

Computer Methods of Analysis of Offshore Structures
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Module - 02
Lecture - 16
Damping Estimate (Part – 2)

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- Example problem,
- Rayleigh damping
- Matlab code for Rayleigh damping

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In both the damping models, the actual behavior of the structure does not depict either of them.

- verified experimentally

found that actual damping is neither mass-proportional nor stiffness-proportional

Rayleigh damping :

$$C = a_0 M + a_1 k \quad \text{--- (F)}$$

(combination of mass & stiffness proportional)

damping ratio @ n^{th} mode is given by

$$\xi_n = \frac{a_0}{2} \frac{1}{\omega_n} + \frac{a_1}{2} \omega_n \quad \text{--- (F)}$$

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Rayleigh proposed damping which includes both C is going to be a 0 m plus a 1 k, therefore it is a combination of mass and stiffness proportional. In that case damping ratio at n-th mode is given by you know the damping ratio for stiffness proportional is equation 6.

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(i) keeping $[c]$ proportional to modal damping ratio,
for a mass-proportional damping system, we get

$$C_n = \alpha M_n \quad (1)$$

damping ratio will be $\zeta_n = \frac{C_n}{2 M_n \omega_n} = \frac{\alpha M_n}{2 M_n \omega_n} \quad (2)$

$$\zeta_n = \frac{\alpha}{2 \omega_n} \quad (3)$$

Damping ratio (ζ_n) is inversely proportional to the natural frequency of the system.
Therefore, α can be selected to obtain a specified value of damping ratio in any mode of design's choice.

Damping ratio for mass proportional is equation 3, therefore damping proportional for a combination of this could be simply the addition of this that is a 0 by 2 1 by omega n because for mass proportional damping is inversely proportional to omega plus a 1 by 2 omega n because for stiffness proportional damping ratio is directly proportional to omega n. I call this equation number 8. Of course, this is my equation number 7 given by Rayleigh.

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$$\xi_n = \frac{a_0}{2} \frac{1}{\omega_n} + \frac{a_1}{2} \omega_n$$

Coeffts (a_0, a_1) can be determined for a specific damping ratio (ξ_i, ξ_j) for its i th modes, respectively.

Pick up ξ in such a manner that it is same for (ω_i, ω_j)

$$\xi_i = \frac{a_0}{2} \frac{1}{\omega_i}$$

$$\xi_j = \frac{a_1}{2} \omega_j$$

$$\xi_n = \frac{a_0}{2} \frac{1}{\omega_n} + \frac{a_1}{2} \omega_n$$

The graph shows damping ratio ξ_n on the y-axis and frequency ω_n on the x-axis. A green curve represents mass proportional damping $C = a_0 M$, which is inversely proportional to frequency. A red line represents stiffness proportional damping $C = a_1 k$, which is directly proportional to frequency. A blue curve represents the combined Rayleigh damping $C = a_0 M + a_1 k$. The intersection of the green curve and the red line is marked with dashed lines extending to ω_i and ω_j on the x-axis, and ξ on the y-axis.

So, therefore, once we said ξ_n as actually $\frac{a_0}{2} \frac{1}{\omega_n} + \frac{a_1}{2} \omega_n$. The coefficients a_0 and a_1 can be determined for a specific damping ratio ξ_i and ξ_j respectively for i -th and j -th modes. So, now, the damping will look like the plot as shown now the original damping which is mass proportional is $\frac{a_0}{2} \frac{1}{\omega_n}$ and this is the ξ_n . The mass proportional damping looks inversely proportional to ω_n increase this is I should say C is equal to $a_0 M$ whereas, the stiffness proportional damping is directly proportional to ω_n . So, I should say in this case C is $a_1 k$ Rayleigh gave a combination of these 2 it says that let it be a combination of mass proportional and the stiffness proportional, this is Rayleigh damping.

So, for any specific value of ω_i and ω_j , this is ω_i this ω_j I should be able to get the same ξ , so pickup ξ in such a manner that it is same for ω_i and ω_j as shown in the figure. So, ξ_i is going to be $\frac{a_0}{2} \frac{1}{\omega_i}$ and ξ_j is $\frac{a_1}{2} \omega_j$ hence ξ_n is $\frac{a_0}{2} \frac{1}{\omega_n} + \frac{a_1}{2} \omega_n$.

