Dynamics of Ocean Structures
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Lecture – 27
Mode Superposition & Truncation

So, we will have the 27th lecture where we will talk about Mode Superposition and Truncation.

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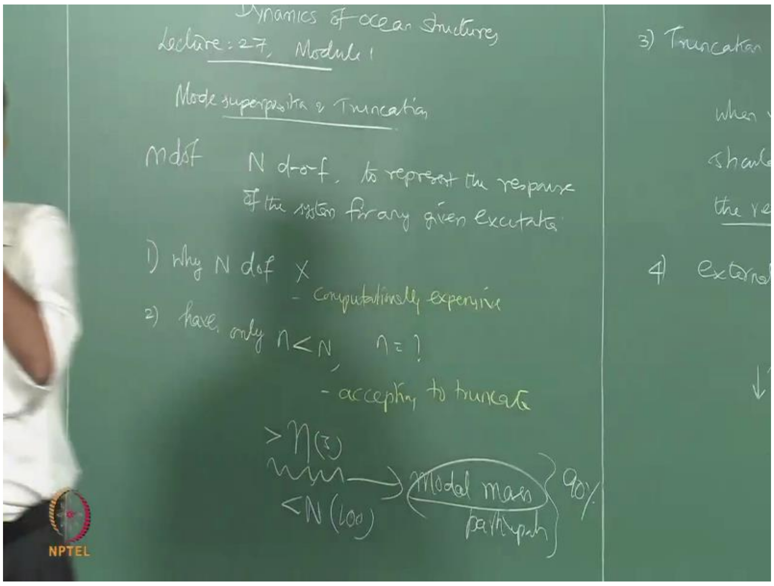

We already know that in a given multi degree freedom system model, we have n degrees n degrees of freedom where we may not require all the n degrees of freedom to represent the response of the system for any given excitation. Now let us try to understand the basic principle behind truncation of modes.

Now first of all let us have few questions why n degrees of freedom are not required he answer is very simple, if you want all the degrees of freedom it will be computationally expensive. Do not talk of a 6 degree freedom system problem like ocean structures, talk of a general problem where the degree of freedom practically goes infinite. In that case if you are looking for all n degrees of freedom to be included in the analysis then it will become computationally very expensive. Two, if you decide not to have n degrees of freedom and have only n, instead of n then n should be how many, I mean how many you want. So, this indicates that we are accepting to truncate.

Now, to answer how much n should be there against capital n let us try to see what is the role of n in the response.

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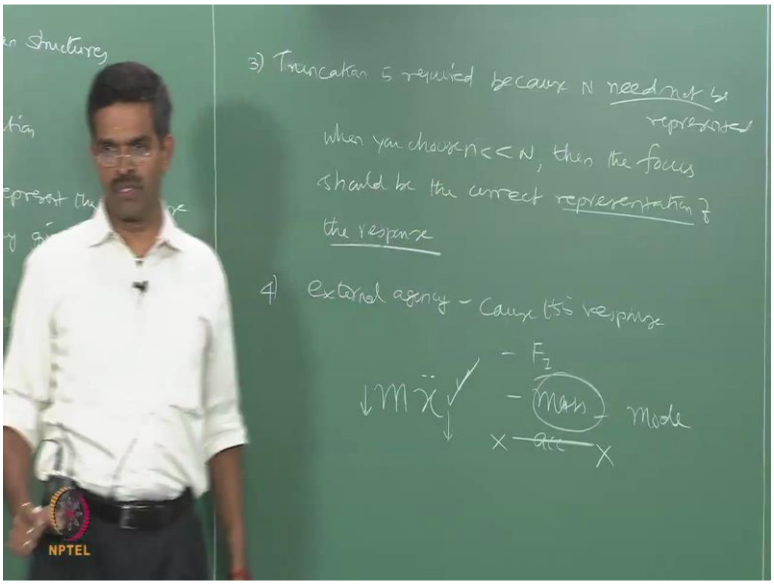

So, the truncation actually is required because n need not be represented I am writing carefully this, n can be because now you have full computational facility you can always find out n that is not a problem; need not be. Now when you choose n which is less than n in fact, one can also say very very less than n, let us say n is 100, but small n is only 3 very very less than n. Then the focus should be the correct representation of the response. So, the truncation is directly related to representation of the response.

