Dynamics of Ocean Structures
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Lecture – 46
Response Spectrum – II

In the last lecture we discussed about the Impulse Response Function and the Transfer Function, which actually connects a loading response in the loading process to that of the response. You already said for the process to remain stationery and Ergodic Stochastic Process the connectivity between the load and the response can be established by a parameter which is time invariant and remains linear.

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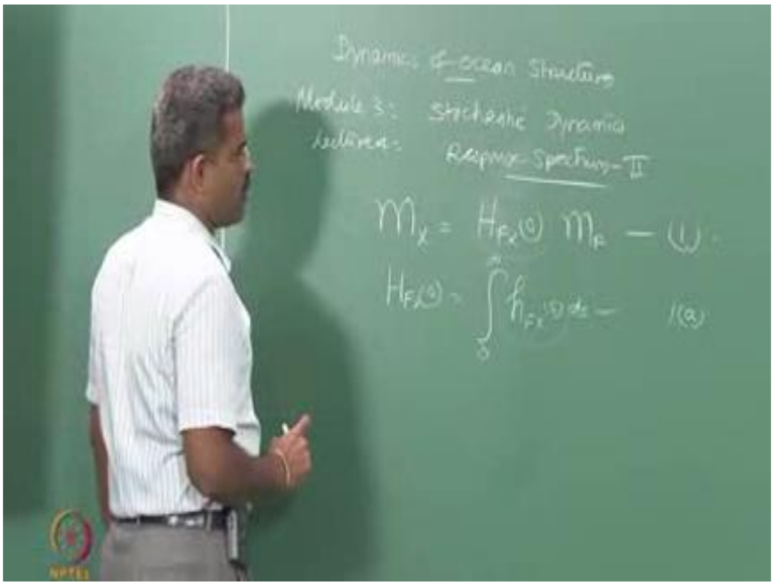

Now, let us try to compare this to get what we consider response spectrum because, now we have only the mean value of the response we do not have the response spectrum which is over for the entire frequency. We will try to get the response spectrum.

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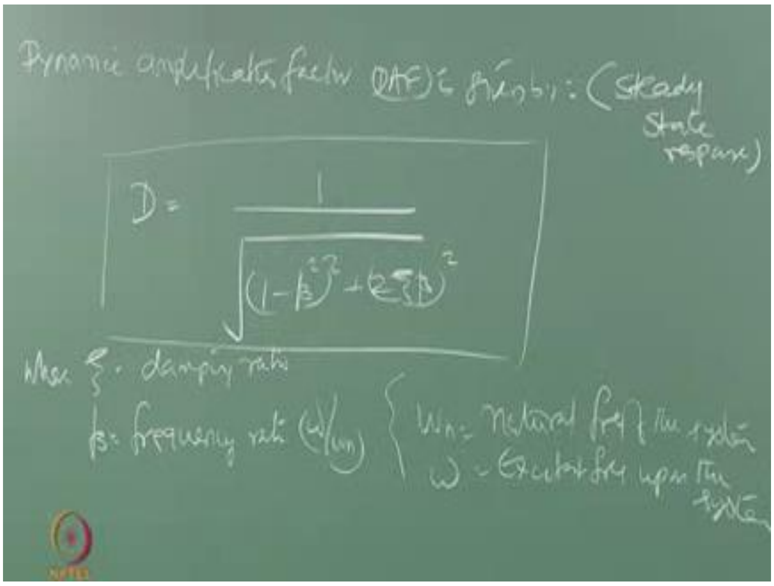

Now, let us compare this with the classical dynamic amplification factor is given by, this is of course, for a steady state response which is given by 1 by root of 1 minus beta square plus 2 zeta beta square where zeta is the damping ratio, and beta is the frequency ratio, which is omega by omega n where omega n is a natural frequency of the system. And omega is excitation frequency applied upon the system, this already we know is an established relationship we know this, just derive this.

Now we want to connect this with the transfer function because, transfer function is subsequently connected to impulse response function with this relationship. Therefore, if I connect this to transfer function I can easily find out an understanding between the loading processes with that of the response process if this equation is proved to be acceptable to all of us. Now H of omega will have 2 components; 1 it should give me a representative value of the amplitude it should give me the representative value of the phase shift, because we know that all the time the response may not be on the same phase as that of the excitation force.

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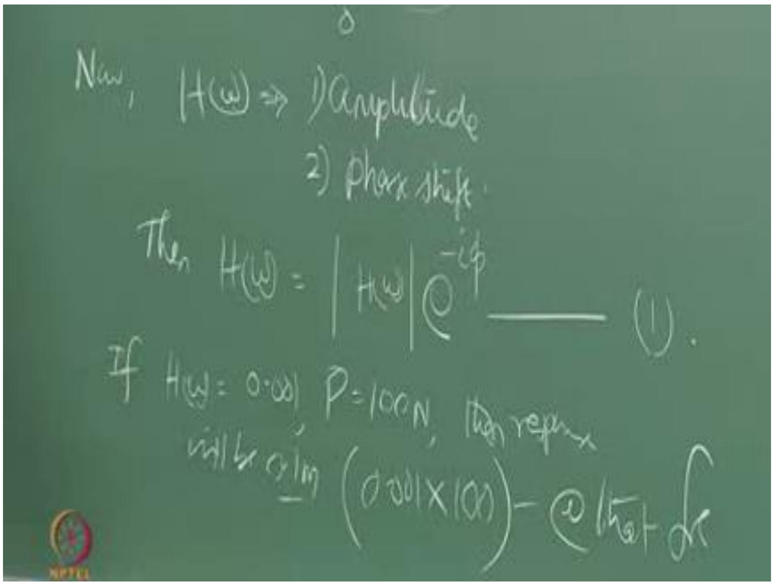

