Electromagnetic Theory
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Lecture -1
Element of vector calculus: Scalar Field and its Gradient

This is going to be about one semester course on electromagnetic theory. The essential content of the course will be to take you through electrostatic and the magneto-static phenomena, and leading to both the integral and the differential form of the Maxwell's equations. At the end of this course, you will have an appreciation of what are the important phenomena and problems connected with Electromagnetism; and later on we will go over to electromagnetic waves and if time permits some discussions on antenna and radiation.

However, this course requires a good knowledge of vector calculus and so we will be spending a bit of time in revising or giving an introduction to essential vector calculus. It will not be rigorous in the way the mathematicians would like it to be, but should be good enough for our purposes. So, the first module, which will consist of approximately 4 to 5 lectures. We will be talking about the vector calculus and its basic applications. The first lecture will concentrate on Scalar Field and it's Gradient.

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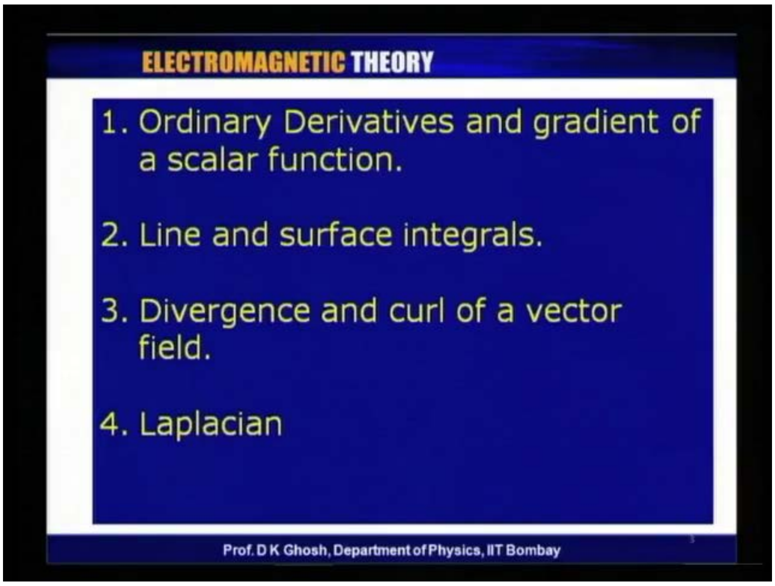

We will start with or rather recalling the concept of ordinary derivatives defines the gradient of a scalar function. In later lectures, we will bring you line and surface integrals concepts and bring out concepts like divergence and curl of a vector field and finally, I will introduce you with the Laplacian.

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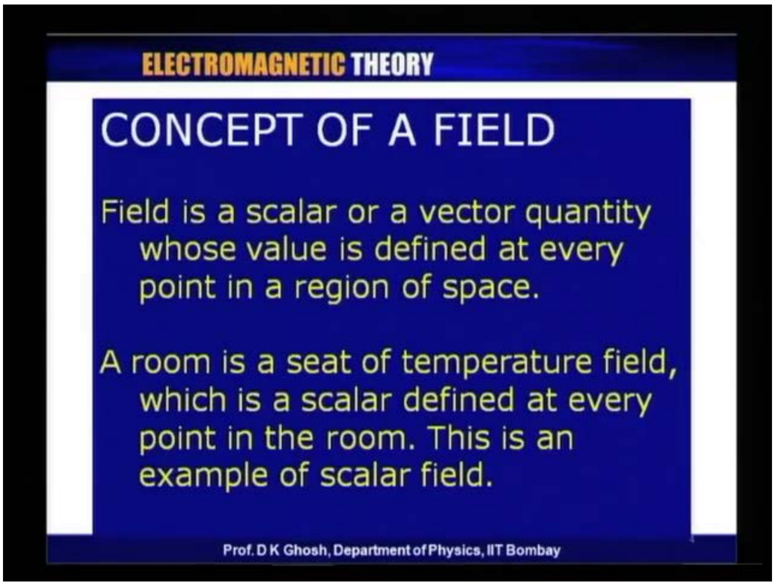

So, let me begin with, the concept of a field. Field is basically something associated with a region of space. So to understand what I mean by that, for an instance, this room in which we can be considered to be a region in which let say a temperature field which happens to be a scalar field exists.

So, whenever in a room, depending upon where you are in the room for an instance, in a room where windows are open, if you are near the window, you might find the temperature there to be higher than the temperature in the remaining parts of the room. Similarly, if you are in the kitchen you are close to a stove or an oven, you will find those areas are hotter than the remaining parts of the room. So, in other words, even though we normally talk about the temperature of a room, it is meant only in an average sense. In practice, what happens is, that every point of the room or the region of space can have a different temperature. So, we talked about temperature which happens to be a scalar quantity.

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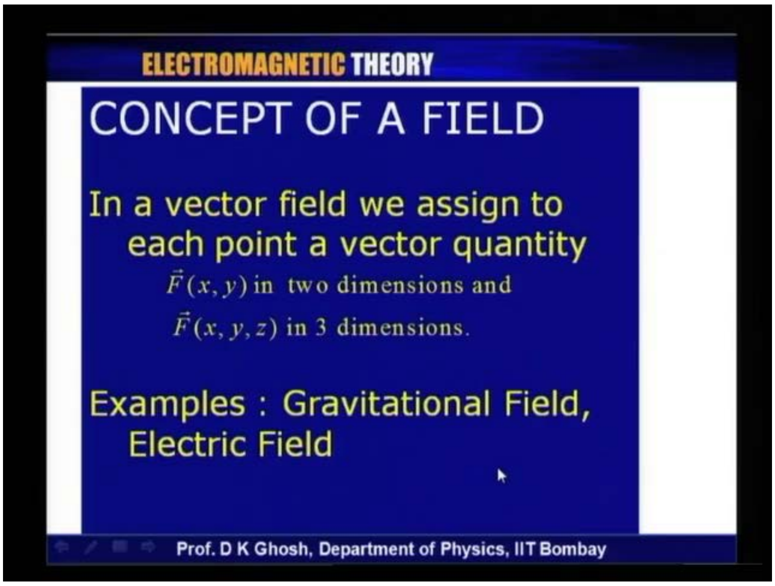

So, basically our definition of a field is a scalar or a vector quantity which can be defined at every point in a region of space and so as I said the room is a seat of temperature field and temperature being a Scalar which can be defined at every point in the room. Now, let us talk about a vector field just like a scalar. As you know, the difference between a scalar and a vector is that, a vector not only has a magnitude, but it also has the direction associated with it.

So, for instance, if I again talk about this room we normally say the gravitational field in this room is this much, which means what is the force that a unit mass which is located in this room experiences? This is connected with the fact. that the acceleration due to gravity in this room is generally taken to be constant, but we know in reality the gravity or the force of the earth on a mass at any point of the earth depends upon the location of that mass on the earth.

So therefore, if I am talking about associated, a force with every point in space in a particular region of space, I am in what is known as a force field which is a vector field. So typically, supposing I am in two dimension then vector f which is a function of x and y in two dimension and similarly, in three dimension I would write the vector f as being dependent on the positions x y and z for our purposes other than gravitational field, the electric field and the magnetic field are examples of a vector field. So, as we said, a scalar field is a number associated with every point. It is very easy, what we could do is for example, take a graph paper and wherever we want we can associate a number we can write down.

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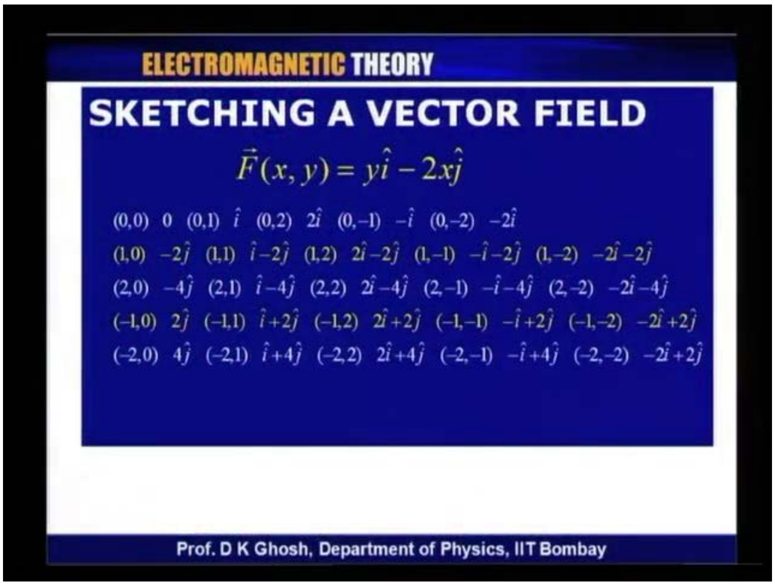

Since, a vector field has both magnitude and direction, it can be very nicely and graphically sketched. I am giving when example of a field written as f of x, y which is two dimensional vector field which is y i minus 2 x j.