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The slide shows the following equations:

$$\begin{Bmatrix} \xi_1 \\ \xi_2 \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} \omega_j & \omega_i \\ \omega_j & \omega_i \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} \quad (9)$$

For $\xi_1 = \xi_2 = \xi$, then

$$A^{-1} = \frac{2}{\left(\frac{\omega_j}{\omega_i} - \frac{\omega_i}{\omega_j}\right)} \begin{bmatrix} \omega_j & -\omega_i \\ -\omega_j & \omega_i \end{bmatrix}$$

$$\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \frac{2(\omega_i \omega_j)}{\omega_j^2 - \omega_i^2} \begin{bmatrix} \omega_j & -\omega_i \\ -\omega_j & \omega_i \end{bmatrix} \begin{Bmatrix} \xi_1 \\ \xi_2 \end{Bmatrix} \quad \xi_1 = \xi_2 = \xi$$

So, I can now say zeta i zeta j is actually equal to half of I am expressing this in a matrix form, I am expressing this relationship in a matrix form 1 by omega i omega i, 1 by omega j omega j of a 0 a 1. One can read this equation and similarly understand zeta i is equal to a 0 by 2 into 1 by omega i plus a 1 by 2 omega i which is as same as this.

So, now, for zeta i zeta j is same as zeta. Then I call this matrix as a matrix let us work out a inverse is very simple to find out that it is going to be twice of omega j by omega i minus omega i by omega j of omega j minus omega i minus 1 by omega j 1 by omega i. I can easily find a 0 a 1 which are the coefficients of the damping matrix as twice of omega i omega j by omega j square minus omega i square of omega j minus omega i minus 1 by omega j 1 by omega i of zeta 1 and zeta 2.

We now say zeta 1 zeta 2 are same zeta that is a condition, you see here omega 1 zeta 1 omega 2 zeta 2 they should be same zeta for this condition to be imply.

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The image shows a digital whiteboard with the following handwritten content:

$$a_0 = 2 \frac{(\omega_i \omega_j) (\omega_j - \omega_i)}{(\omega_j^2 - \omega_i^2)} \zeta$$

$$a_0 = 2 \zeta \frac{\omega_i \omega_j}{(\omega_i + \omega_j)} \quad (10)$$

$$a_1 = 2 \frac{\omega_i \omega_j}{\omega_j^2 - \omega_i^2} \left(\frac{1}{\omega_i} - \frac{1}{\omega_j} \right) \zeta$$

$$= 2 \zeta \frac{\omega_i \omega_j}{\omega_j^2 - \omega_i^2} \frac{(\omega_j - \omega_i)}{\omega_i \omega_j}$$

$$a_1 = 2 \zeta \frac{1}{(\omega_j + \omega_i)} \quad (11)$$

At the bottom, a matrix is defined as $C = a_0 M + a_1 K$ with the note "Rayleigh". To the right, a bracket groups a_0 and a_1 and labels it as C .

I can now say a_0 is going to be twice of $\omega_i \omega_j$ into $\omega_j - \omega_i$ of ζ divided by $\omega_j^2 - \omega_i^2$. I am just finding out this value a_0 is twice of this ζ 1 this ζ 2, there is no ζ 1 ζ 2 there are ζ divided by this that is what I am writing $\omega_i \omega_j \omega_j - \omega_i$ ζ minus $\omega_i \zeta$. I can see that is what I am writing and so on, which can be simplified as 2ζ of $\omega_i \omega_j$ by $\omega_i + \omega_j$ where this can be denominator expanded as a square minus b square I get the product I can cancel this I will get this value that is my a_0 equation number 10.

Similarly, reading from the second row of this equation a_1 will be equal to twice of $\omega_i \omega_j$ by $\omega_j^2 - \omega_i^2$ of $1/\omega_i - 1/\omega_j$ by $\omega_i \omega_j$ because this ω_j is minus is negative j of ζ . This says 2ζ $\omega_i \omega_j$ into $\omega_j - \omega_i$ by $\omega_i \omega_j$ of $\omega_j^2 - \omega_i^2$. So, this goes away I can now say it is going to be 2ζ 1 by $\omega_j + \omega_i$, I got a 1 equation 11.