So, we should have 2 components represented in the transfer function, these are mandatory let it should give me indications about the amplitude and the phase shift if this is agree then H omega can be rewritten for our understanding as H omega e minus alpha there is a very specific reason why, we have not taken either the cos or a sin function, but you have taken an exponential function of phi, phi. I will tell you this after I come to the derivation there is a very classical advantage of this. Let us call this equation number for the current class as 1 this is already we have derived in the last lecture. So, I will retain it for our understanding. So, this is the first equation what we will have for this class of course, this equation is classically known to us. So, is not to be numbered.

So, now this will have 2 idea; 1 the amplitude will be given by this. For example, let us say to understand this physically if the transfer function is a magnitude 0.001 and the foreseen function p let us say is 100 Newton then the response, will be 0.1 meter. How do we get this? This is the transfer function multiplied by the amplitude, at that frequency. So, it is a very interesting linear relationship directly 1 to 1 correspondence. If you know the amplitude if you know the transfer function value I will be able to predict the amplitude of the response, but this only predicting the amplitude there is no phase. So, phi value is important you have to capture this.

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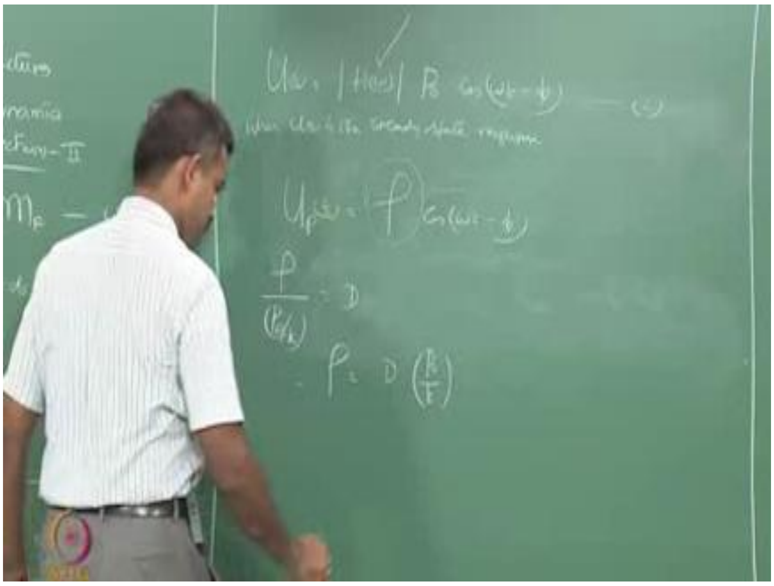

Now, the response in general can be given by in general. Let us say the transfer function amplitude, let us say cos omega t minus phi. Now this has got all components into it, 1 is the amplitude of the transfer function with a mode value. So, that there is no sine protection in this, but the sine will automatically come into representation of u in terms of phi, that is the equation number 2 now that is a general function.

Where u t is the steady state response, where u t is the steady state response of the system. Now let us compare this with the generic expression because, this is the transfer function well generic equation is not like this the generic expression is we are looking only for a particular integral, when we are looking only for a particular integral that is the steady state solution. Generally I express this as some amplitude with that of the foreseen, function frequency with the phase shift that is the general expression we have derived already in the first module now this amplitude of the steady state is generally considered to be equal to P0 sorry rho by P0 by k, is generally considered as the dynamic amplitude factor d, where P0 by k is the static response.

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So, on the other hand this can be dynamic amplitude factor into P0 by k. Therefore, Up of t will be DP0 by k of cos omega t minus phi which is 1 by root of 1 minus beta square plus 2 zeta beta square. P0 by k cos omega t minus phi which is Up of t I call this equation number 3.

Now, comparing 2 and 3; that is 1 by root of 1 minus beta square square plus 2 zeta beta square P0 by k cos omega t minus phi is compared with H omega P0 cos omega t minus phi. So, from this we can easily infer that H omega is nothing, but 1 by k of the H omega is called the transfer function where H omega is called the transfer function.

So, very interesting, if I know my dynamic amplification factor for a given system which actually a plot of frequency ratio versus the ratio of the amplitude of the static response I will be able to get my transfer function very quickly using this expression if you know my response of the system for the static roll. 1 by k is nothing, but response of the system for a static load for a unit load. This is why it is called response spectrum, we will come to that point now I am going to derive the response spectrum based on this, but now it is a very interesting connectivity, if I know my daf for unit load the static response of the system is what I am connecting to my stochastic response of the given system. I have connected this now, but still H of omega in this case is giving only the amplitude there is no phase shift.