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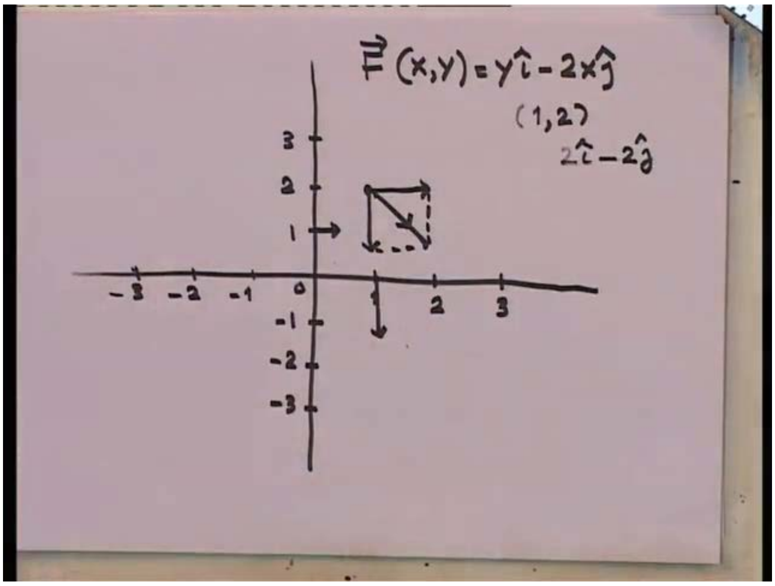

Now, let me illustrate how does one represent the two dimensional field on a graph paper. For instance, so let me try to draw a graph, I will take some arbitrary units, this is the origin, this is 1, this is 2, this is 3 etc and similarly on this side, I have minus 1, minus 2, minus 3 repeat that on the y axis. Now, let us look at some specific points. Now notice that at the origin my value of x is 0 and y is 0 so as a result the Field is 0 there, so I will since 0 does not have a vector direction associated with it, so we just put a point there. Now, let us look what happens for example, at the point 0, 1 so notice 0, 1 is this 1. Now, if it is 0, 1 since x is 0, y is 1, the value of the vector field at this point is simply y. Now which means, it is in the x direction, in the x direction having a magnitude of 1.

Now, what I do is, I take an arbitrary scale. So let us suppose, I decide that this is my 1 unit vector from here to here. Since it is an i direction, I just write it here and I can scale it like that. Now, let us come for example, to the point 1, 0 since x is 1, y is 0, .You notice this is minus 2, but in the j direction, minus means in the negative y direction and since its length is twice as much, I represent it by a vector by twice the length of this. Similarly, let us take the point 1, 2 which is the point here. So, let us look at, what is 1, 2. Since y is equal to 2 this is 2 i and x is equal to 1 it is minus 2 j.

So, what is 2 i minus 2 j. It is a Vector, which I was first take 2 units along with x direction and another 2 units along with y direction and take complete the perpendicular and find the resultant so therefore, the field at this point is represented by a vector along this. So, what I have done is this, that you could go on, you could take various representative points on a graph paper and sort of try to draw your own grade some of the values on a graph paper are given here, but you could do it yourself.

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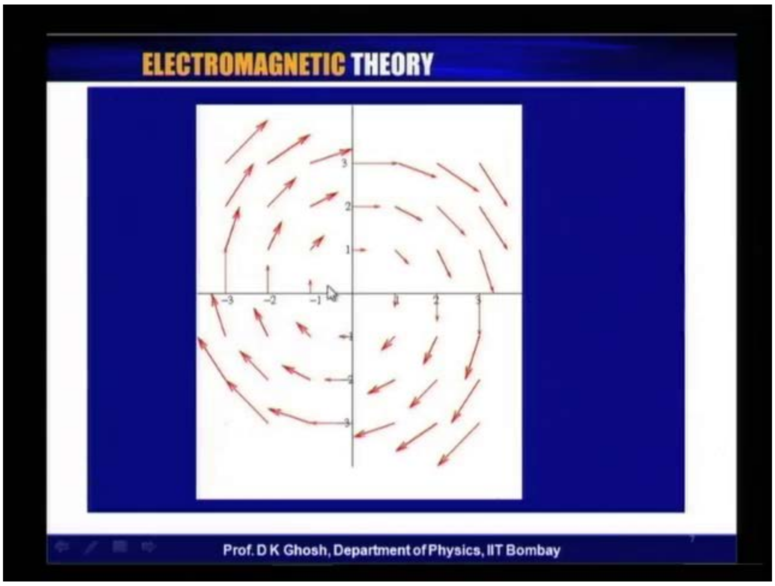

Now if you take a large number of points and draw this according to this, you will find the vector field has a pictorial representation of this type. It is it is pictorially represented like this. Of course, if you are doing it yourself manually there are just so many vectors you can draw, but if you want to get an idea of what it looks like, you could take some computer package like for example, Mathematica.

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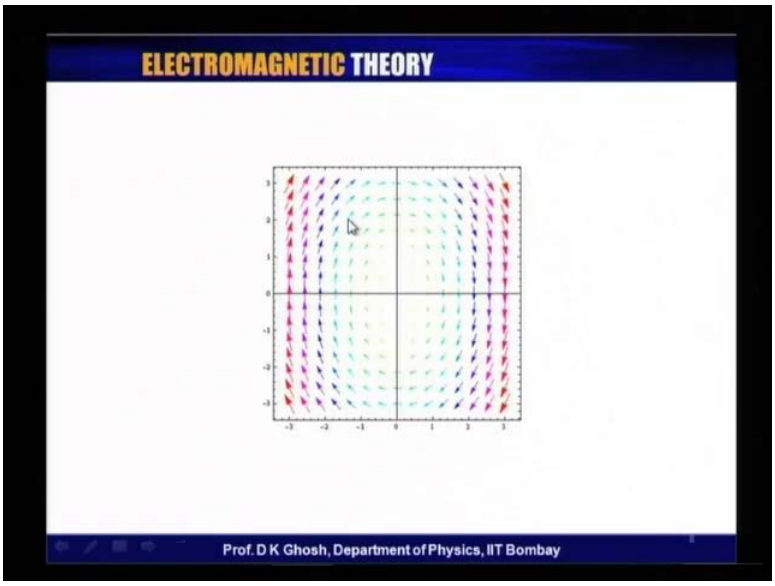

So, this is what, the Field would look like if you have or you are going to draw it over a very large number of points. I am not going to be able to draw it in three dimension, but on the other hand, it is possible to get a computer simulation of vector fields in three dimensions as well. Well, since we will be primarily concerned with electric and magnetic field.

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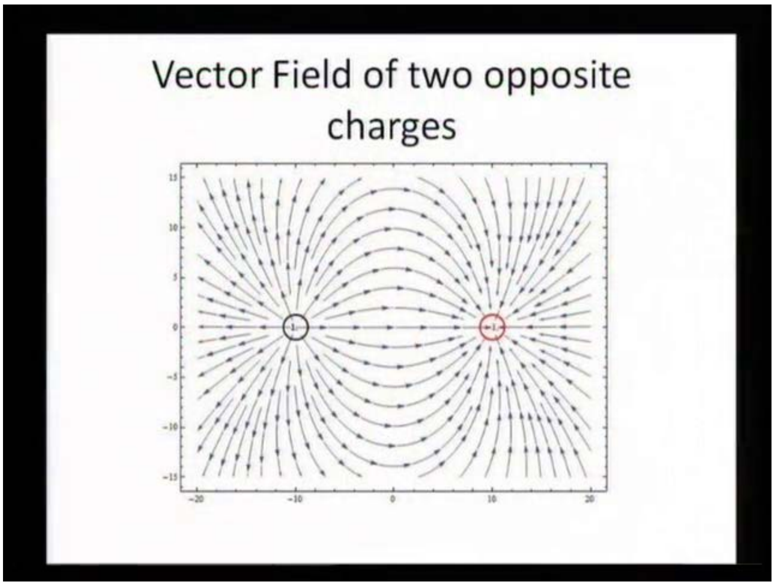

Let me illustrate, the vector field concept with something that you already have learnt in your schools. Here what we have done, I have a positive charge which is represented with this and a negative charge which is represented in this red color. Now, supposing I have to draw the vector fields these are incidentally called sources. As you know, that if you put a, if you have a positive charge and you put another small positive charge near it that charge is repelled. In other words, the direction of the force supposing you put it here; the direction of the force on that charge will be from this charge to that charge and it will be repelled and if the charge happens to be attract negative then it will be attracted.

