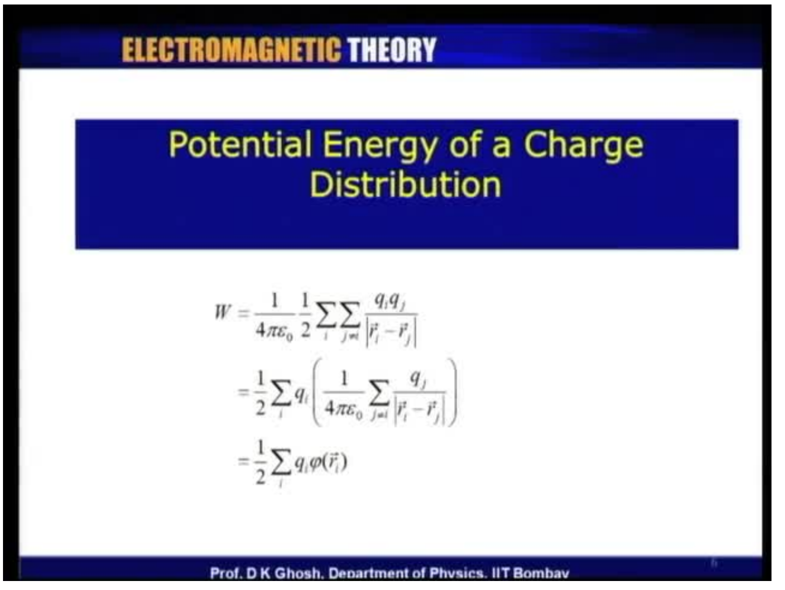

But, notice in this sum there are some restrictions, the restriction is because r i cannot be equal to r j, because a particle does not exert force on itself.

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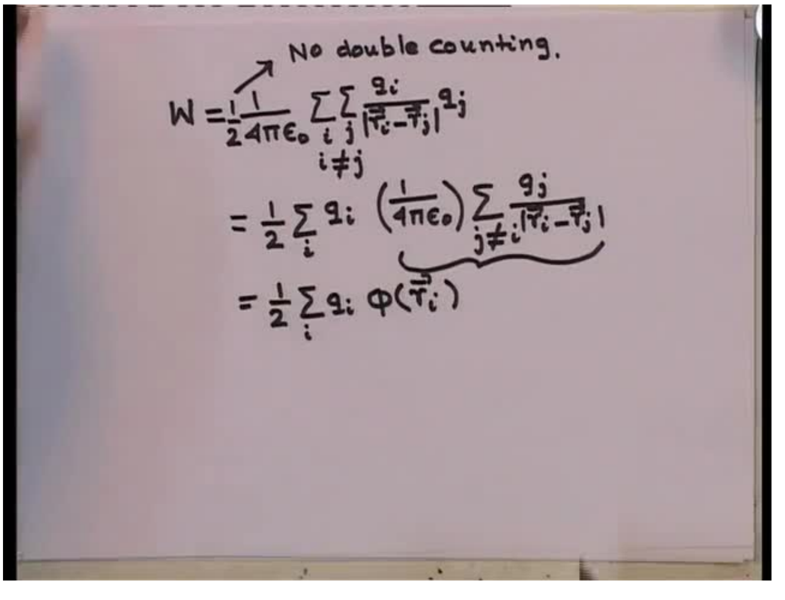

So, let me let me write that i is not equal to j. There is another factor that have to worry about, in this way of writing the sum I have done it double counting. Because I am counting i and j, because there is a double sum, so each term is counted twice. It is not true that there is a separate term for interaction between i and j and that between j and i, so to avoid this double counting I need to put in a factor of half in this sum as well. So, this is the factor to avoid double counting.

But, now you notice this quantity is equal to half, sum over i, q i. Now, let me write this 1 over 4 pi epsilon 0, there sum over j not equal to i, q j divided by r i minus r j. But this term here is nothing but the potential seen by the charge q i at the position i due to all other charges which are located at the position j. Therefore, this can be written as sum over i, q i times phi the potential at the position r i, so this is an expression for the potential energy of a discrete charge distribution. Now, what happens if I have a continuous charge distribution?