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So, strictly speaking H of omega should be 1 by k of roof of 1 minus beta square square plus 2 zeta beta square of e minus I, phi I call this equation number four now why e I phi? Why e I phi we could take sine we could take cos there are 2 reasons for this, the first reason is d by d t of e I phi could be I d by d phi by d t of e I phi. What does it mean it means that the derivative of e I phi will remain with the clay same fashion or the same style or type.

The second thing is e I phi 1 product e I phi 2 will give me e I phi 1 plus phi 2 which is e I phi 3, which they remain again the same type. So, even if you multiply even if you derive or take a derivative the type of the result will still remain same. This will give you many advantages when you do the because you will have to take a derivative you may have to get a product expression in autocorrelation function in co variance function you have to do the product and so on. So, the type of using e I phi will give you the same type back even when you do this operation. So, that is an advantage why we are using e I phi here.

Now, we are making this statement as H omega now with the transfer function I am able to protect the amplitude of it and I am able to also protect the phase shift of this and it is very easy provided, I have the value of the phase shift between the loading and the response, I can easily find out this the response. If I know the loading along with the phase shift or with every frequency because you know beta is frequency dependent while plot it later and show you. Now I am going to use this further to actually derive the response spectrum.

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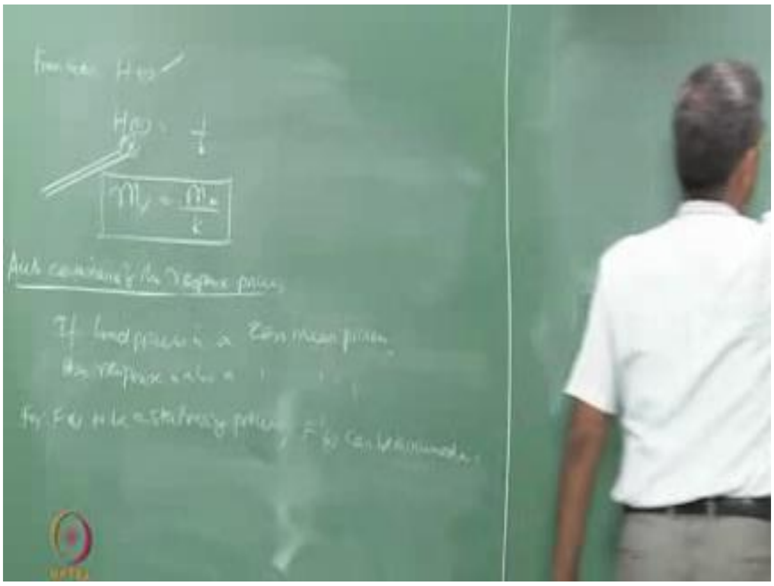

So, from equation four H omega is known. Therefore, H0 which I would specifically say as s H Fx 0 this Fx nothing to do with any non-linear time invariance this Fx is indicating qualitatively this is a transfer function connecting the load and the response. T0hat is the meaning. So, if the moment I put 0 here I will get actually 1 by k is it not. So, m x is nothing, but m of by k is not it very simple and linear see if you know the mean value of my foreseen function or the process I will get the mean value of the response which is equivalent to the static value of this response actually. So, it is time invariant and it is liner which we wanted in a stationery Ergodic process that is what we got here. Now from this how do you deduce the response spectrum? So to do that, we will move forward to find out the autocorrelation of the response.

Let us say I want to find autocorrelation of the response. One may ask me a question why the moment, I have an autocorrelation I will take a fast (Refer Time: 14:30) transform of this to find the response spectrum. So, first let us done the expression for autocorrelation now of the response. So, I am able to read it of the response process of the response process. So, I am looking for autocorrelation with the response process or to be very specific I will equate this later. I will call this auto covariance, now I will equate this later. Auto covariance, because we have an expression for covariance already we will equate it later.

So, this expression tells me, if load process is a 0 mean process; then response is also a 0 mean process. That is why is called H Fx 0. So, this is 0 mean processes. Now for F of t to be a stationery process, for F of t to remain as a stationery process F dash of t can be assumed as a new function. If F of t is a stationery process a new function F of F dash of t can be assumed as below provided it is a 0 mean process.

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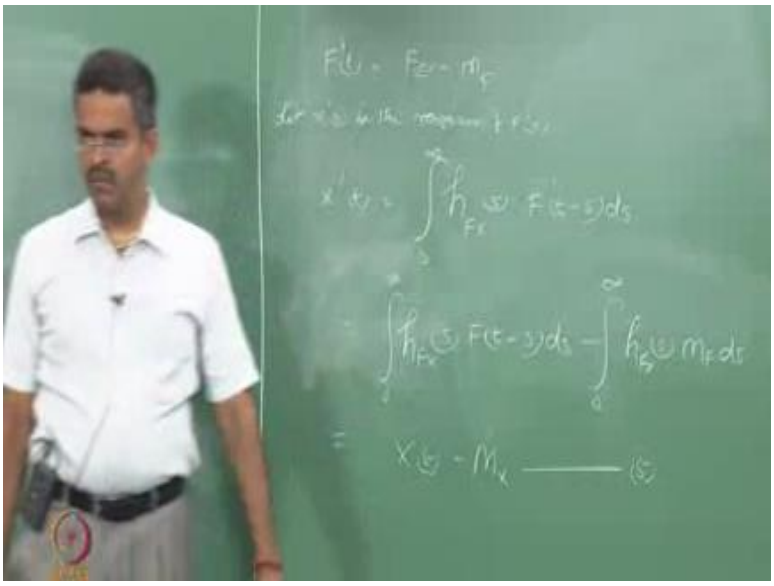