Now, the question comes if you want to represent the response, let us say the response is invoked under what external action. The external action or the external agency which is going to cause this response essentially is inertia force. Now one may ask me a question sir there are other kind of forces also available drag lift etcetera, we already said that for a system to dynamically excited inertia force very very important. Now one may say sir inertia force is mass x double dot that is what the inertia force is - the inertia force should be predominantly present, significantly available in the system to qualify the system for a dynamic analysis.

Now we are talking about structures which has got lower mass, it means the top side is compromised why talking about floating structures, installation cost is very high compared to fixed platforms therefore, the mass is compromised. But on the vice versa I see the floating structures will have a very high acceleration. So, the inertia component is still dominant. In earlier cases like fixed data structures, the acceleration component is very very low because it is more or less fixed, but it is massive therefore, the inertia component anyway which is the product of these two is significantly present in the system for which you do a dynamic analysis.

Now, the external agency which is responsible to cause or create this response is the inertia force. Therefore, I must try to associate the inertia force with two parameters one is the mass other is the acceleration, amongst these two acceleration is a boundary condition property, mass is a system property because mass is not dependent on boundary condition where as acceleration will depend on boundary condition if it is fixed free floating etcetera. Therefore, let us not talk about boundary condition because you cannot design a system which can operate only on a given boundary condition for example, you have platform A - the platform A is suitable only when it is fixed at the bottom, the platform B is suitable only when it is floating I do not think any platform can be ever designed for accommodating the platform for a specific boundary condition. Platform should accommodate any boundary condition for which it should sustain it should be generic.

Therefore we cannot focus acceleration component in inertia to check the truncation of modes then we are left over only with one point that is mass. Mass contribution always comes from modes we already said that in the previous lectures also therefore, essentially it is not truncation of frequency, it is truncation of mode shapes we are truncating the mode shapes. Now, since mode shapes and frequencies are paired, they are coupled when you truncate the mode shape you will also automatically truncate the frequencies. So, n here refers to number of mode shapes not number of frequencies - the number n. Though it is referring to degree of freedom, degree of freedom will directly relate to number of frequencies in general, but our understanding should be always focused at the mode shapes and not the number of degrees of freedom related to frequencies alone.

However they are connected together when you truncate the mode shape obviously, frequency will also get truncated. Now the whole exercise gives me a very clear picture that - yes, I agree now I must truncate the modes. Even though I can compute n, n need not be calculated because the higher representation of n or greater than n with less than n this value need not significantly or will not significantly contribute to the response. So, if I truncate n against capital n as 100, if I say 3 - I must show in the calculation if I consider 4 5 6 etcetera, the contribution from 4 5 6 etcetera is not significant compared to that of 3 alone. So, there should be a justification of truncation. So, truncation is not simply a number, whatever you want to get. So, there should a truncation that comes from modal mass participation, again you see here even the truncation is decided upon the modal mass participating in the response.

Now, there is a very big question here which people generally get confused or annoyed. Now we are talking about response, response is displacement whereas mass is not associative displacement it is only inertia and we know inertia and displacement out of phase for a given system maybe sinusoidal cosine etcetera displacement and acceleration are out of phase because they are negative, right. So, in that case when we talk about truncation of mode shapes will it affect the mass contribution in opposite manner - that question will automatically come. So, we just judicially select the mass contribution in such a manner that the participation of mass alone in absolute terms should be there. So, what is that number all almost or most of the international codes say that your modal mass participation should be close to 90 percent.

So, no code tells as on today that you must include 100 percent mass in analysis - no code, every code says close to 90 percent. Now your question comes here - why codes have compromised 10 percent of mass is it not important. Now, please understand we need not compromisation of mass, it is compromisation of modal mass participation it means when all the mass getting displaced in one side in the first mode, when the second mass in the second mode they are getting compromised, in the third mode they are further getting compromised it is adjusted within this 3 modes that the modal mass participation in the response is almost significantly accounted for. It is not the most truncation it is not the mass alone, it is the mass participation on the response.