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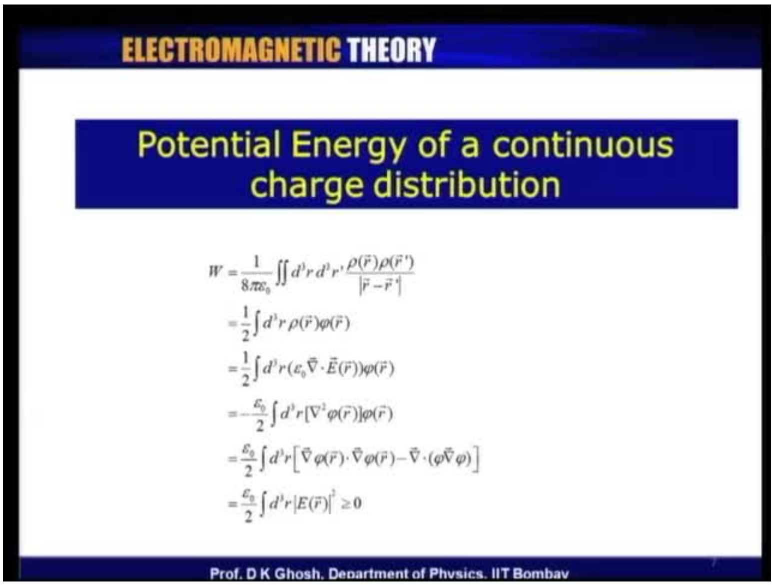

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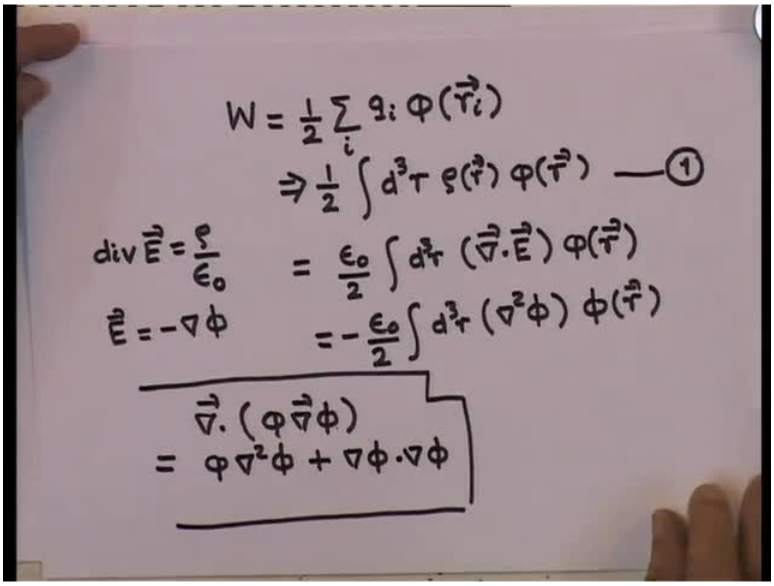

So, for the discrete case let me write it down, for the discrete case, the work done is half, sum over i q i potential at the position r i. Now, if I, it is easy to transform this, it is easy to transform this to the case of continuous charge distribution. For a continuous charge distribution this sum over i will go over, I will take a volume and locate a small volume element at the position r and the charge in that will be rho times the volume element, the charge density times the volume element. So, as a result the natural extension of this expression is integral d cube r over the volume rho r, so this is the volume the charge that is contained in that volume element and of course, the potential at the position r.

This this expression is one of the standard expressions for calculation of the electrostatic energy of a continuous system.

Now, let me do some algebra, so firstly I, you recall that according to Gaussian law divergence of the electric field e is rho by epsilon 0. This tells me that this rho there I can write as epsilon 0 times divergence of e, therefore let us put that epsilon 0 by 2 integral d cube r del dot E times phi of r. But recall that e itself is minus gradient of phi, so as a result I get minus epsilon 0 by 2 d cube r, so del dot of e gives me del square of phi times phi, well there are all these dependences there.

Now, I do a little bit of asthmatic with this, and what I will do is this that, I will look at this expression and notice the following that, del dot phi times del phi, so divergence of a scalar times vector is phi times del square phi. So, the scalier times divergence of whatever vector is there, del dot del is del square plus gradient phi dot the second term which happens to be also gradient phi, so it is gradient phi dotted with gradient phi. So I use this expression, I use this expression to simplify this term there.

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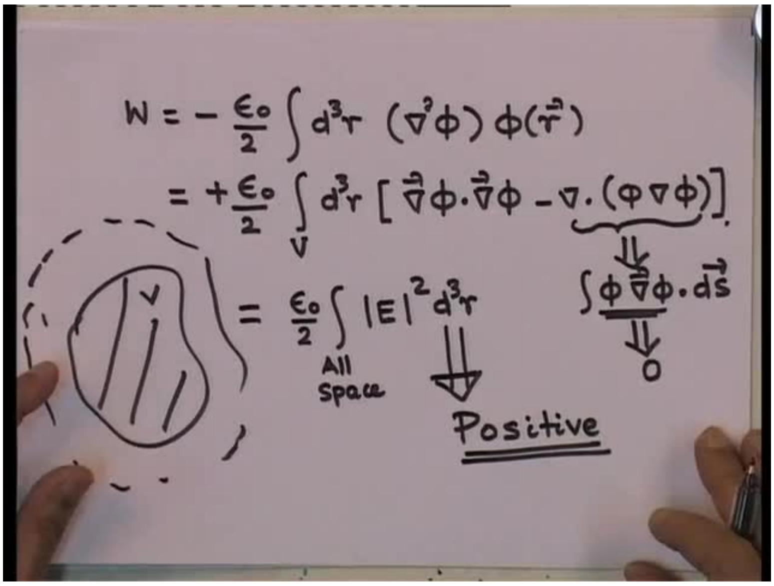