So, that is F dash of t can be equal to F of e minus m of f. Because it is a 0 mean process then, let x dash of t be the realization of the response of a F dash of t because corresponding there will be a response also when you have got F dash of t. Now than can be written as, impulse response function H Fx s impulse response function H Fx s F dash t minus s d s, this is the standard equation what we already have. That is why it is called impulse response function which is connected to my transfer function by an integral we already know this. Of course, we have only positive realization because the negative value of this I has no consequence. Now my F dash I am talking about capital x here. Therefore, I should say capital x please correct it here this is capital x is a full set this is also a full set which is original.

Now, F dash is already F of t minus Mf, therefore, I can write this expression as 0 to infinity Fx as F t minus s d s minus integral H Fx s m F d s infinity. Which is nothing, but x of t and we already know this is mean of response, from the last derivation of the last class we already know this that is mean of the response m x now I call this equation number maybe 5. This also shows a very interesting relationship if the new process loading process is 0 mean processes you will automatically. Get the response also to be a 0 mean process actually is matching the same.

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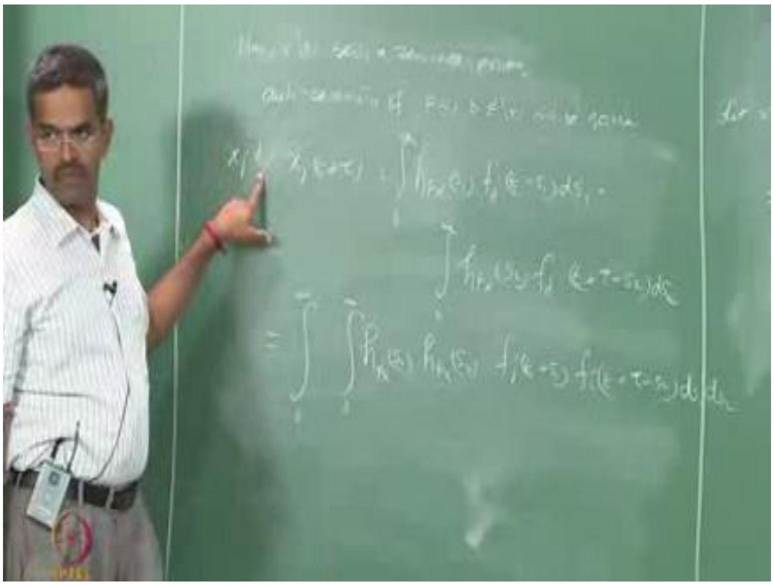

Now, x dash of t be a 0 mean process auto covariance of 2 processes t and F dash of t will be same that is a deduced logic from statistics. Now, how to write this auto covariance for these two; let us write down that slightly in a different manner let us say my x j of t which is the realization values of x of t and the new value of x j plus c plus tau the tau is the interval because I am taking the mean about the n symbol not the time domain, that is the stochasticity or that is the Ergocity in the given process.

Which is given by because we already have this equation I am expanding it slightly in a different manner using these 2 algorithms, which is given by H Fx s1 Fj t minus s 1 t s 1 integral infinity, into in H Fx s2 F j t minus s 2 d s 2 integral 0 to infinity this is d s 2. Which can be rewritten as double integral of 0 to infinity, H Fx s 1, H Fx s 1, H Fx s 2, F j t minus s 1 F j t minus t plus tau minus s 2 ds1 ds2. Now, I am going to bring this as an expected value as similar to this, first let us understand how the expected value of this will look like.

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So, expected value of x of t, x of t plus tau, will look like limit n tends to infinity 1 by n of x j of t x j of t plus tau. Hence I mean we know I will write we know even if you do not know this now you at least know please note this. We already have explained this actually summation only of n symbol there is no time dependence only your tau is in control we have already explained in the last lecture. So, let we know at least this now this expression will become, because I want to rewrite this back in this form. So, that I will use the same algorithm for Fx, which I will say double integral of so, now, I can using this relationship I can extend this as s and d s square which can be then simplified as the covariance function c F which is having the value variables as tau plus s 1 minus s

2 ds1 ds2.

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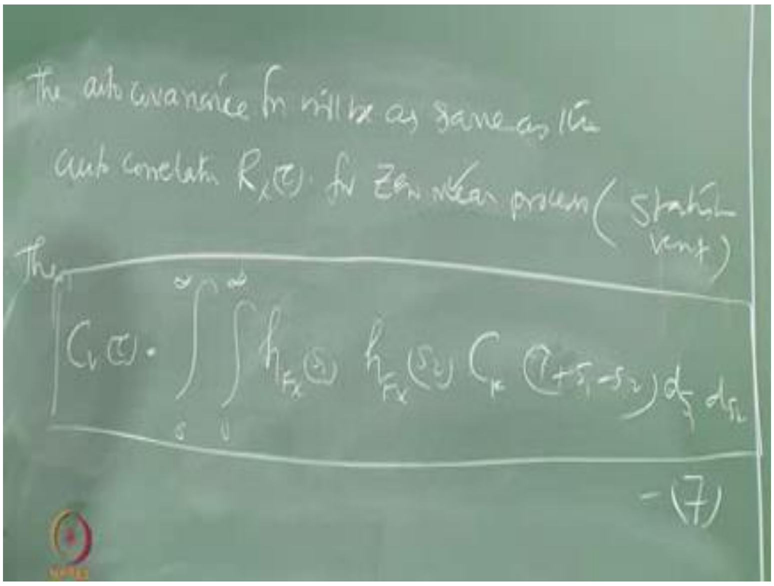