So, when you include the first mode and second and third modes you will automatically see the 90 percent of this will be equivalent to 100 percent contribution of the mass considering all the n degrees of freedom that is what people have said and therefore, they have made an ordinance in the regulations of design code saying that do not have to consider more than 90 percent. But unfortunately in the entire exercise it always leave for confusion that to check whether 3 is sufficient or I must have the 4th value. If you want to know whether 4 is sufficient, I must have the 5th value, it means there is an ambiguity here if you want to truncate the 3 you have to work beyond 3, also otherwise you cannot be sure that 3 is sufficient.

Now, this ambiguity will be taken away in this lecture today. But this is not addressed generally in dynamic analysis books because this portion of ambiguity whether tree 3 or n is sufficient how do you prove that. Therefore, there is a justification in modal mass truncation which we will discuss now. What I will do is, I will not bother about 100 minus 3 parts of the modes, I will only bother about zero to the first 3 part of the modes, but I will find out the effect of 100 minus 3 using this only. So, that is a very intelligent way of doing dynamics.

So, I will not look at 4 to 100 at all I will look only from 1 to 3, but I will account for 4 to 100 with 1 to 3 that is the part what we are trying to do today. So, modal truncation is mandatory, contribution of mode on participation of response is mandatory, mass is important because we are talking about inertia. Inertia has got two components acceleration is boundary dependent we cannot use that as our truncation rule we use mass and mass is associated with mode therefore, we say modal truncation not mass truncation - I say here modal mass participation.

So, please understand we are not truncating the mass, we are not compromising any mass at all; we are using 100 percent mass on the top side for my analysis. What we are trying to do is what is the effect of that mass within 3 modes, within 4 modes, within 5, modes given there are 100 modes in the system we are trying to do that only. So, more mass is not compromised only the participation of mass in the final response is compromised. Let us try to understand this statement very clearly. Now let us do how this can be done is there any questions here, is it clear why we are doing the truncation. How we are going to do it.

So, my justification in the truncation in this lecture should terminate at the justification of why not use 4 to 100, if I am able to tell that for example, then you are convinced that 3 alone is sufficient then we can demonstrate this with an example and show where that 3 alone we can reach close to 90 percent. Just remember this number is only a guideline if I get 89 it is all right even you get 91 it is still alright. So, exactly 90 you will never get. It is only a guideline 90 percent. Remember we are not compromising the mass at all, we are using the full mass you will see that I am using the full mass matrix, I am not truncating any mass at all; I am truncating only the higher mode shape contribution of the mass that statement is very very clear that is what we call as modal participation.

You will always see in the dynamics term mode and mass are always coupled like Eigen value and Eigen vector, frequency and mode shape, like mode and mass are always coupled they always get together. Whenever you say mass, mode will always play a role right, because modes are nothing but physical interpretation of relative position of mass at any frequency of vibration that is what the mode shape is. Therefore, we have to understand this in dependency of these two terms.

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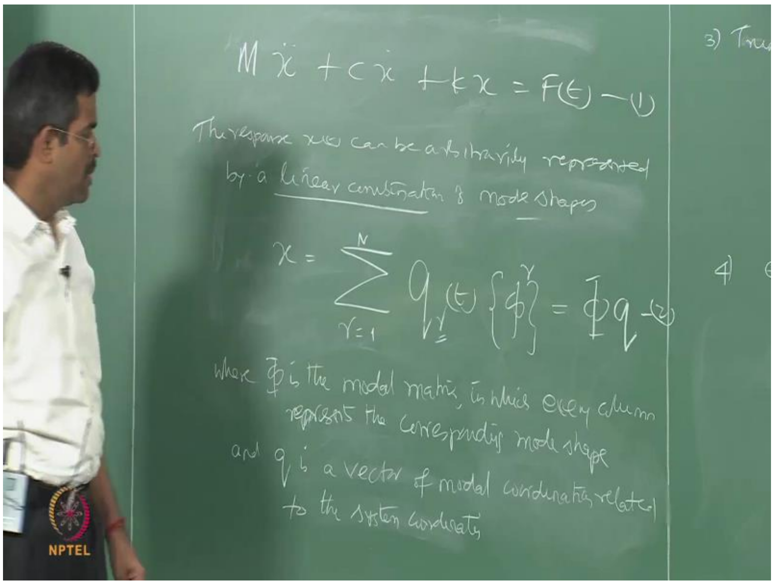

Let us take a multi degree freedom system where everything is a matrix and a vector. The essential outcome of this equation of motion should be x of t, a vector - I want x as a solution for this equation of motion that is what we are trying to find out I can use either a classical Eigen solver, I can use iterative schemes ultimately I am interested in finding out x of t frequency of mode shapes.