So, this is what the vector fields would look like for a region which has a positive charge and a negative charge. Now, these lines are so close or the lines along a particular direction are so close that will normally connect them by a continuous curve and these are generally known as lines of force. So, in other words, that if you have drawn the lines of force, if you want to know what is the direction of the vector field at this point, so you have to take draw a tangent to the field of that point and that will give you the direction of the vector field.

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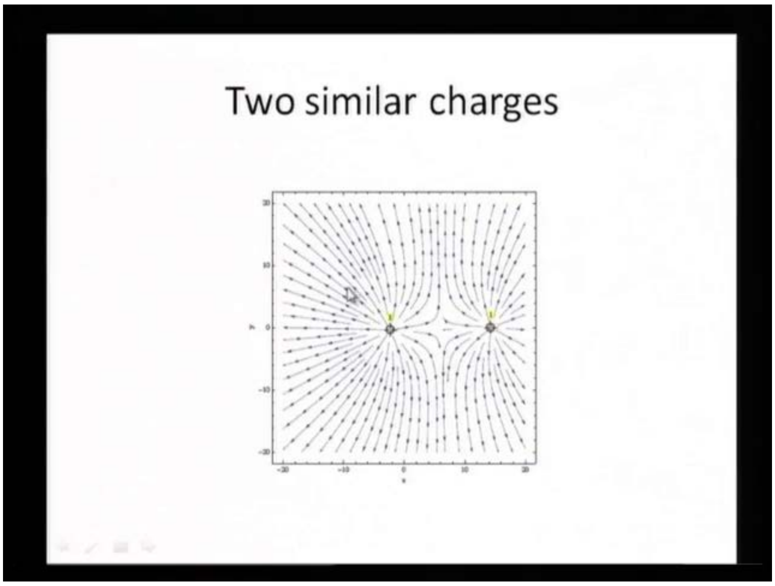

Well, the next picture simply shows what happens if both the charges are similar, so in this case both the charges are positive. Therefore, you know that the it is repelled from both the charges. We will of course, comeback to discussion of lines of forces later when we look at the electrostatic field. So, let me now begin the discussion of calculus that we talked about, but before I do that, let me remind you of the definition of an ordinary derivative of a function.

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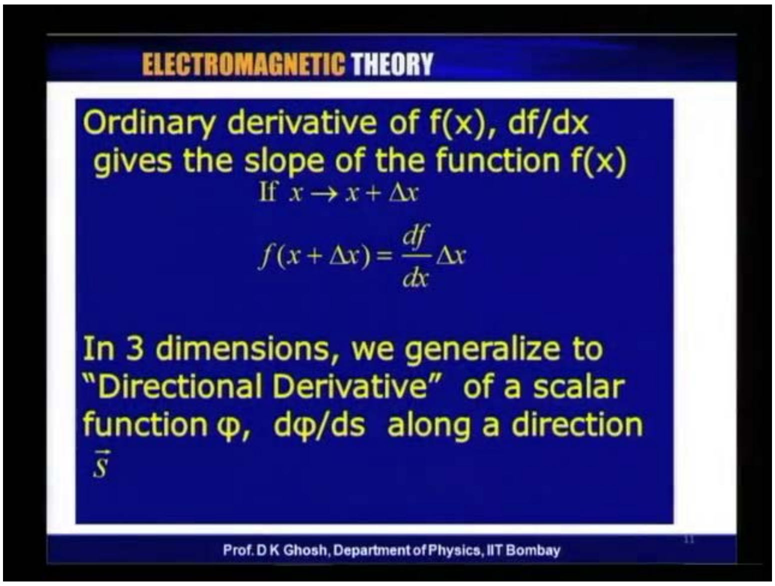

Let us look at how you define ordinary derivative in your school.

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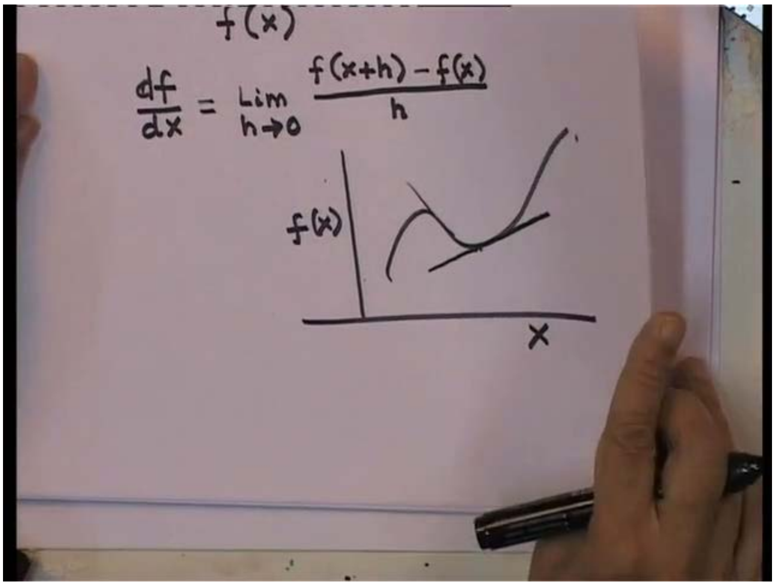

If you recall that is f is a function of a single variable x then, we define df by dx at the point x as limit of the following quantity. I take the value of the function f at the point x plus h where h is a small quantity subtract from it. The value of the function at x and divide it by the increment h and when you take the limit of h going to 0 this is my definition of a derivative.

Now, what I have done here, is to actually re-write so that if this is the definition this means that the value of f at x plus h is simply equal to its value at x plus h times the value of the derivative of the function at the point x. Here I talked about h because that is the way we normally talk about in school and here I have used instead of h a delta x so it tells me that the value of the function at x plus delta x is equal to df by dx times delta x.

Now, the question is this, suppose I want this happens in one dimension, what happens if I want to extend it to two or three dimension? Now, we have a problem, when we with one two so let me first tell you how does it interpret. Supposing, this is the graph of function with x so I can draw an arbitrary graph you notice that if I am at a given point and I draw a tangent to it that is the definition of the slope at that point geometrical interpretation.

In other words, at this point the slope is positive which means the value of the function is increasing on the other hand, if you are here for instance and then you notice that the slope is negative, so slope essentially gives us an idea about the rate of increase or decrease of the function. Now, if I want to now extend this concept to higher dimension so let me try to extend it to two dimension, first. Now I cannot obviously draw the function in two dimension.

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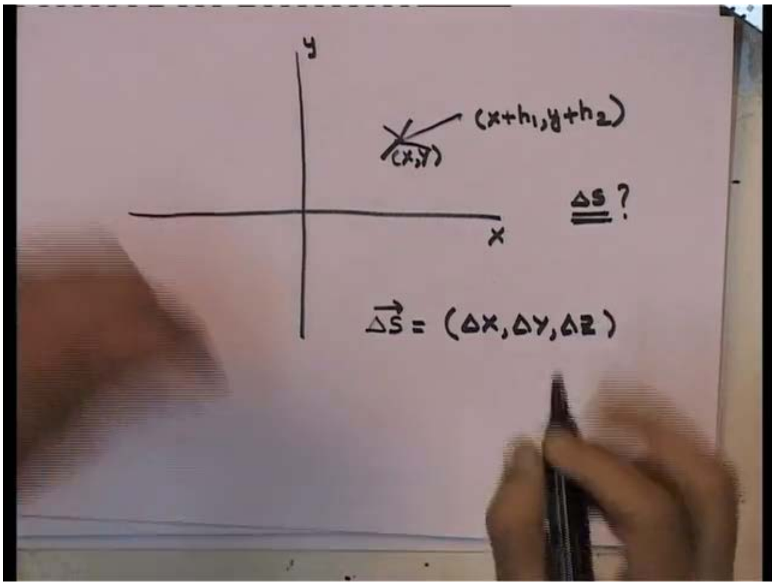

The reason is I need x and y and if at all I am going to draw the function it has to along the z axis which I do not have, but let us not worry about it right now. So, supposing I am at this point some value of x, y and I want to find out some value of f at this point x, y and let me say I want to find out the value of the same function so this is my point x, y and this is the point x. Let us say plus well if you like h1 and y plus h2. Now, the point is the following, that if I want to go from this point to that point this of course, since I have two points the direction is clear, but you notice that if I tell you, what happens to the value of the function? What happens to the value of the function? As I go away from this point x, y by an amount delta s for an instance?

In one dimension delta s could be in either of the two directions just positive or negative, but here once I am told that I have I have to move by an amount delta s, I can move in any direction. In other words, it would depend upon the direction in which I move and we introduced what is known as the directional derivative.