Now, the auto covariance function will be as same as autocorrelation function if, for 0 mean processes, this is actually statistically verified. This is a verified statement it is a verified statement. So, I can then say c v of tau which is similar to this from this, using a fast Fourier transform I can always find out the response spectrum of I call this equation number 7. Somewhere down the line will be 6, somewhere here may be we can call all of them as 6, this 7 because I will refer this further let me check.

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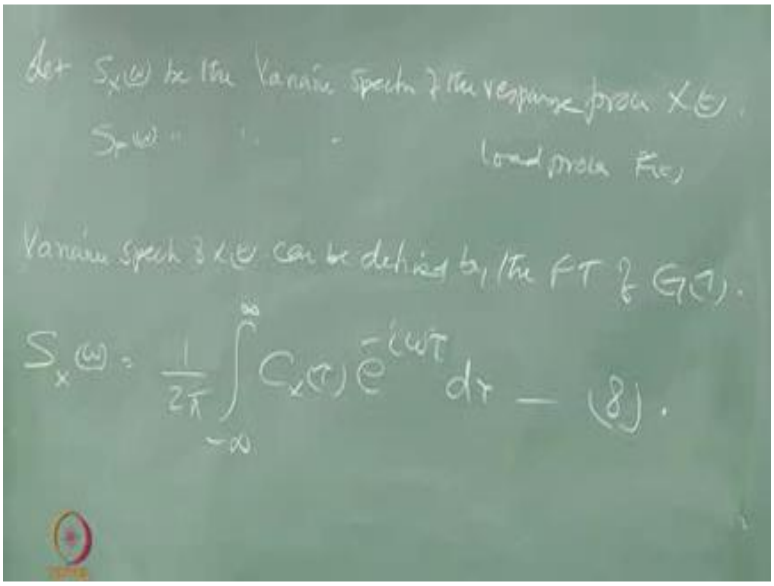

Now, let us connect this to my response spectrum. Let Fs x omega be the variance spectrum of the response process x of t S F omega be the variance spectrum of the load process F of t. Now the variance spectrum of x of t can be defined using the Fourier transform of autocorrelation function of this by the Fourier transform of equation 7. So, what does it mean is s x omega will be 1 by 2 phi of minus infinity to plus infinity. I am taking the Fourier transform of this which is c x tau. Now this c x I think be careful this is r x c x tau this is not c b c x tau c x tau of because I have used capital tau. Here, we are using everywhere capital tau, I suppose capital tau this 1 is yeah, so c x tau e minus omega t d t equation number 8.

Now, substitute for c x tau from here. So, there are 3 derivatives integrals 1 2 integrals 0 to infinity for these 2 and 1 from minus infinity to plus infinity for this.

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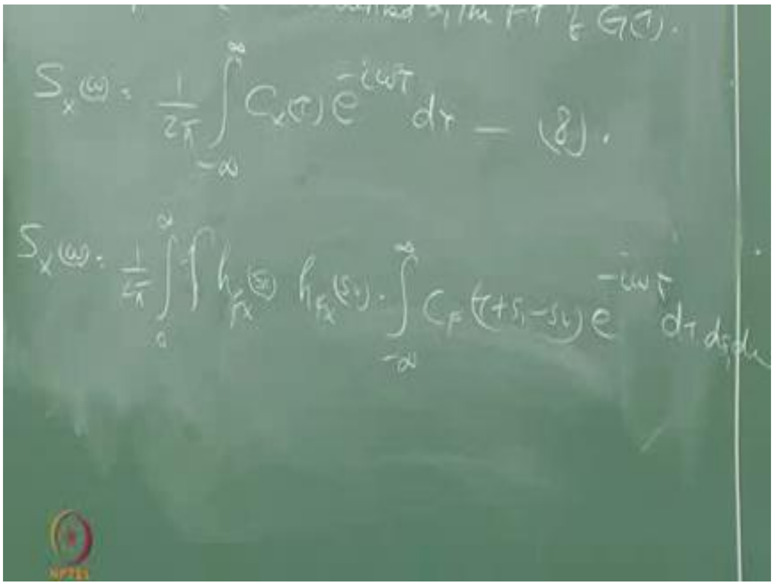

Let us write that. So, the response spectrum is omega is 1 by 2 phi of 0 to infinity of H Fx s 1 H Fx s 2 minus infinity to plus infinity of c F tau plus s1 minus s2 e minus omega tau d tau ds1 ds2.