Now, C is actually equal to $a_0 M + a_1 K$ which is given by Rayleigh. So, for the known values of a_0 and a_1 one can find C matrix because M and K are already known.

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The image shows a screenshot of a presentation slide with handwritten text. The text is written in black ink on a white background with horizontal lines. The title is 'Observations (Rayleigh damping)'. The main text reads: 'modes (i & j) with specified damping ratio should be chosen. In the chosen modes, for different frequencies same ξ is applied'. A bullet point follows: '- In order to check, examine ξ in other modes for the same ratio'. Another bullet point: '- 3 dof model, ③ modes need to be included in the analysis, then ξ in all the three modes should be roughly same (desirable)'. The slide includes a toolbar at the top with various icons and a footer with the NPTEL logo and the number 13.

There are some observations on this condition some observations are - in applying this procedure of Rayleigh damping the modes i and j with specify damping ratio should be chosen. The condition is in the chosen modes for different frequencies same damping ratio is applied.

So, you should be able to choose 2 different frequencies which has the same damping ratio. Further in order to check examine damping in other modes for the same ratio. For example, if it is a 3 degree freedom system model 3 modes need to be included in the analysis then the damping ratio in all the 3 modes should be roughly same, I should say the word roughly same because you cannot get exactly the same ratio which is desirable to apply this method.

Let us take an example problem which we already solved.

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The slide shows a three-degree-of-freedom mass-spring system with three masses m_1 , m_2 , and m_3 connected in series by springs k_1 , k_2 , and k_3 . The bottom mass m_3 is supported by a fixed base. The handwritten notes provide the following data and calculations:

- Let $m_1 = m_2 = m_3 = m = 3500 \text{ kg}$
- $k_2 = 1.5 k_1$ and $k_3 = 2 k_1$
- $k_1 = k = 1500 \text{ kN/m}$
- Natural frequencies:
 - $\omega_1 = 0.57 \sqrt{k/m} = 0.57 \sqrt{\frac{1500 \times 10^3}{3500}} = 11.8 \text{ rad/s}$
 - $\omega_2 = 1.414 \sqrt{k/m} = 29.27 \text{ rad/s}$
 - $\omega_3 = 2.163 \sqrt{k/m} = 44.778 \text{ rad/s}$
- Mode shapes:
 - $\phi_1 = \begin{Bmatrix} 0.68 \\ 0.32 \end{Bmatrix}$
 - $\phi_2 = \begin{Bmatrix} 1 \\ -1 \\ -1 \end{Bmatrix}$
 - $\phi_3 = \begin{Bmatrix} 1 \\ -3.68 \\ 4.68 \end{Bmatrix}$

The example problem is let us say this is m_1 , m_2 and m_3 ; this is k_1 , k_2 and k_3 . Let m_1 and m_2 and m_3 be a constant value which is 3500 kg, k_1 , k_2 and k_3 be a constant value not constant let us take it like this k_2 is 1.5 k_1 , and k_3 is 2 k_1 , and k_1 is k which is 1500 kilonewton per meter let us say this data is given.

So, now, the mass matrix and k matrix are known to us which is required I can apply any standard procedure what we discussed in the last lectures and find out ω s. Let us say I determined ω by influence coefficient method and the values are ω_1 is 0.57 root k by m which is 0.57 root of 1500 kilonewton divided by 3500, let us say 11.8 radians per second. And the corresponding ϕ_1 is 1.68 and 0.32. Similarly ω_2 is 1.414 root k by m . So, we substituting we get 29.27 radians per second and the corresponding ϕ_2 is 1 minus 1 and minus 1.

Similarly, ω_3 is 2.163 root k by m which equals to 44.778 radians per second and the corresponding ϕ_3 is 1 minus 3.68 and 4.68.