So, the response x of t can be arbitrary represented by a linear combination of mode shapes that is an authentic statement. So, let us try to in mathematical form x can be simply a linear summation superposition that is why some people call this as mode superposition method. It is available; it is true, it is correct because we are talking about linear contribution. One may ask me a question sir if the mode shapes, straight line, mode shapes, straight line some people draw mode shapes like this, some people draw mode shapes like this, is this linear, is this non-linear these are all rubbish it is only just representation we have no interpretation of the nature of the curve joining this point, this point, this point and this point. The nature of the curve is not affected here at all, it is only the related displacement of the mass positions at any frequency we are not bothered about what is the nature of this curve. So, it is not that the shape will qualify linear and non-linear.

So, it is a linear combination therefore, people call in literature this as linear superposition, mode superposition etcetera. Superposition is always valid when you have got linear combination. So, I should say here summation of degree of freedom r equals 1 to n that is a hypothetical statement because x should combine all the modes let us say q r of t and phi r where r is the mode number. I can call this now as simply a mode shape multiplied by q - equation number 2; where phi is the modal matrix in which every column represents the corresponding mode shape, first column means first mode shape, second column means second mode shape and so on; is a vector, vector of modal coordination related to the system coordinates.

So, this number will indicate which mode? Should be multiplied with the modal matrix because this is a full matrix this will be a contribution from the corresponding vector, I will call this equation number 2.

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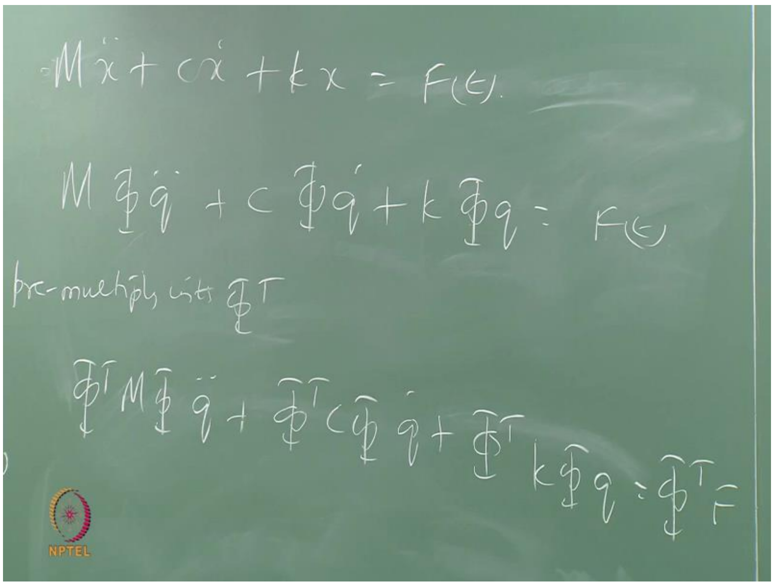

Then, having said this let us say m as double dot plus cx dot plus kx is F of t when x is actually equal to phi q, I can always rewrite this equation as m phi q double dot plus c phi q dot plus say k phi q as F of t pre multiply this with phi transpose. So, phi transpose m phi q double dot phi transpose c phi q dot phi transpose k phi q phi transpose F. Now we already have specific meanings of these terms - let me come back to those meaning.

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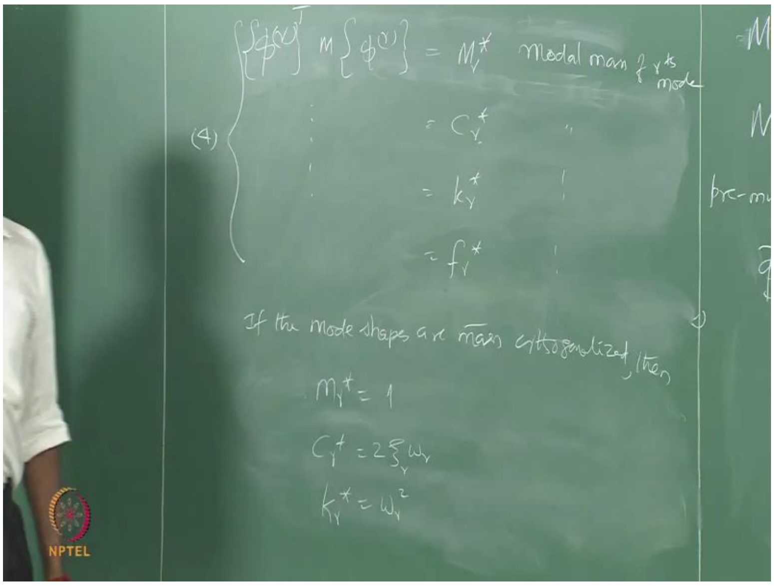