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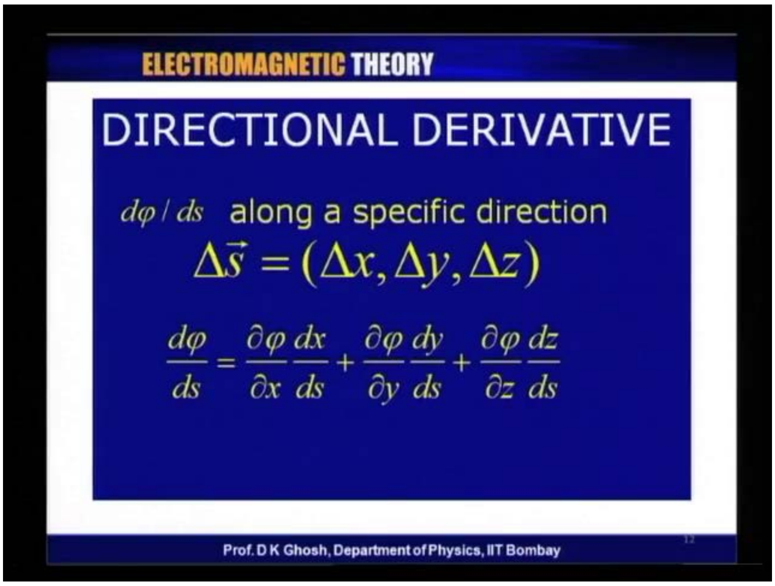

So what happens is this, that I am then asking that suppose you move by an amount delta s move by an amount delta s along a specific direction along a specific direction which we will represent by delta x, delta y and delta z, this is my directional derivative. So, look at what does the directional derivative imply, see I have, d phi by d s. Now in going along this direction, I could actually go.

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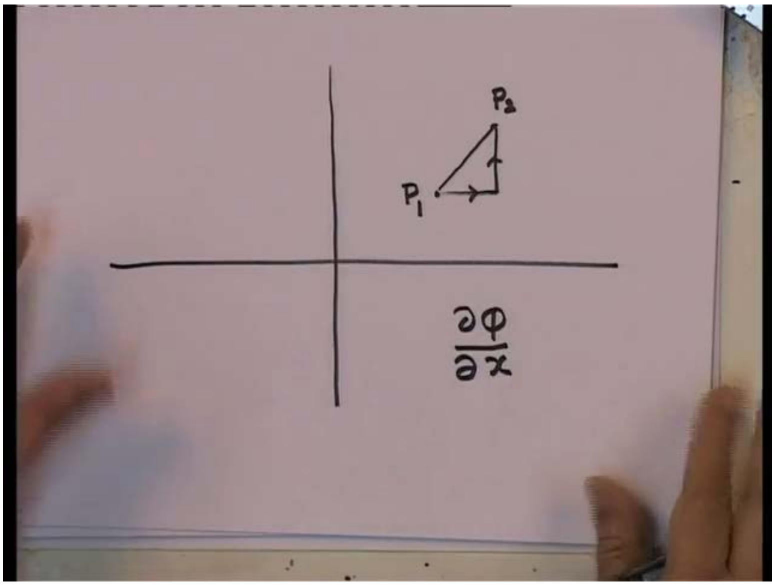

Suppose I am going from this point, let say this point. So this is point number 1 to point number 2. Well I need to go like this, this I can calculate in many ways, what I could do is to say that well I will go like this and then like this, along this y is constant, along that x is constant, so what we do is to say look d phi by d s d phi by d s I take a special type of derivative which is not written as d phi by d s, but it is written as del phi by del f, del x this is some people call it partial phi with respect to x.

So, this is partial phi with respect to x, meaning move along the x direction, keeping y and z constant. So, d phi by d s is partial with respect to x and of course, then I need to multiply with d x by d s, partial with respect to y d y by d s, partial with respect to z d z by d s.

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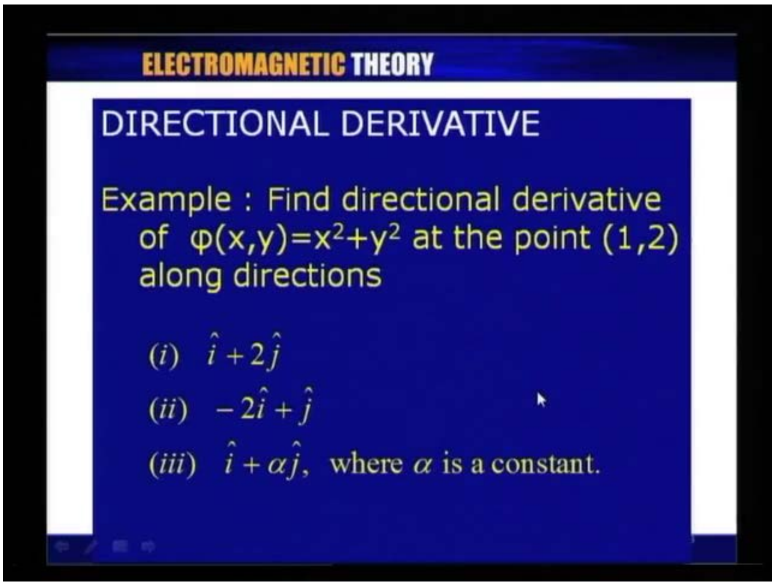

Now, so that is my definition of a directional derivative. Now, what I am going to do is, I am going to illustrate the calculation of directional derivative by taking a specific function phi, this is I will take it in two dimension for convenience, you could repeat it in three dimension.

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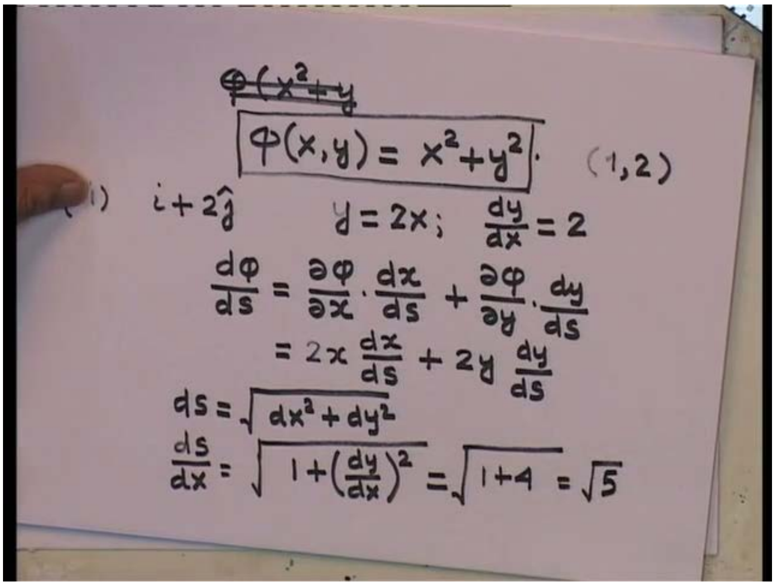

So, phi x, y equal to x square plus y square and I will calculate the directional derivative at the point 1, 2, but along three different directions, but let me first illustrate the first one that I want to calculate directional derivative along i plus 2 j, let us do that. First let us look at what does this function look like, so this is a picture which is, which comes from certain computer plot so here, I have x, y plane these are the coordinates and the function is plotted along the z axis. Now, if you plot phi x, y which is equal to x square plus y square in such a three dimensional plot what you find is a cup like structure a cup like structure, so this is what it is.

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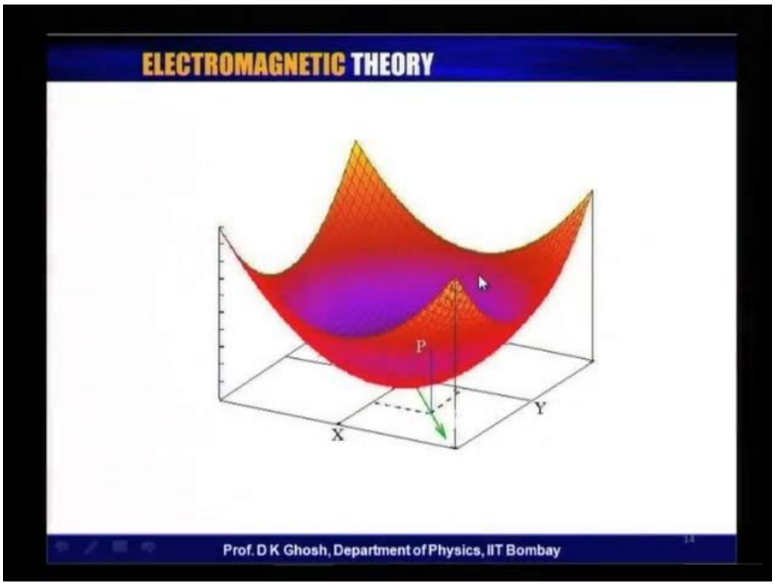

So, what are we are trying to say is this, suppose I have I know the value of the function at let us say, this point this is x y plane and this I know is this much. Now, what happens if I go by an amount delta s let us say in this direction? So, I have to climb up put a perpendicular here and know what the value is. Let us come back to the formulae that you wrote down. We said d phi by d s, I will cut out the third direction is partial phi with respect to x, d x by d s plus partial phi with respect to y, d y by d s. I do not have the z component. So, I told you that partial derivative means treat the other variable as constant.