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The image shows a handwritten slide with the following content:

$$[M] = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad k = k \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2.5 & -1.5 \\ 0 & -1.5 & 3.5 \end{bmatrix}$$

$$= 3500 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad k = 1500 \times 10^3 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2.5 & -1.5 \\ 0 & -1.5 & 3.5 \end{bmatrix}$$

$$a_0 = \frac{2 \omega_1 \omega_2 (0.05)}{\omega_1 + \omega_2} = \frac{2 \times 11.8 \times 29.27 \times 0.05}{11.8 + 29.27} = 0.841$$

$$a_1 = \frac{2 \xi}{\omega_1 + \omega_2} = \frac{2 \times 0.05}{11.8 + 29.27} = 0.0024$$

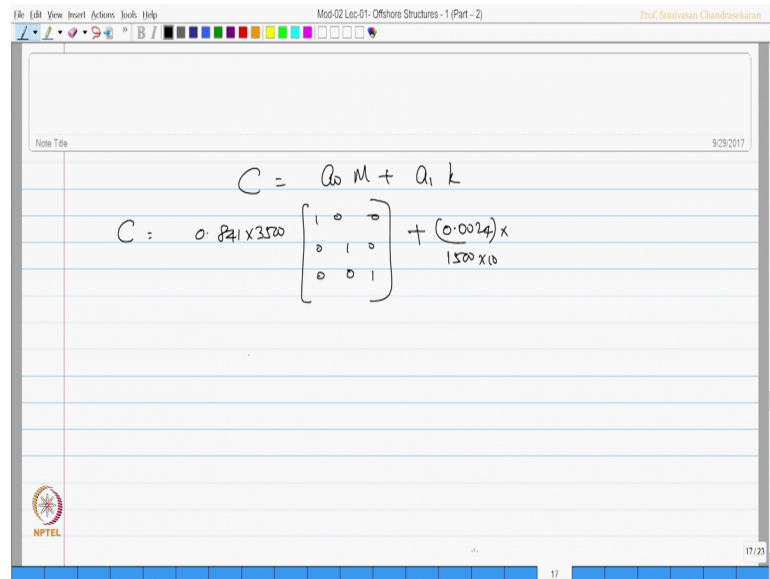
The slide also includes a small NPTEL logo in the bottom left corner and a slide number '16' in the bottom right corner.

So, the mass matrix is now known to me which is m times of 1 0 0, 0 1 0, 0 0 1. From the equation of motion by force Newton's force method I can also find the k matrix I leave this is an homework to you, k will be actually equal to 1 minus 1 0, minus 1 2.5 minus 1.5 and 0 minus 1.5 and 3.5 that is my k matrix now. So, m matrix can also be said as 3500, so much 1 0 0, 0 1 0, 0 0 1 in kg and k matrix can be now said as 1500 into 10 power 3 of 1 minus 1 0, minus 1 2.5 minus 1.5, 0 minus 1.5 3.5.

So, now let us quickly compute a 0 which is 2 omega 1 omega 2. I am taking first 2 modes and I am assuming zeta to be 5 percent 0.05 divided by omega 1 plus omega 2 that is the equation for, you can see here for a 0 which is 2 zeta omega 1 omega 2 and sum of this and a 1 is 2 zeta by this sum. Let us do that which is going to be 2 into 11.8 into 29.27 into 0.05 divided by 11.8 plus 29.27 which gives me as 0.841.

Similarly, a 1 is 2 zeta by omega 1 plus omega 2, 2 0.05; 11.8 plus 29.27 which is 0.0024. So, now, I can find the C matrix as a 0 m plus a 1 k.

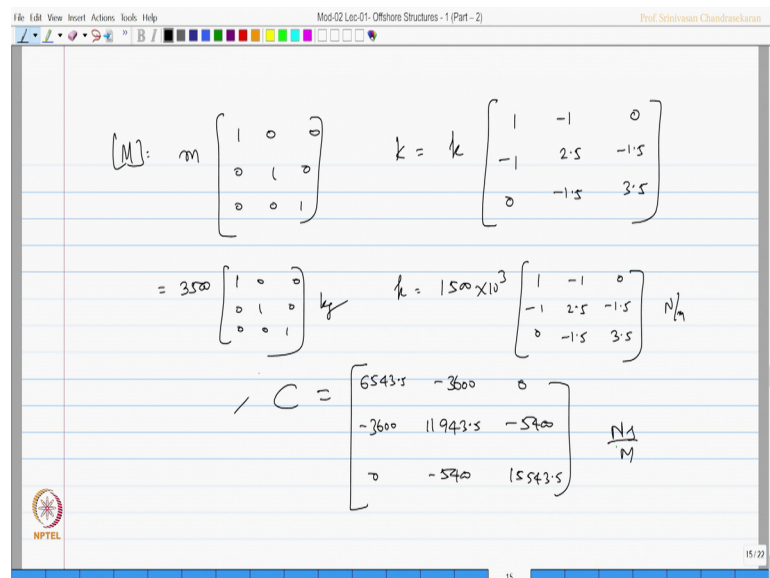
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A screenshot of a presentation slide showing the calculation of the stiffness matrix C. The equation is written as $C = a_0 M + a_1 k$. Below this, it is expanded to $C = 0.841 \times 3500 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{0.0024}{1500 \times 10^3} k$. The slide includes a toolbar at the top and an NPTEL logo at the bottom left.