Phi transpose m phi at the rth vector let us say, I will write like this - because this summation is going to be applied because q is actually a vector which is getting linearly superposed for different mode shapes. Let me write the equation 3 in the slightly in a different form to expand this term separately. So, we say rth mode of transpose that is a vector, post multiplied by rth vector. I will call this as m r star; let us call the modal mass of rth mode. Similarly, c r star k r star and F r star which will be the representation of damping, stiffness and force at rth modes. The corresponding components of the equation 3, I call this set of equation now as equation 4.

Now, if the mode shapes are orthogonalized let us say to be very specific - mass orthogonalized then, m r star will be unity we already know that, c r star will be 2 zeta omega, k r star will be equal to omega square - I think we have proved this.

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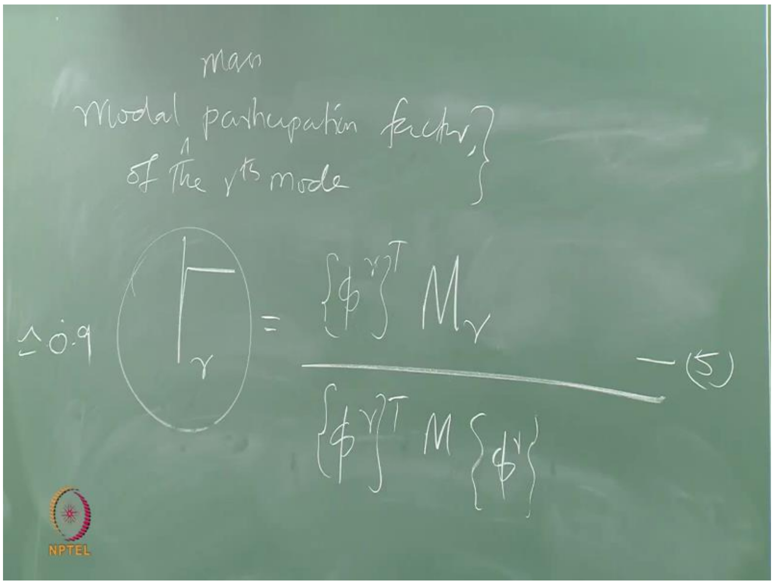

And the modal participation factor which I want to find out now of the rth mode will be given by a number which is equal to the rth mode which can be phi r transpose m phi r sorry m r by phi r transpose m. What does it mean? Let us try to understand physically this statement, let us say this is a number sub factor, let us say this factor is 10 percent - point one, what does it mean is the total mass pre multiply, post multiply and the rth mode will have 10 percent contribution on the rth mode of the mass that is the physical meaning of this.

Now, I can easily find out depending upon this weightage which mode has the maximum contribution. If I have this value, if the first value comes to be 0.9 - let us say first value itself is 0.9 I do not have to work out the higher modes at all because 90 percent is what the codes recommend. So, this will give me simply a index how much mode should I go ahead and modal truncation is done here itself from the participation factor and this is purely based on mass only there is no other term here except mass, and mode shapes which we have also discussed about few minutes back that is how we will truncate the modes.

So, this will give me a number a factor rather which is indicating the modal participation you may wonder it is a modal participation, I would say it is modal mass contribution. However, you want to understand this modal mass contribution in the whole response.

So, this factor is very efficiently used by us to truncate the modes, right. So, I do not have to actually work out all the modes, I first work out the first mode because I am working only for the rth mode shape this is not capital phi matrix rth mode shape, first mode shape I work out that. Find to get this factor, if it is not 0.9, I would go to the next mode and next frequency; similarly third mode and third frequency and so on.