So, let us rewrite it W is equal to minus epsilon 0 by 2 integral d cube r del square phi phi of r. And this is then by using this identity, what you have written down is given by there are two terms now. So, come back here, so del square, phi times del square phi which is I have here, is del dot phi del phi minus del phi dot del phi okay. Therefore, this term becomes equal to there is a, I will take the minus first, so that this is integral d cube r grad phi dot grad phi minus del dot of phi grad phi.

Now, notice this term, now this is this is over, this is the work done in assembling a continuous charge distribution. Therefore, the this volume over which this integral is taken, is the volume of whatever system that you have, for example, this could be your volume v. But notice something interesting, I need not confine myself to his volume v, I could as well take any imaginary volume, for instance this one… the reason why I could do that is, because if you look at this expression here number one, you notice that it is a product of rho with the potential and if I take the region to be expanded like this, in the intervening region there is no charge density charge density is 0, so the contribution to the volume integral will become 0.

Now, if I take the volume over which I do this integration to be infinite volume, even then there would be actually contribution coming to that equation one, which I showed you just now from the physical volume because rho is 0 in the other case anyway.

Now, however if I do that then the this term which is, del dot so this is d cube r del dot phi del phi which by divergence theorem gives me phi grad phi dotted with d s. In this I need to calculate, I need to plug in the potential or its derivative, namely the electric field only on the surface. If I take the surfaces going to infinity then of course, this term goes to 0. Therefore, I need to concentrate only on this term and since minus gradient of phi is the electric field, this expression gives me epsilon 0 by 2 over all space modulus E square d cube r. And you notice this is an integral over positive terms only therefore, this quantity is by definition positive.

This is the point which we will need to make comments on as we go along. Let me make a quick comment here and I will come back to this issue a little later. The energy could be both positive and negative for example, supposing I am putting one charge at one position and it is an opposing charge in another position. The potential energy is negative, but this being a, an integral over square of the electric field it is always positive.

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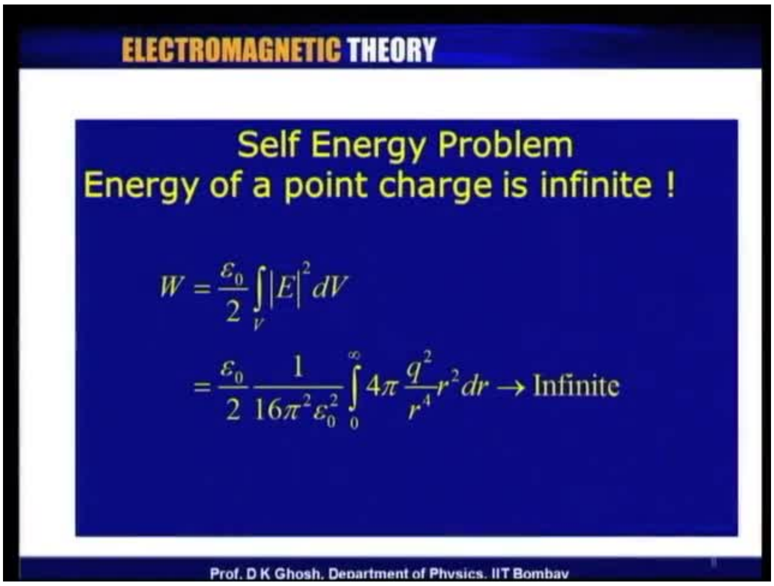

Now, the problem has its resolution which will be talking about little later, is because when we took the point charge we assumed that no work was done in making a point charge. In other words, point charge was given to me and as a result what I am actually doing is not calculating the total energy in case of the charge distribution, but only the interaction energy between the charges.

So, in discrete case the work that is done to make a charge is not included. Now, we can actually calculate how much it is? Because we have just now seen the work done is epsilon 0 by 2 by absolute square d V. And now, let us let me assume that I am talking about a to create a point charge. So, this quantity then is e square, so it is q square by r to the power fourth and r square d r.