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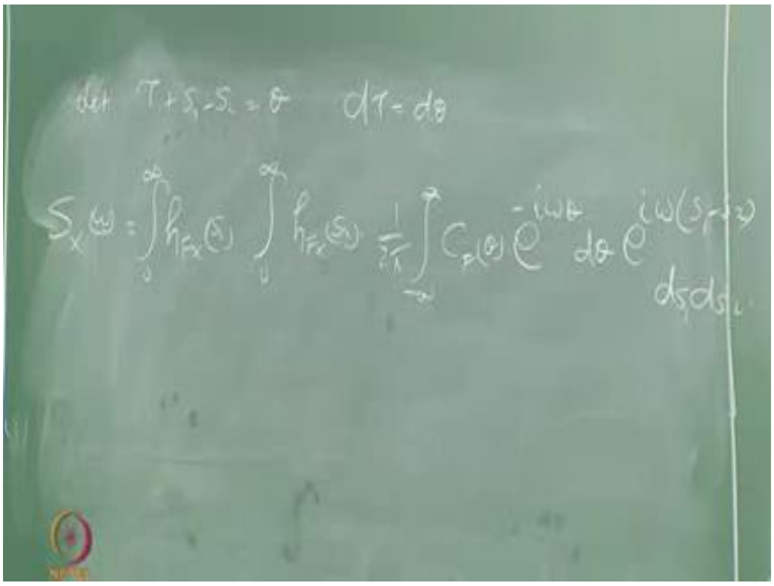

Now, in this let us substitute tau plus s 1 minus s 2 as theta. So, d tau will be d theta spectrum 0 to infinity, H Fx s 1 that is the first 1 0 to infinity H Fx s 2 the second one, 1 by 2 phi minus to plus c F theta because that is the autocovariance or corelation of the foreseen function c F theta. Now e to the power of minus I omega theta b theta that is how this is complete, but I am left with still s 1 and s 2 because actually the integral is not with e, but it is with the tau, but to make it complete because is theta, here I am writing like this. So, I have to add some more terms to make it equal to that of equation 7 a. So, I will do that here as e to the power of I omega s 1 minus s2. Of course, ds1 s2 of their, let me check this equation s1 s2; 1 by 2 phi c of theta I omega theta. Yeah, of course, this will be actually please see this is change of integral order s 2 and s 1 because I am integrating s2 and s1.

Let me separate this, what I will do now here is integration of 0 to infinity H of x is 1 ds1. In fact, e I omega s1 ds1 integral 0 to infinity H Fx s 2 e I omega s 2 there is a minus sign here minus ds2 and the 1 which is left over this is done of course, 1 part is here this is done 1 part is here which is minus sign here the 1, which is not done is this I call this as the variance spectrum of the load because this is actually is the load which we call this as s F omega.

Now, 1 is on the positive other is for the negative, we all know already if this is my impulse first response function for the given loading I can write this as H omega this is the transfer function because, it is integral 0 to infinity if we look back the equation 1 of today we have this with us to be very specific we should say H Fx omega the other 1 could be H Fx minus omega of s F omega there is no x. Here it is actually the input spectrum the load spectrum it is no transfer function these are transfer functions 1 is on a positive hand of omega 1 is the negative hand of the omega. Now I want to show that the value in both the cases will remain almost same in the process this is stationery and mean 0 mean process.

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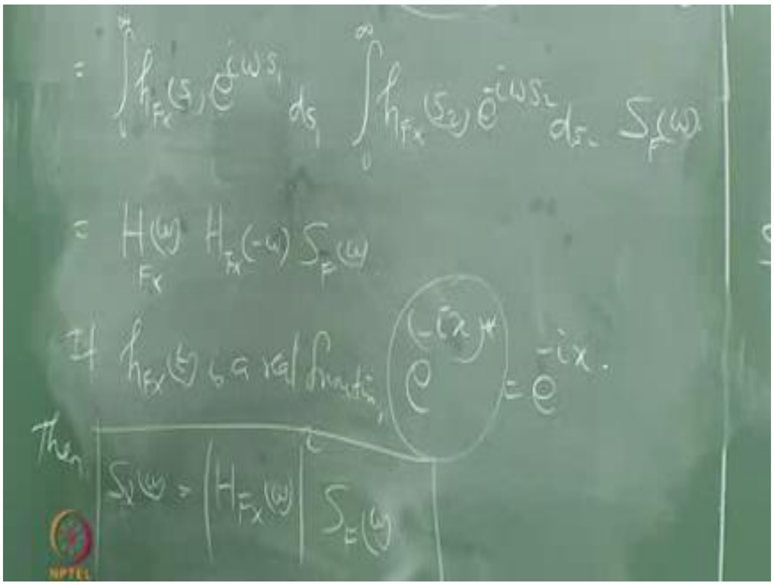

Now, if H Fx omega is a real function let us take it as a time is the real function e minus I x star will be e minus I x we give this as star, then s x omega can be written as H Fx omega H x omega star Fx omega what I call this as H Fx omega.

Mode square S F omega which is my equation for response spectrum, if I know the input load spectrum and we already know the transit function is related to the linear static response of a dynamic controlling factor of a given system. So, we can easily find out the response for this for every omega.