So, I do not have to actually work out till n degrees of freedom to really know where is my 90 percent standing, I can easily find out from this factor where this 90 percent to be extended, is this clear. Having said this equation number let us say this is I think 5 because that series of equations of 4, this is equation number 5.

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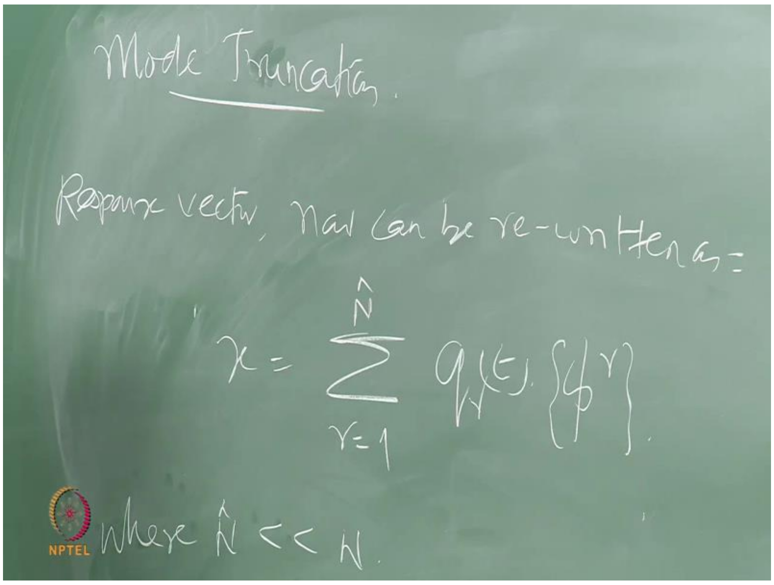

Now, let us talk about Mode Truncation. The response vector now can be rewritten as x is linear summation of r is equal to 1 to n hat, q r of t of phi r. The same equation of one, I am rewriting where n hat is lower than n. Now intelligently engineers asked a question now you have truncated the modes till n, do we have to make any compromise of the missing mass in the x of t because I have now truncated the n hat? I really do not know what would be or would have been the contribution of higher modes on x of t. So, do I have to make any correction on this?

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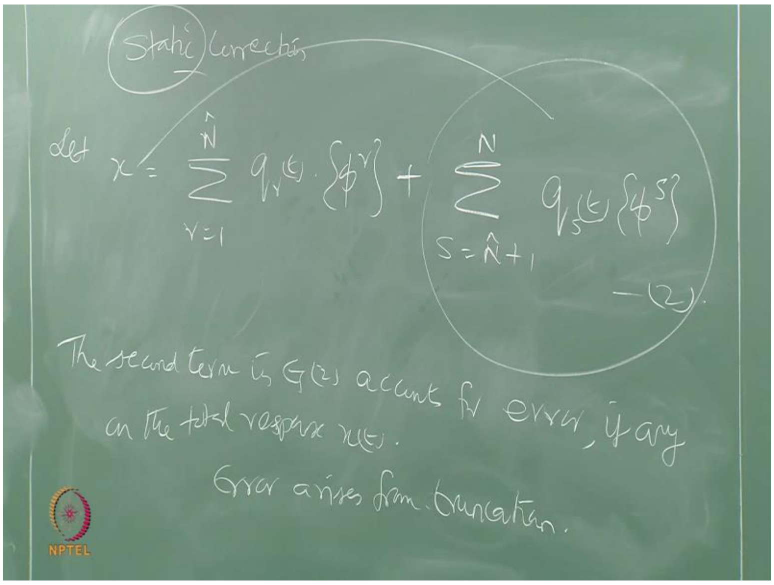

Static correction, I will come to this point why it is static correction why this word static is being used in the literature I will come to that point, but it is called static correction. Let us see what is the correction, let the response be written like this. I am going to do this like this I will write the response in two parts – one, from 1 to n hat, the next is n hat to n. I can write the response in two different parts, I can do that because it is linear superposition. So, let us write x of t or x as summation of r equals 1 to n hat of q r of t phi r plus summation of s is equal to n hat plus 1 till n. I am using a different count just to avoid confusion q s of t phi s this can be written, right - equation number 1 and the equation number 2.