So, you notice this that this one… This expression this expression blows up at the lower limit. In another words, the there is an infinite contribution to the energy and this is normally referred to as the self energy. When we calculated the energy of a collection of point charges, we never included this self energy. The calculation that we have done now, we have not yet actually done a physical calculation, but this expression and the corresponding discrete expressions differ by an infinite amount. This is what is known as a divergence problem in physics. We will not be making much of a comments on it, but we will make a few comments as we go along with our calculations.

Now, let me let me now give illustrate the calculation of the total energy of the charge distribution, taking a very specific example. And I will do this calculation of the electrostatic energy by four different methods.

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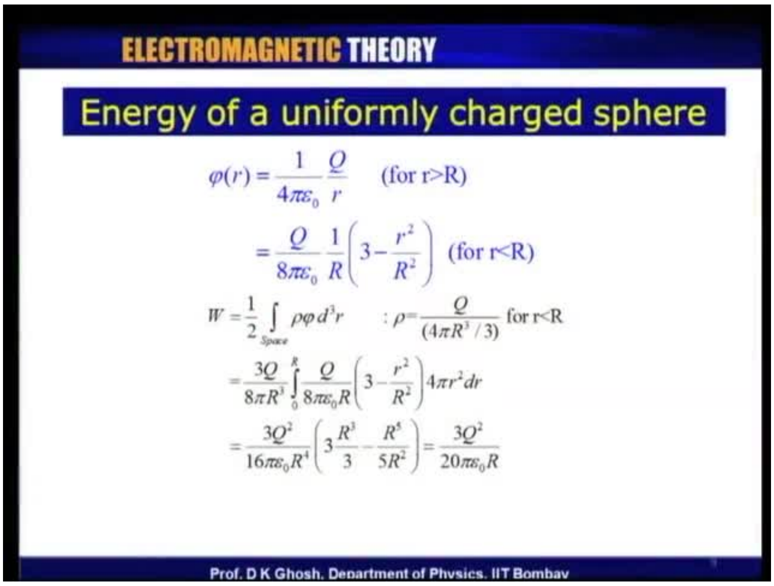

The problem that I have taken, simple problem is let me calculate the energy of a uniformly charged solid sphere. In the last lectures we had worked out the potential due to the charge sphere, so let me recall it for you again.

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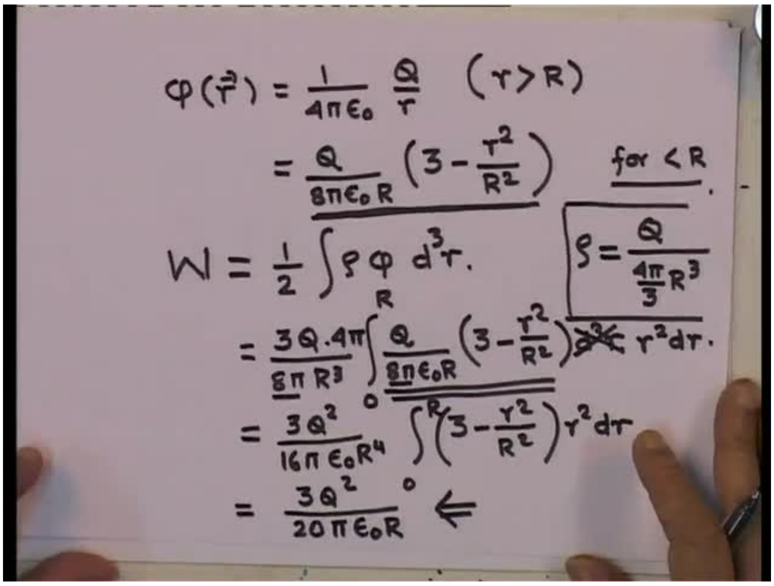

So, phi of r is 1 over 4 pi epsilon 0, Q by r for, r greater than the radius of the sphere. And we had obtained an expression like Q by 8 pi epsilon 0 R into 3 minus r square by R square, for r less than R. In my method one, I want to start with the calculation of the energy, by my first definition which is, this is the integral over the volume of rho phi d cube r. Of course, I know what is rho, so it is a uniformly charged thing it is simply Q divided by 4 pi by 3 R cube. So, uniform charge, so charge Q is uniformly distributed.

Now, the charge density outside even if you have taken a bigger volume is 0, so this, in this method the contribution to the integral does not comes from the whole space, but comes from the physical space. Therefore, it is only form the potential expression that I have to take should be the second expression here. So, if I plotted it in, so I am since rho is constant it will come out, and I am left with 3 Q over 8 pi R cube integral, now only my integration is from 0 to R, the expression for phi there, which is Q by 8 pi epsilon 0 R and 3 minus r square by R square d cube r. Of course, this integral does not have any angle dependence, so as a result I get 4 pi from the angle, so which I must plug in here. And then my integral is simply over r square d r.