So, that is C is going to be 0.841 into 3500 of 1 0 0, 0 1 0, 0 0 1 that is my mass matrix plus 0.0024 that is my a 1 into 1500 into 10 power 3.

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A screenshot of a presentation slide showing the calculation of the mass matrix M and the stiffness matrix k. The mass matrix is given as $[M] = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and the stiffness matrix as $k = k \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2.5 & -1.5 \\ 0 & -1.5 & 3.5 \end{bmatrix}$. The mass matrix is then scaled by 3500 kg, and the stiffness matrix is scaled by $1500 \times 10^3 \text{ N/m}$. The final stiffness matrix is $k = 1500 \times 10^3 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2.5 & -1.5 \\ 0 & -1.5 & 3.5 \end{bmatrix} \text{ N/m}$. The resulting stiffness matrix C is $C = \begin{bmatrix} 6543.5 & -3600 & 0 \\ -3600 & 11943.5 & -5400 \\ 0 & -5400 & 15543.5 \end{bmatrix} \text{ N/m}$. The slide includes a toolbar at the top and an NPTEL logo at the bottom left.

Summing them all we get C matrix as 6543.5 minus 3600 0, minus 3600, 0 11943.5 minus 540, minus 5400 15543.5 in so many Newton second per meter because k matrix is in Newton per meter mass matrices is in kg, C matrix is this.

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Check ζ_3 !

$$\zeta_n = \frac{a_0}{2} \frac{1}{\omega_n} + \frac{a_1}{2} \omega_n$$

$$\zeta_3 = \frac{a_0}{2} \frac{1}{\omega_3} + \frac{a_1}{2} \omega_3$$

$$= \frac{0.841}{2} \frac{1}{44.778} + \frac{0.0024}{2} (44.778)$$

$$= 6.31\% \approx 5\% \text{ All 3 modes contribute to the response}$$

Now, I want to check what is my zeta 3, that is checking for the third mode zeta 3 I want to check. So, we know zeta n is a 0 by 2, 1 by omega n plus a 1 by 2 omega n. So, zeta 3 is a 0 by 2 omega 3 plus a 1 by 2 omega 3; a 0 we know is 841 by 2 by 44.778 plus a 1 is 0.0024 and omega 3 is 44.778 when you estimated this I get this as 6.31 percent which is very close to 5 percent hence we can say all 3 modes contribute to the response.

So, Rayleigh damping is very easy to obtain a problem we will take the numerical example of this through a computer code.