Now, I want to actually know what would be the contribution of this specific component on x that is what I am interested in. Anyway I have to find out this, I am not omitting this. So, the correction if at all I want to do it should compromise or it should compensate for this part. So, I must focus here anyway I am evaluating this in my x of t. So, the second term in equation 2, accounts for error if any on the total response x of t, where is error coming from? The error arises from truncation; it is coming from truncation if you do not truncate - no error.

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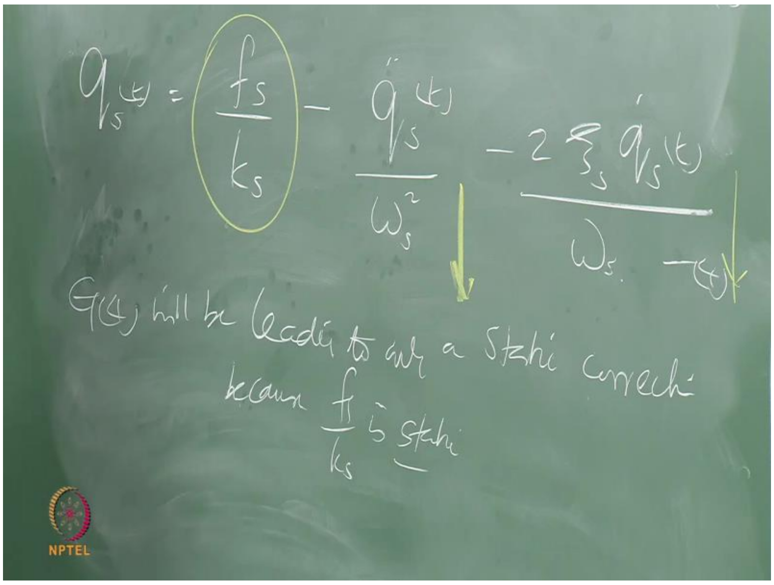

Now, let us go back to the equation of motion m s q s double dot of t plus c s q dot of t plus k s q s of t is F s. I am using the subscript s because these are the equation of motion with degrees of freedom which is mentioning the number beyond n hat, up to n hat is r I already have that in my contribution. What I am trying to do is, I am trying to find out the solution of this equation of motion with degrees of freedom from n hat to n see what is the x of this and let us see are we going to neglect this x if this x is very high compared to the original x then I should not neglect this. So, if at all I want to find the error I must always compromise the error only for this part. If I find the x for the full r system which is varying from 1 to n completely without separating from n r to n I will not be able to find the contribution of the missed mass on x of t, I can never do that because leave total response to me. So, I must find out only the contribution of this part.

Now let us quickly look at this equation and try to find out q s of t from here because that is the response actually is it not, that is the response - q s of t is the response which can be given by F s by k s you can quickly rearrange the terms F s by k s minus - I will take the mass term q s double dot of t omega s square because k and m minus 2 zeta s omega s that is the classical damping not omega s - q s, because I am dividing it by k separately. So, q s dot of t by omega s. Now you see this equation separately this is equation number let us say 3, this is equation 4.

Look at equation 4 the contribution of the second term on the response is inversely proportional to frequency, as the frequency keeps on higher and higher the contribution of this term on q will go down. We are actually talking about higher frequencies, therefore, the contribution of this term the dynamic contribution of this term on the response for higher frequencies will become lower because it is inversely proportional to omega square the damping term will also will not have a significant contribution because it is again inversely proportional to omega.

Now the one which I left out in the contribution is F s by k s this is nothing but a static response; that is why this correction is called static correction, there is no dynamic component involved in this. The contribution on the dynamic components on the response at higher modes will become negligible. So, equation 4 will be leading to only a static correction because F s by k s is static it is a static response.