Now, this integral is very straight forward to work out, so I can plug in here is 3 Q square, I had 8 pi into 8 pi that is 64 pi square there 4 pi goes, so that is 16 pi epsilon 0 and of course, I have got a R cube R, so it is R to the power 4 and I am simply integrating from 0 to R 3 minus r square by R square, r square d r. Very trivial integration to do and you can plug it in very easily and get a result 3 Q square over 20 pi epsilon 0 R. So, this is the energy of a collection of charges which give me a uniform distribution of charge in a sphere.

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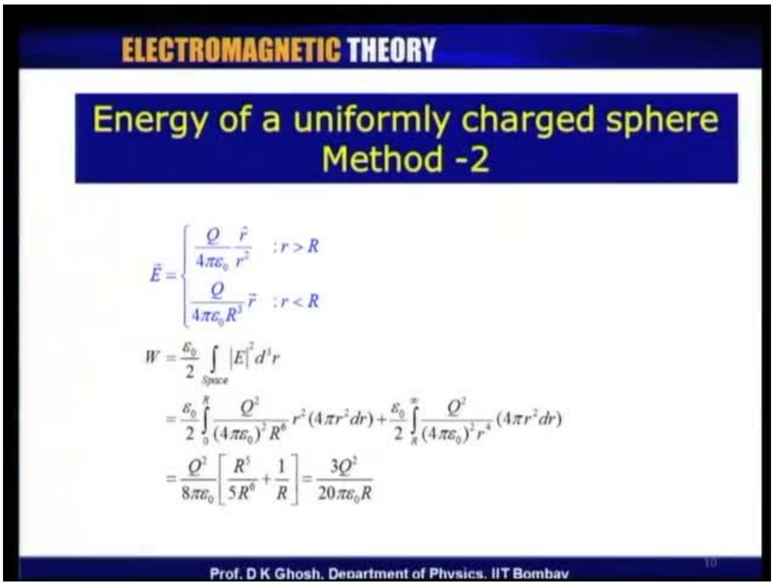

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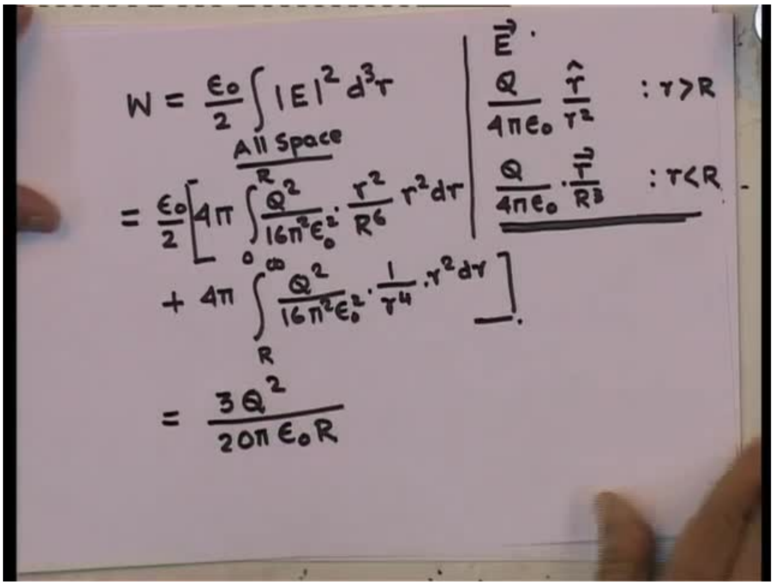

Now, let us proceed with this, this time I will use the expression that we derived at the end, that is by method two, which is work is epsilon 0 by 2 integral absolute E square d cube r. And I told you that because the surface term was thrown out, I had taken the range of integration to be infinite. So, therefore this integration is over all space, not just the physical space. If you wanted to confined to physical space which we will do later, we will have to keep the surface term as well, because the surface term was thrown out by taking the integration to infinite volume and of course, we found the that is perfectively legitimate because the charge density excepting in the physical volume was 0.

Now, in another words I need the expression for the electric field, now this we know was trivially calculated by Gauss's law, the electric field expression for r greater than R was as if the entire charge was concentrated at the centre of the sphere, so it is vector r by or unit vector r by r square coulombs law, and that is for r greater than R, for r less than R. Because only small r cube by capital R cube amount of charge is enclosed, so the field is linear and this was 4 pi epsilon 0 and this was vector r by R cube and that is for r less than R.

So, I need to take both these contribution, so within within the sphere I have this expression outside the sphere I have that expression. And I get epsilon 0 by 2, once again all these expressions do not have any angle dependence, so my volume integral always gives me a factor of 4 pi integral e square, so let me take that that is Q square over 16 pi square epsilon square, r dot r is r square by R to the power 6 and of course, r square d r which comes from the volume element. And this integral is from 0 to r, the second integral is 4 pi, R to infinity once again this Q square by 16 pi square epsilon square and this time it is one over r to the power 4 and r square d r.