So, y is constant so this is nothing but, 2 x, d x by d s plus 2 y, d y by d s. Now, let us compute we know we know that d s is nothing but, square root of d x square plus d y square . So, if I am calculating d x by d s. I will actually calculate d s by d x and take the invert it, so d s by d x is square root 1 plus d y by d x whole square and this is okay. This quantity is what we are interested in. Now notice, that I have a particular direction i plus 2j. Now, along that direction if you look at i plus 2j direction. The relationship between y and x is, y is equal to 2x. Now if y is equal to 2x, then d y by d x is just equal to 2. So, this is simply, 1 plus 4 which is equal to square root of 5.

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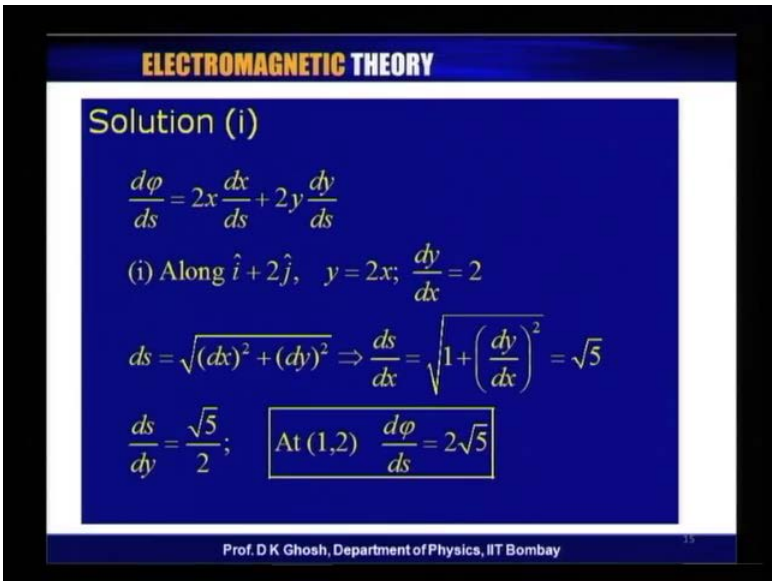

In an identical way I can compute what is d s by d y and of course, I need d x by d s, which is then 1 over square root of 5 and this of course, is 2 by square root of 5. So, plug these things in there and compute what is d phi by d s which will work out to 2 root 5.

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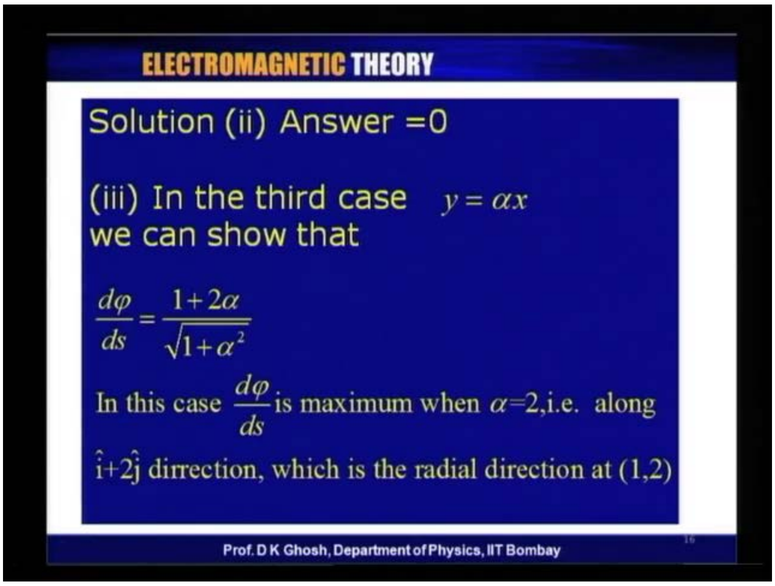

The second part of the problem I will not do, but I would expect you to do it an exercise, but I want you to look at the third problem a little carefully. Here what we have asked?

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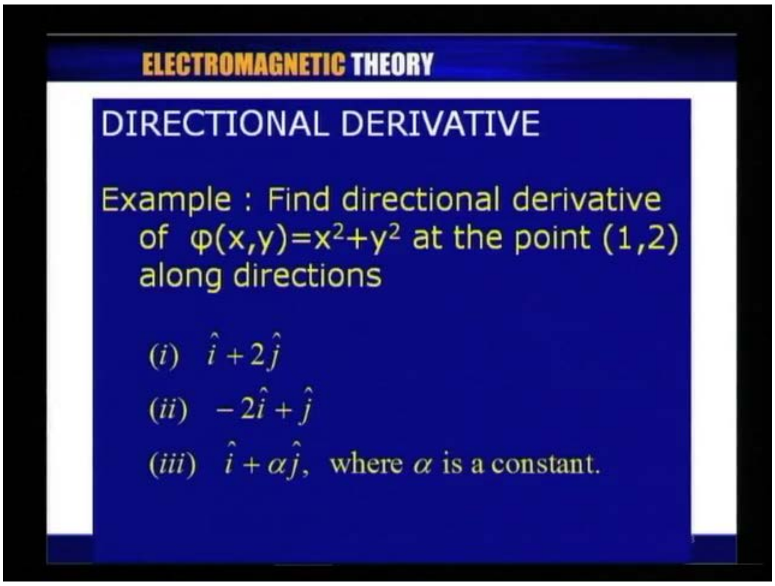

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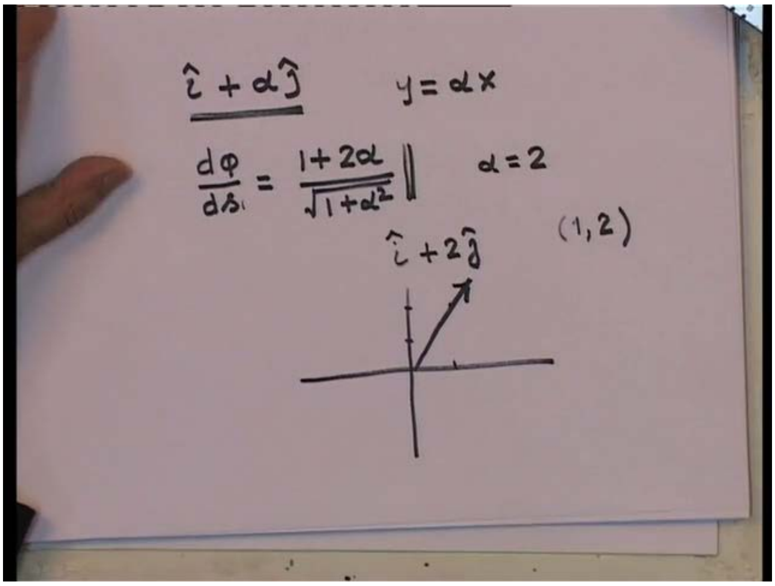

We asked that we wanted along the direction i plus alpha j where, alpha is a constant. Now, along this line y is equal to alpha x. Now, you can essentially repeat the same calculation that I did for the first part of the problem and you can show that the directional derivative of the function phi, d phi by d s is along this direction is given by 1 plus 2 alpha divided by root of 1 plus alpha square.

Now, if you examine this function you will find that the directional derivative takes a maximum value when alpha happens to be equal to 2, you of course, know how to do it. You know that I want maximum of this so I differentiate this one put it equal to 0. Evaluate what is alpha? Now, if you take alpha equal to 2, I am saying that the directional derivative is maximum in the direction i plus 2j, but notice I was calculating this this directional derivative is calculated at the point 1, 2. Now, if you look at, if you look at the graph the point 1, 2 is this point at the direction i plus 2 j the direction i plus two j at this point is a radial direction. So, therefore, the directional derivative for this function takes the maximum value along the radial direction.