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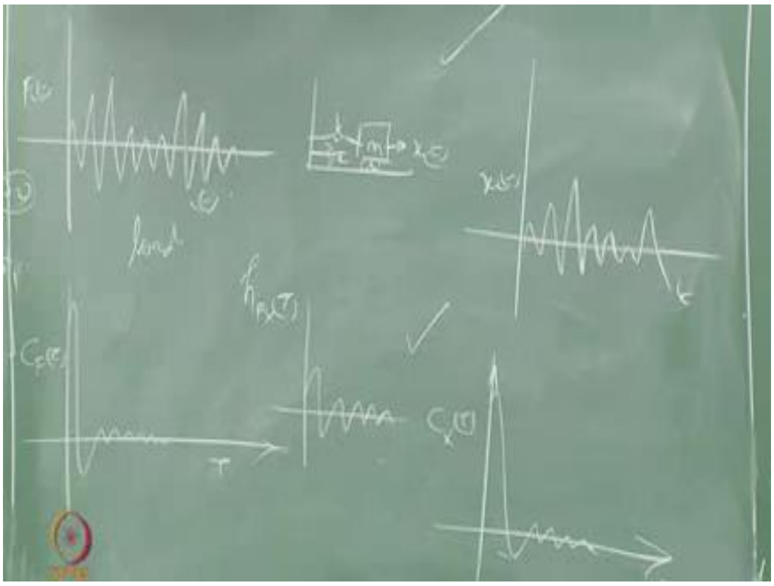

If we try to plot this let us graphically understand this equation is very important part of this lecture. Let us try to graphically understand this is very important, let us say I have a row, I have a load; let us say this is my load t, F of t. I have a system, maybe single degree what I get is the response of the system, which will be in the same. This is a known fact. If I have the auto covariance of the force which plots like this, which is c F tau for a value of tau then, I will get c x tau that is the response for the value of tau if I know the impulse response function, which is H Fx tau which will be a plot like this and this. This is also established in the last lecture. Now these 2 put together will lead to a graphical understanding, which is equivalent to this equation which will be if I know the input spectrum.

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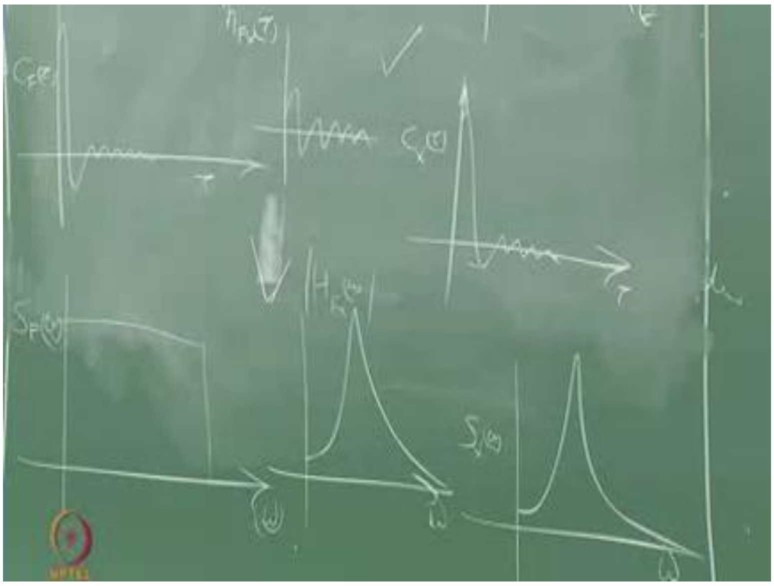

If you know the input spectrum maybe, whatever may be the spectrum I will get my output spectrum response spectrum using a connectivity which is m x the mean value of the response can be given by the transfer function mean, value of the force we already know this.

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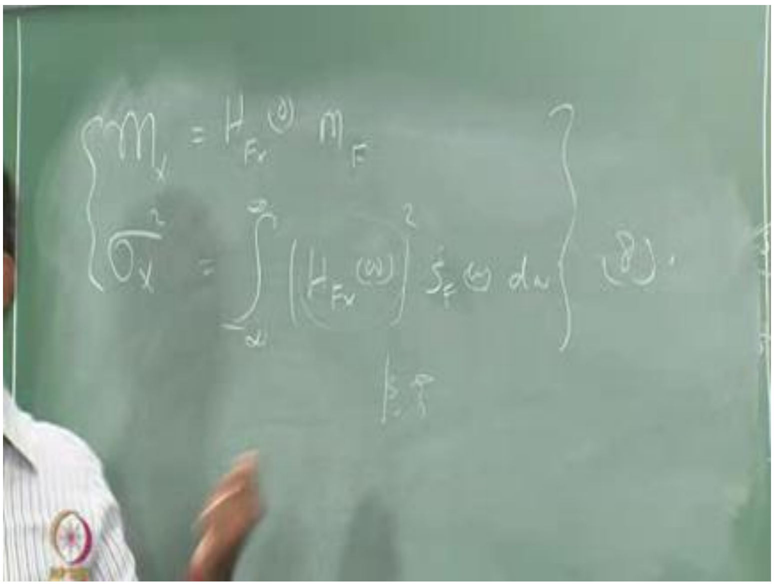

Now, the variance of the response both are what we call first order statistics is now minus infinity plus infinity H Fx omega s of omega d yeah call this equation.

Now, this transfer function has a component of 2 1 beta, other is zeta this is actually nothing, but a proportion of or a multiplier of dynamic amplifier factor which is not these 2 values now for very low value of zeta.