Once again with these integrals are fairly straight forward to do, because this is just an r forth integral which gives me r to the power 5 by 5, this is one over r square which gives me one over r on integration and the infinite limit will go away, because one over r goes to 0 and only this term will be there, you can add it up and find out that the result is same as before namely 3 Q square by 20 pi epsilon 0 R.

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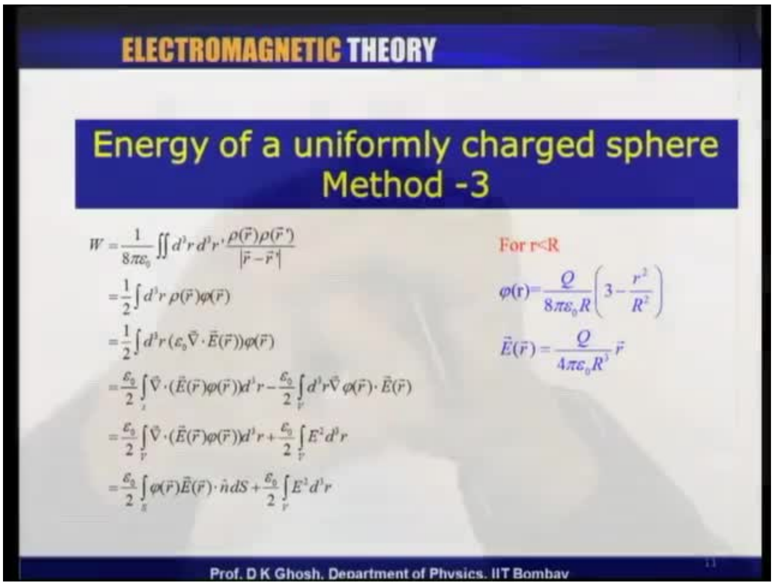

Let us look at a third alternative, the third alternative that we talk about is to include this surface term, that is do not take the range of integration to infinity, just confine yourself to the original expression. If you recall this is half d cube r rho r phi r and we had changed this we had changed this to get an expression of this type, that epsilon 0 by 2 phi r e dot n d S. This was surface term which we have thrown out and of course, the volume term which is this. Now, what is the difference between this expression and the expression that we worked out just now? Here, the surface is the physical surface which is the bounding surface of the physical volume, so in this case I have to take the surface and the volume to be the physical volume.

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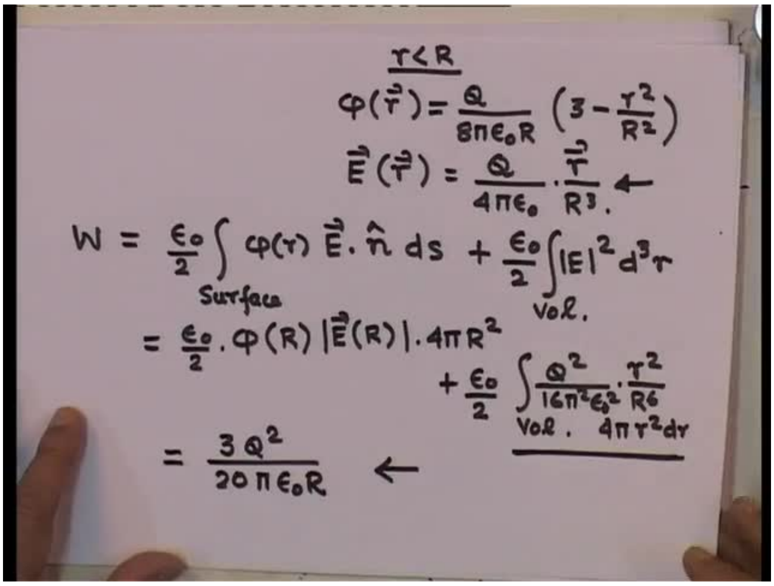

And once again I need both the expression, both potential at r and I only need I only need for r less than R and this we had seen is given by Q by 8 pi epsilon 0 R, times 3 minus r square by R square, this we had used a little while back and the electric field which you also use just now is given by Q by 4 pi epsilon 0 and this was vector r by R cube. So, let us look at how does it go? My work done has two terms, one is epsilon 0 by 2 integral over the surface, this time my physical surface, phi of r vector E dotted with n d S and plus epsilon 0 by 2 integral E square d cube r over the physical volume.