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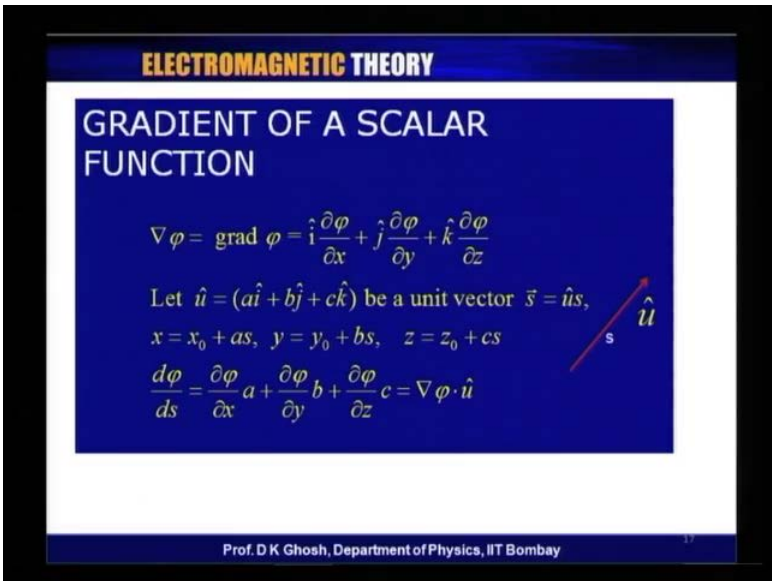

Now, let me now define what is meant by Gradient of a function.

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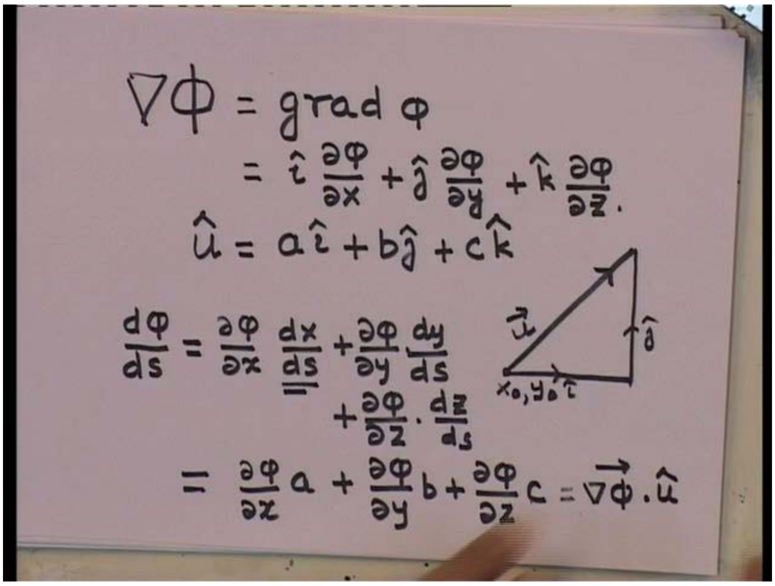

Now, this Gradient is written using a symbol which is called a grad. It is a inverted triangle like thing, you also write is we call it as del phi, we also write it sometimes in word as grad phi. We define grad phi, we will comeback what it means a little later, as unit vector i partial of phi with respect to x plus unit vector j partial with respect to y plus unit vector k partial with respect to z.

So, let us suppose, I have a unit vector u which has components a i plus b j plus c k. Remember I have said, u is a unit vector, in other words, a square plus b square plus c square must be equal to 1. Let let us, just represent it by a vector like this. Now, we are at this point this is the direction of the unit vector as shown here, this tells me that if I am at the point x 0, y 0 and I want to go to a point let say x y, then my x is equal to x 0 plus a s, s is the total length by which I am moving in this direction y equal to y 0 plus b s, z equal to z 0 plus c s.

Trivial to see it in two dimension because supposing this is s supposing this is this is s then this is the direction of the unit vector this is the direction of i this is the direction of j. So, clearly, this is nothing but, the cos theta which is the angle between the unit vector and the x direction and therefore, this falls. Now, with this d phi by d s which if you recall we had written partial with respect to x, d x by d s etc. This is the gradient phi this is the gradient phi definition this is the gradient phi definition and u is a unit unit vector like this so this is nothing but, d so what is this quantity since we have seen what is x is x 0 plus s, y is y 0 plus b s etcetera etcetera etcetera. This tells me that this d, d x by d s which is what I need here x 0 is a constant is nothing but, a.

Similarly, d y by d s is b z by d s is c. Therefore, I write this as d phi by d x, a plus d phi by d y, b plus d phi by d z, c. But a, b, c are components of a unit vector along the three direction d phi by d x, d phi by d y and d phi by d z are component of the vector grad phi. So this is nothing but, the scalar or the dot product of grad phi with the unit vector u.

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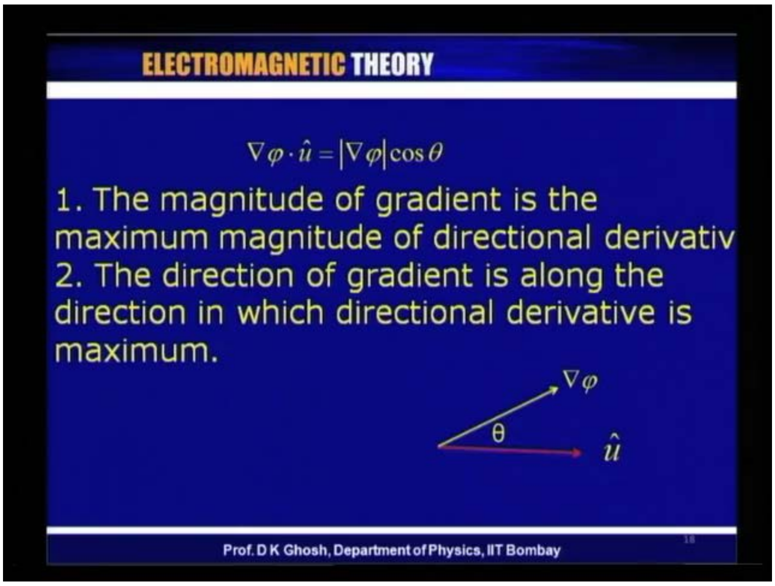

Now, once we have derived this, let us look at what does Gradient mean, u is a unit vector. As a result, grad phi dotted with u is magnitude of grad phi magnitude of u is one times cosine of the angle between the gradient direction on the unit vector direction. Unit vector, is the direction in which you want to move.

So since, the maximum value of cos theta is 1, it tells me that and the cos theta becomes 1 when theta is 0 which means the direction in which you are moving is along the gradient direction therefore, the magnitude of gradient is maximum magnitude of the directional derivative and the direction of the gradient, is the direction in which the directional derivative is maximum this is what illustrates this.

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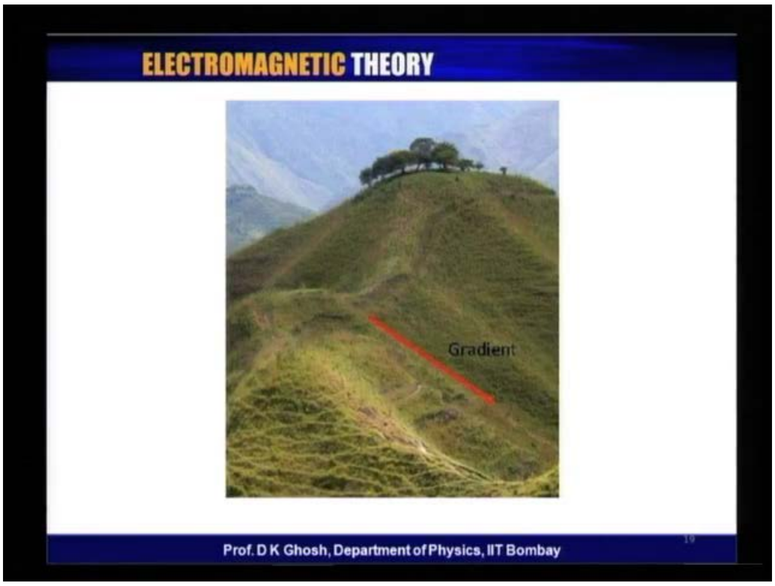

Well, what does it actually mean, physically supposing you are in a hill and you are not x quite at the top, but let say somewhere in the middle and you want to come down. Now, there are many ways of coming down the slope, but let us suppose you want to move by a given distance if you want to move by a given distance, the fastest you will go is if you move along the direction of the steepest slope the steepest and that is the direction of the gradient. That is the direction of the gradient.