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```

1 RAYLEIGH DAMPING
2 % Program for finding damping matrix using Rayleigh method
3 clear;
4 % Enter dot
5 dof=3;
6 %Enter Mass matrix
7 m=[3500 0 0; 0 3500 0; 0 0 3500]; % mass in kg
8 %Enter Stiffness matrix
9 k=[1500000 -1500000 0; -1500000 3750000 -2250000; 0 -2250000 5250000]; % Stiffness in N/m
10 %Enter assumed damping ratio
11 drp=5; %damping ratio in percentage
12 c=drp/100;
13 fprintf('Mass Matrix\n');
14 disp(m);
15 fprintf('Stiffness Matrix\n');
16 disp(k);
17 % eigen values and eigen vectors
18 [modu,w_square]=eig(k,M);
19 [freq,w_square]=eig(-w_square);
20 for i=1:ndof
21     wn(i)=freq(i,i);
22     moden(i,i)=modu(i,i)/modu(i,i);
23     fprintf('Frequency: wn = %6.2f rad/s \n',wn(i));
24 end
25 fprintf('Modal Matrix x = \n');
26 disp(moden);
27 % Rayleigh damping
28 a0=2*c*wn(1)*wn(2)/(wn(1)+wn(2));
29 a1=2*c*(wn(1)+wn(2));
30 c=[a0*M+a1*I]*c; %damping Matrix
31 fprintf('Damping Matrix\n');
32 disp(c);
33 % Check for damping ratio in third mode
34 d3=(a0/(2*wn(3)))+(a1*wn(3)/2);
35 drp3=drp*(100);
36 fprintf('Damping ratio in third mode = %6.2f \n',drp3);

```

Mass Matrix

3500	0	0
0	3500	0
0	0	3500

Stiffness Matrix

1500000	-1500000	0
-1500000	3750000	-2250000
0	-2250000	5250000

Frequency: wn = 11.721 rad/s
 Frequency: wn = 29.277 rad/s
 Frequency: wn = 44.783 rad/s

Modal Matrix

$$x = \begin{bmatrix} 1.0000 & 1.0000 & 1.0000 \\ 0.6794 & -1.0000 & -3.6794 \\ 0.3206 & -1.0000 & 4.6794 \end{bmatrix}$$

Damping Matrix

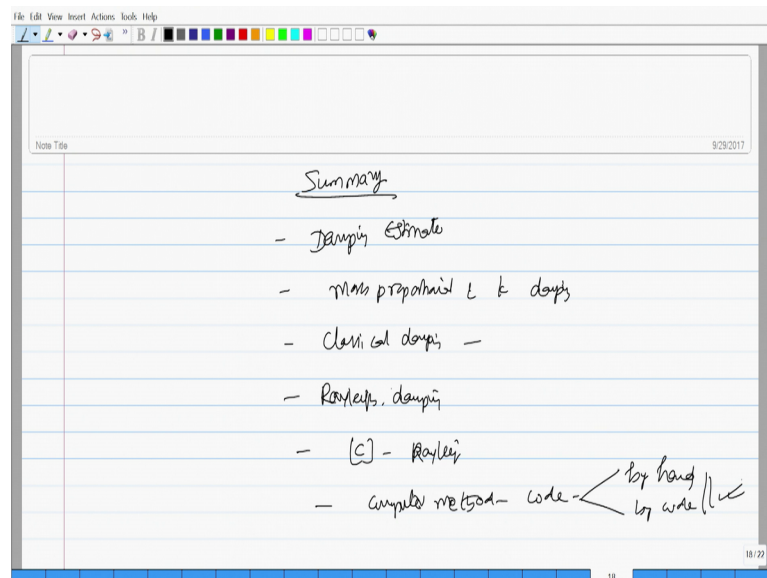
$$1.0e+04 \begin{bmatrix} 0.6588 & -0.3659 & 0 \\ -0.3659 & 1.2076 & -0.5488 \\ 0 & -0.5488 & 1.5735 \end{bmatrix}$$

Damping ratio in third mode = 6.396 %

The computer code in this program now shows how the code is written for Rayleigh damping this is for a Rayleigh damping. You have to enter the degrees of freedom, you have to enter the mass matrix in our case is the mass matrix. Similarly one can enter the stiffness matrix, we have entered the stiffness matrix then enter the damping ratio we have taken 5 percent. Once you do that then the Rayleigh constants are evaluated and then the damping matrix is evaluated and it is printed and check for the third one. So, the sample output looks like this, this is my mass matrix, this is my stiffness matrix and these are the input frequencies which we already got 11.721, 29.277 and 44.783 by computer program.

The modal matrix what I get is what I gave you this is my first mode; this is my second mode this is my third mode. And the damping matrix is what I get here 3 by 3 which is exactly same let us say 6588 I by hand I got 6543. So, that is what, it is going to be same. And check for the third mode damping ratio I get 6.396 whereas, we got third mode as 6.31. So, the computer program exactly solves the problem in the same style as we did by hand a sample output is shown to you. And let us see the summary now.

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So, in this lecture friends we learned how damping estimate is important in offshore structure, what is mass proportional and stiffness proportional damping, what is classical damping, what is the problem with the classical damping and why Rayleigh damping is appropriate to offshore structures and how to solve see from Rayleigh damping. We also

learned the computer method, we also have the access to the computer code, we solve the problem both by hand and by computer code and we found the answers are agreement closely with them.

I hope you will be able to program this in MATLAB and do couple of problems and realize how easy and convenient is Rayleigh damping for damping estimates in offshore structures.

Thank you very much.