Now, notice that in this expression in this expression I need the values of quantities on the surface and the electric field is directed along the normal, because I have a sphere. So, as a result this term is epsilon 0 by 2 phi on the surface, magnitude of electric field on the surface, times the surface area which is 4 pi R square plus epsilon 0 by 2 and this is the integral only of this part. So the integral over the volume of Q square by 16 pi square epsilon 0 square and r dot r which is r square by R to the power 6 and of course, 4 pi times r square d r. Once again this integral is straight forward which we have done some time back and here phi of r you notice if you put r is equal to r this is simply 2 q by 8 pi epsilon 0 R and e of r is of course, Q by 8 pi epsilon 0, 1 over R square capital R square.

So, you can plug this numbers in and once again show that, this is equal to 3 Q square by 20 pi epsilon 0 R, so this is a third method of doing it. The difference between the first second and the third method is, that in the third method we confine our integrations both over the volume and over surface of the physical sphere, where as in the previous method we had conceptually or we had imagined that, let us take the integral over all space and in that case the surface term vanished. But there was a cost and the cost was that the, if the volume integral has to be done by taking the electric field all over the space.

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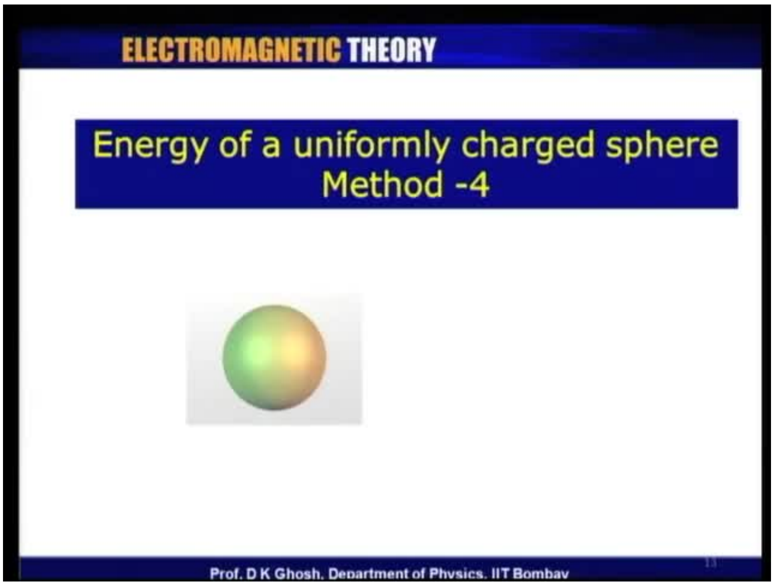

Thus, still a finer finer way of doing this and this is a little interesting way of doing it that, let us do it from the first principle let us do it from the first principle and what is first principle? The first principle is, I have a sphere which is being built up gradually, that is I bring an amount of charges d q, from infinite distance and I bring spread this charge over uniformly over a sphere that has already been created. Suppose at a given instant of time, I have already a sphere of radius, let us say small r. It has been created by the same process as I am going to describe now. Now what I do is this, this is a sphere of radius small r, now I have bring in charge, small amount of charge d q form infinite distance and spread over this sphere uniformly, thereby making a second sphere a sphere or thereby increasing the radius of sphere by an amount d r and the charge on that sphere by an amount d q.

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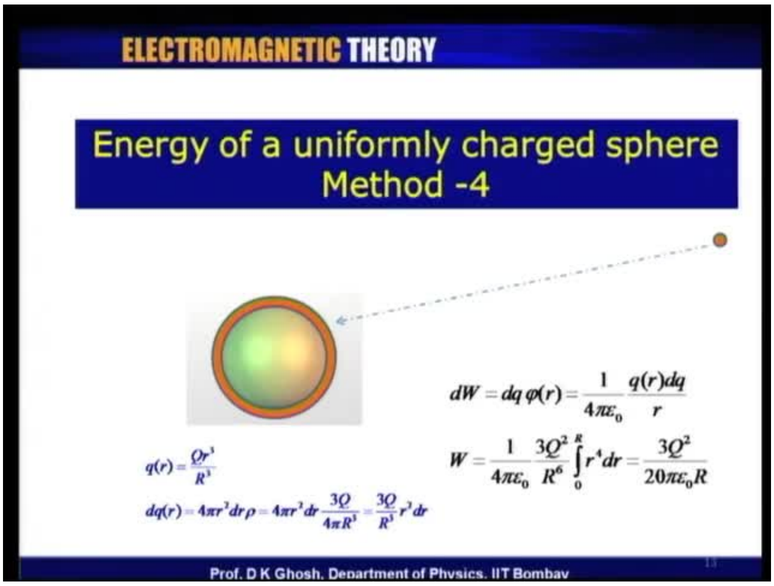

So, this is what we do, so there is the charge which is at infinite distance and its going to come there and once it has come there I am going to spread over this by increasing by an amount d r.