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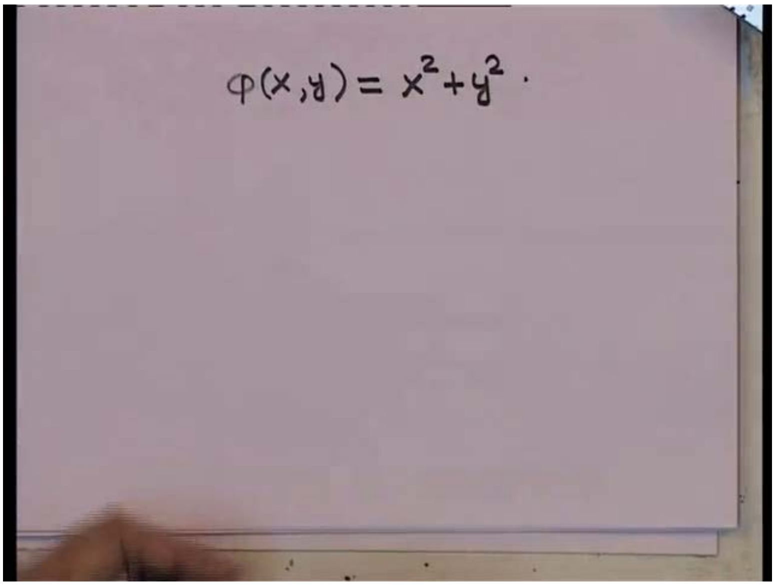

So, let us return back to that function phi x y. Well we do not really have a z there so equal to x square plus y square.

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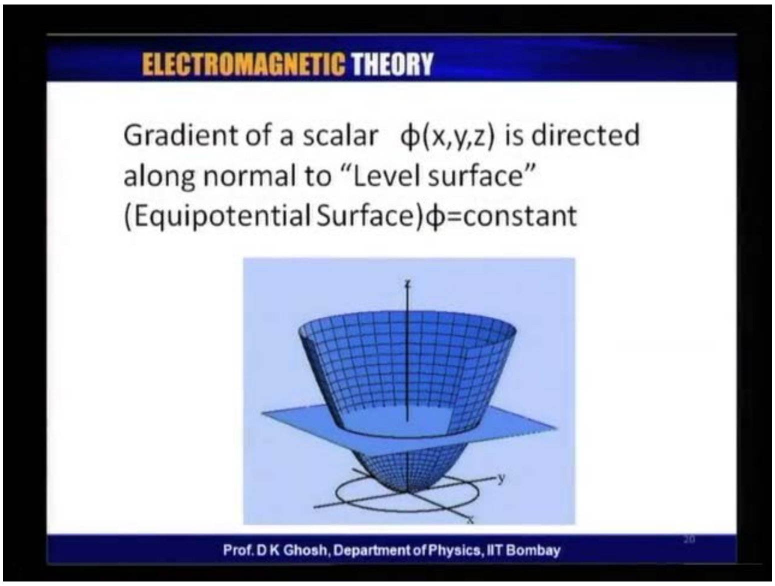

Now, notice that, the gradient we just approved is in the direction in which the slope or that amount of change the rate of change is maximum. Therefore, it follows that that if I have a surface phi x, y, z, the gradient must be perpendicular to the surface along which the value of the function does not change. So, let us the same picture let us look at it slightly differently.

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Now, this is the picture that we drew of the function x square plus y square the access etc era are removed there. Now, if I have to look at the intersection of this surface with a plane, so this is a plane. Now, obviously a plane on which the value of the function will remain constant will cut this surface in a circle so that is my level surface.

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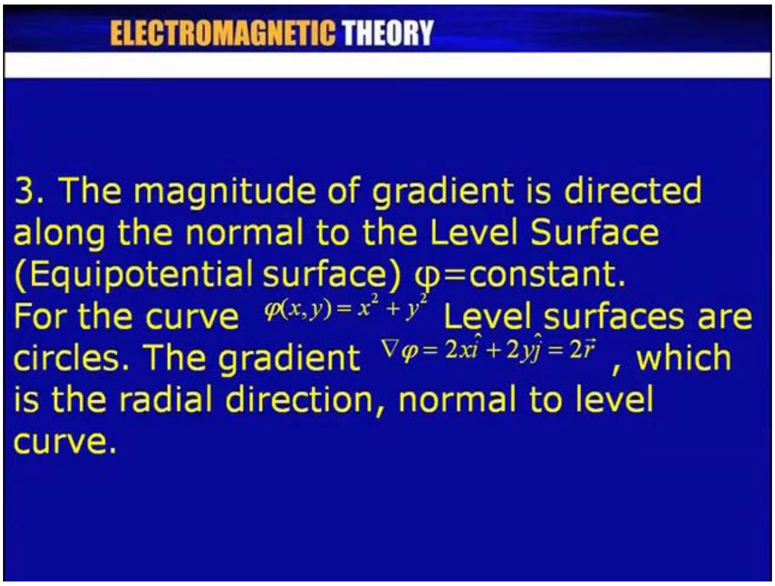

In physics, we are familiar more with what is known as an equipotential surface, that is a surface on which the potential is constant. So, what we have proved is this, the magnitude of the gradient is directed along the normal to the level surface or in our case the equipotential surface so this is the surface on which phi is constant.

Now, we know that for the curve that we have been talking about for the curve we are talking about phi x, y, x square plus y square. x square plus y square equal to constant will give me the level surface, but what is x square plus y square equal to constant this is nothing but, family of circles and these are the circles in which the flat plane cuts that picture of that I showed you some time back.

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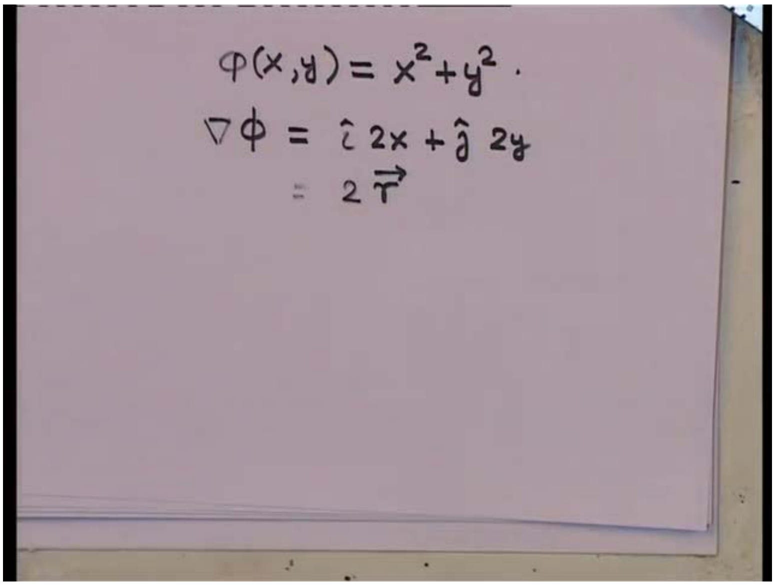

Now, so gradient of phi of this function remember, i times d phi by d x is partial phi by partial x is 2 x plus j times 2 y, which is nothing but two times the radial vector radius vector radial vector. So, this is nothing but, along the radial direction and which is normal to the level curve.

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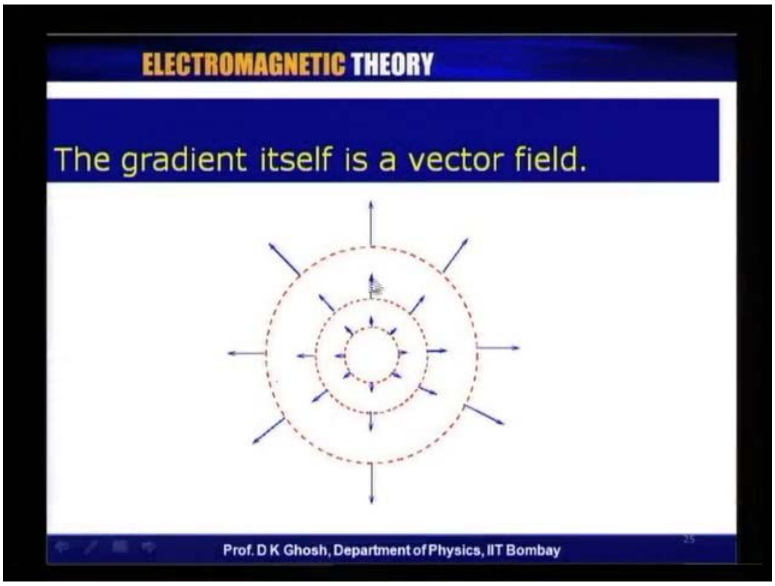

So, this is what is shown here that I have chosen three circles as represented, notice as the radius of the circle increases the gradient increases and as a result what I have done here in this picture is to show these as radial vectors, but when the circle is smaller my vector magnitudes are smaller, they are radial as the circle becomes bigger and bigger the arrow lengths become bigger and bigger.

So, in other words, gradient itself is a vector field so please understand this gradient of a scalar function is a vector. Gradient of a scalar function is a vector it is a vector because its direction its direction is normal to the level surface. So, this is a quantity which has both magnitude and direction and as a result it is a vector field.