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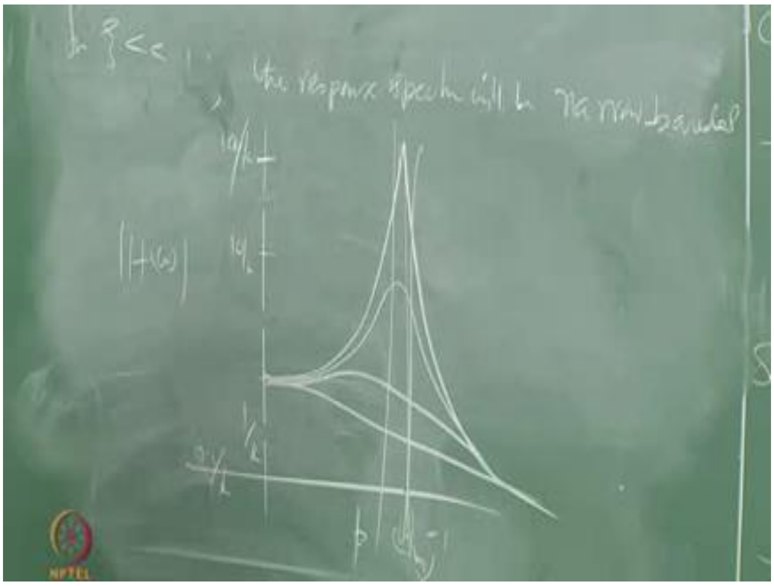

Let us express it very low value of zeta the response spectrum will become narrow banded. Let us plot this for different values of let us say this is my plot which is beta is omega by omega n and I am plotting here not the daf, but I am plotting here the H omega on the other hand is nothing, but daf k. So, my first value should be 1 by k and so on. So, I start from here point 1 by k, then 1 by k ten by k let us say I will take 1 by k here I will take ten by k here and I will take 100 by k for my plot. So, we all know it will start from 1 because daf also starts from 1 and as I go omega equal to omega n that is here it goes up and for some value of zeta otherwise it is unbounded for some value of zeta it becomes very narrow and comes down. And similarly as increased zeta always start from here and so on.

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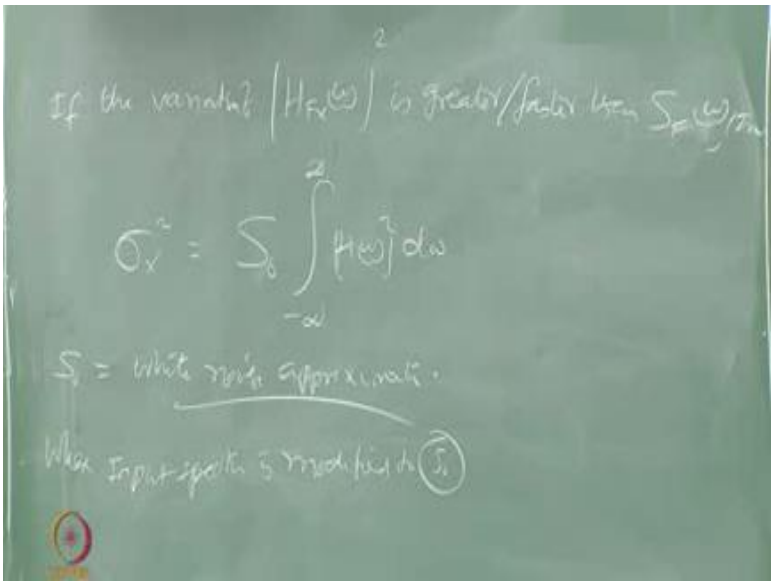

So, for very low values of zeta the response spectrum actually becomes narrow banded. So, people have made some (Refer Time: 41:18) in statistics or stochastic response call this white noise approximation. What is the white noise approximation? If the variation of if the variation of HFx omega square is greater or faster than s of omega. What does it mean? The variation in s of omega is actually not very serious mode or less linear, but this is not so.

If it is greater or contrarily if s of omega variation is lower that is this is the input spectrum the load spectrum if the varies in load spectrum is lower compared to that of the transfer function then, the approximation is slightly different, then we can say the variance sigma x square because here this variation is much faster, much sensitive compared to this variation that is what we are saying here much sensitive. I approximate this as since the variation is faster I will call this as constant out of the integration and say represent this, sensitivity taking it out from the integral where s0 is called white noise approximation, it is called white noise approximation. The variance of white noise variation approximation, becomes sigma square s naught is equal to s naught of this, where the input spectrum, where the input spectrum is modified to s naught the input spectrum is modified to s naught.

So, in this lecture that we have summarized that we are able to connect the transfer function to the classical dynamic amplification factor which we already know in dynamic advance response. We have also said that they are time invariant and for a 0 mean process if we know the mean of the foreseen function or the foreseen process I can easily find the mean of the response by simply linearly connecting them, and if I know my daf the dynamic amplification factor I can easily find the equivalence static response of the system for unit load which will give me my transfer function. For the given system, further the variance which is also 1 of the important statistical information required for stochastic analysis can be simplified using a white noise approximation using this relationship which we call as equation number 9. Where s naught is modified spectrum of the input data.

In the next class we will talk about the return period, and then will talk how this can be used for my Fatigue Analysis for a Dynamic Systems. We will stop there for the course.