So, the question is this. How much work was done in bringing this charge d q, form infinite distance and spreading over the, already existing sphere. So, you notice that I had a sphere which already had, which had a radius small r, as a result it already had a charge which is q of r. And I know that is since the total final charge will be capital Q the amount of charge q r, q as a function of r is q small r cube by capital R cube. What is delta q? What is d q? d q is the extra amount of charge that is contained in a shell of which the spherical shell, which lies within radius r and r plus d r.

So, therefore the amount of charge d q is the surface 4 pi r square times d r that is the volume of the shell times the density rho. And if you put all this thing in, you find d q is 3 Q over R cube r square d r. Now how much work is done? The work that is done is, d W which is equal to d q times potential at r and I know the potential at r is 1 over 4 pi epsilon 0 times q of r by r.

Remember that, this shell is put outside the sphere. So as a result, it is as if, the field is, field appears as if the entire charge q r is concentrated at the centre. This is 1 over 4 pi epsilon 0 q r d q by r, now so what is my total amount of work done in assembling this entire charge distribution. This W is then simply I have to simply integrate over the sphere form a radius 0 to a radius capital R. And this is all I need to do this is all i need to do. This q r I plug in form here and then of course, now remember that this is simply the integration over d q and there is just an integration is to be done over d r and this gives me again 3 Q square by 20 pi epsilon 0 R.

So, this is assembling the charge layer by layer, this is assembling the charge layer by layer. What you have seen is the following that the charge, if I have a continuous charge distribution I can calculate the energy in different ways. In fact I have shown you four different ways of doing it. But let me now come back to this question, let me now come back to this question which we talked about in the beginning and that is, there is a difference between the discrete charge distribution and the continuous charge distribution.

We had seen that in case of continuous charge distribution the integration is over modulus of electric field square, which is always positive. On the other hand for a discrete charge distribution, the energy could be both positive and negative. Now, this is this is an anomaly and we have to give an answer to that question. The answer lies in the fact that, as I mention earlier that for a discrete charge collection I never question how the discrete charges were made, so as a result supposing I am, I go back and look at this expression that epsilon 0 by 2, my energy is epsilon 0 by 2 E square d cube r.

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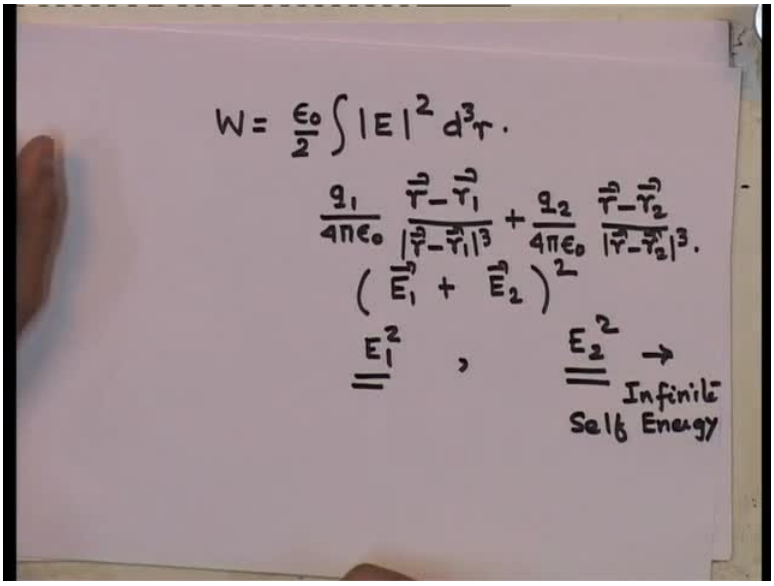

Now, let me say that I have just two charges let me say that I have just two charges. Now, I know what is the field at any position r due to these two charges. Supposing I have q one, so I have q 1 by 4 pi epsilon 0 r minus r one divided by r minus r one cube plus that due to the second charge, which is q 2 by 4 pi epsilon 0 r minus r 2 by r minus r 2 cube. Now, notice this is the electric field due to charge q 1 or q 2 at the position r.

Now, if I take the square of this, when you take the square of this, notice that electric fields satisfies the superposition principle. But square of that does not, so I will have terms which are E 1 square and I will have terms which are E 2 square and if you calculate these two they will turn out to be infinite, which is what I have been calling as the self energy. It is the cross term, which if you calculate it properly will give you the energy that we calculated for the interaction energy of the two charges before. So the difference between the two methods is due to the fact, there is an infinite contribution to the self energy which is neglected in the, our first method of calculating energy due to discrete charges. We will continue with this comment little more elaborate on it do this calculation next time.