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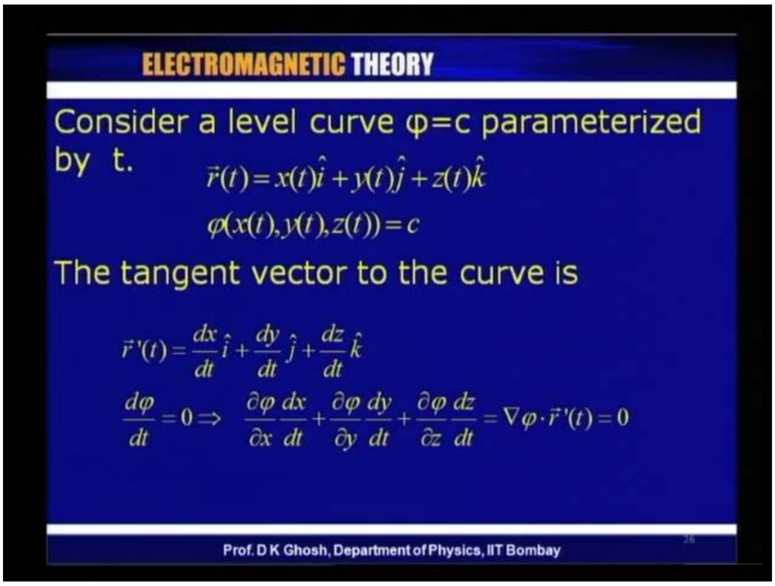

I want to prove this formally.

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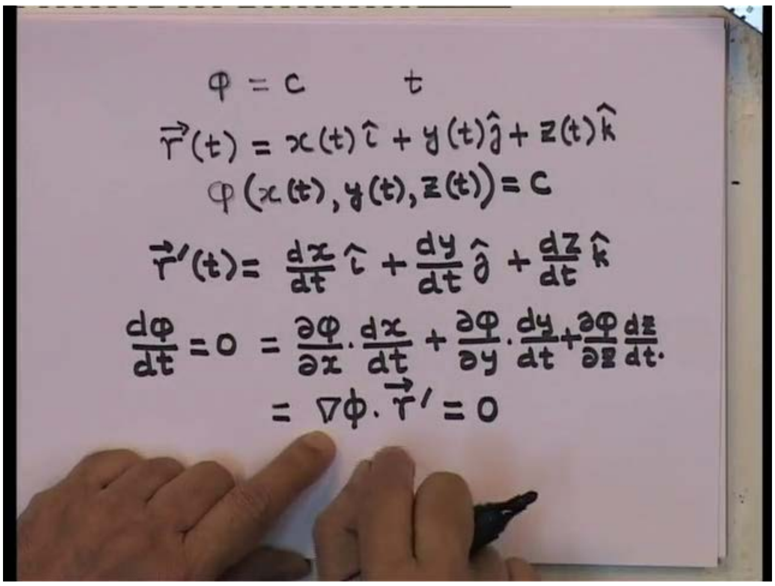

So, we have talked about a level curve well in the example that I gave you the level curve or the circle, but supposing I am looking at a level curve for which phi is equal to c, which is the constant and let us say that this curve is parameterized by a variable t. Therefore, on the curve I write down r of t equal to x t, i plus y t, j plus z t, k and my function phi which is a function of x, y and z can then be written a s phi x t, y t, z t equal to the constant c.

Now, let us write down, the tangent vector to the curve let us write down the tangent which is written as let us say r prime of t. So, this is because it is parameterized by t the tangent is nothing, but the derivative of this quantity with respect to t so this is (d x by d t i plus d y by d t, j plus d z by d t k. Now, I want when is d phi by d t is 0, because I am looking at a level surface. The value of the function is constant so d phi by d t is 0, but d phi by d t is according to our earlier discussion partial with respect to x, d x by d t plus partial with respect to y, d y by d t plus partial with respect to z, d z by d t.

Which is if we look at what is r prime and the definition of the gradient is nothing but, is nothing but , the dot product of grad phi with the r prime vector and this must be equal to 0 this must be equal to 0. So, this tells me that the gradient of phi the gradient of phi is normal to the tangent vector is in other words it is normal to that curve.

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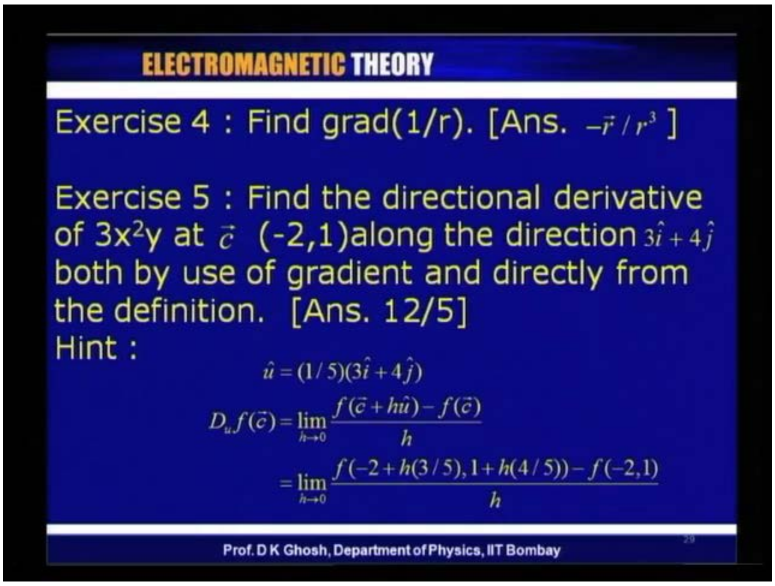

Now, one could try to do some of these things from first principle.

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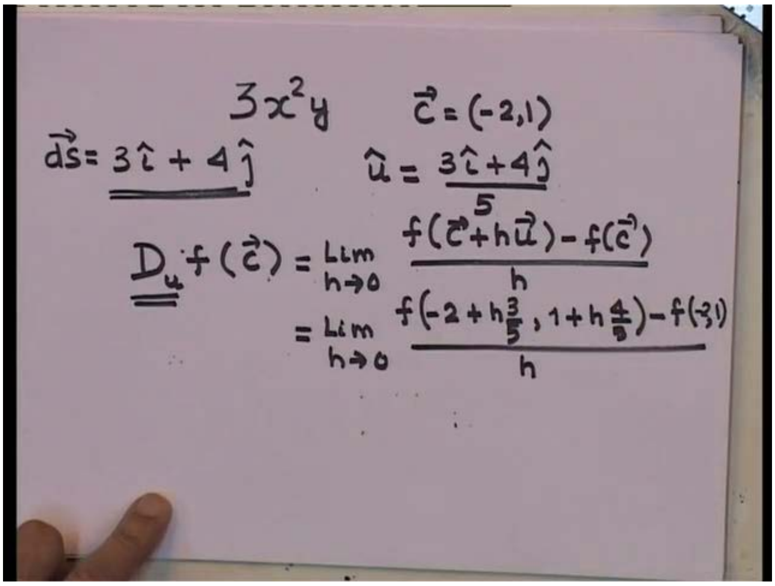

For example, I am going to illustrate a partially do a problem that, let me take a very simple function 3 x square y and I want to you find out the directional derivative of this, at a point c which is minus 2, 1 along the direction 3 i plus 4 j. Now notice, this is my, this is not a unit vector this is what I represented earlier as d s. The direction s in order to make it a unit vector and call it u according to our previous notation I must divide it by the magnitude of this vector namely square root of 3 square plus 4 square so which will be 3 i plus 4 j divided by 5.

Now, what is the directional derivative of f at this point c. This is this is notation which mathematicians use, directional derivative along the direction u of the function at the point c. Now, from first principle, this is limit h go into 0, value of the function at c plus h u minus the value of the function at c divided by h. Now, this is a vector so I could rewrite this as limit h going to 0 of f at minus 2 plus x direction is 3 by 5 so it is h times 3 by 5 , 1 plus h times 4 by 5 minus f of c which is f of minus 2, 1 and divided by h. You can now of course, plug in this, and you all this is a very simple derivative to calculate.

So, let us look at what have we done today, we started with a review of elementary derivatives as we have learnt in school and extended it to two and three dimensions. In doing so, we also introduced the concept of a Field, we found that we can have either a field could be a scalar or a vector.

A scalar field, for instance, a temperature field is one where the region of space at every point in a region of space at every point you have a scalar quantity associated with it. In case of a vector field, we in a region of space at every point we associate a vector. Now, having defined the scalar and the vector functions, we concentrated primarily on scalar field today and defined what is meant by a gradient and we found that gradient of a scalar is a vector which is directed along the normal to the level curve or level surface, and it is the direction along which the change in the value of the function is maximum. In the next lecture, we will be talking about quantities associated with a vector Field and go ahead from there.

Thank you.