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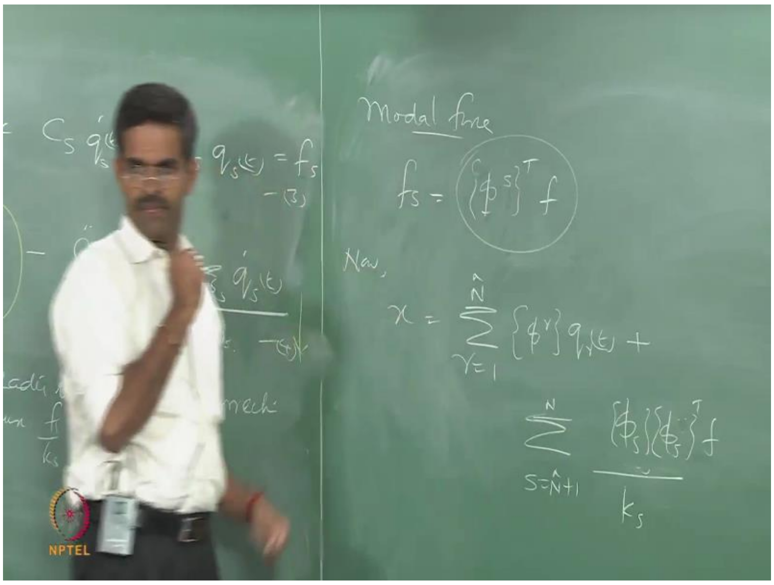

Now, let us look at the modal force. Now, one can have a question here we are not applying a modal force to the system we are only applying an inertia force to the system or the load is applied the structure is vibrating, causing displacement velocity and acceleration it is generating in inertia force. Where is the modal force coming into play? Modal force is invoked to the system by the relative displacement of mass back to the system. For example, the mass is fixed at this point if a system is vibrating if the mass is made to move here there is a force required to make this mass move from this point to this point that corresponding force in the specific mode is called modal force because without the force you cannot move the mass. So, we are talking about the modal force now.

The modal force of F of s can be expressed as phi s transpose F which is borrowed from the original equation of motion after pre multiplying phi transpose, let us see the equation back again we have this value there - the right hand side of this equation. What I am doing is I am extending that only for the s-th mode which is beyond n hat; n hat is a truncation mode for me. The total modes are n, I am truncating at n hat where n hat is much lower than n. I am now looking at those modes beyond n hat. So, I am only picking that force, why I am picking that force? Because this is the force offered on the corresponding mode. So, it is a modal force.

Now, I have to apply for this correction, apply for this correction back in x of t. Now x of t is nothing but the linear superposition of two values – one, from zero to n hat then, from n hat to n. From n hat to n, I do not want to include that I want to apply the correctioned weight because this may cause an error. Now my x will become superposition of r to n hat phi r q r of t plus s n hat plus 1 to n; please make correction - F is actually equal to phi s transpose F by k s and I already have a multiplier, let me rewrite this. The modal mass, the modal force is representation of this value and the contribution of x is again a modal participation of that force in the system that is all we have already said - which will amount to x as r equals 1 to n hat phi r q r of t plus summation of s n hat plus 1 to n, F s of f where F s can be 1 by k s of phi r phi r of t.

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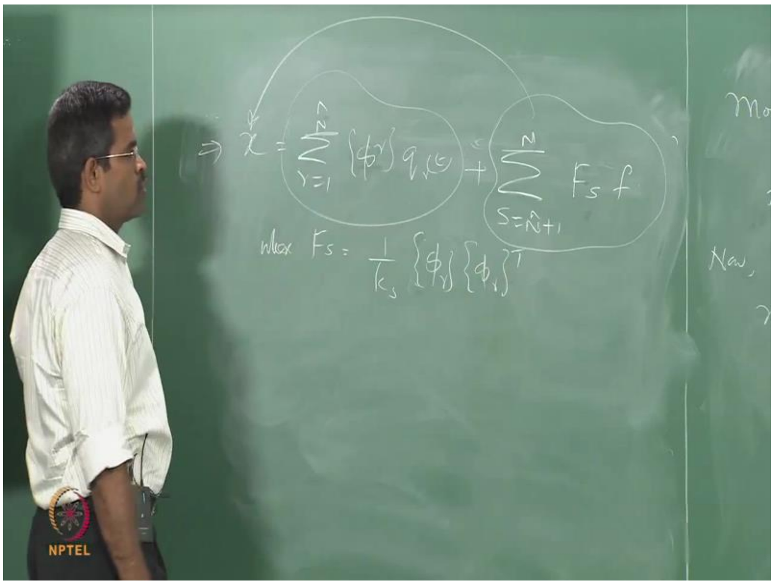

So, this becomes by additional term which is contributing to my correction on x, there is of course, the original term which is only truncating the modes till n hat.

We will expand this equation later. I have tried to show you an example in the next lecture I will show you how the contribution of the static correction will implement on x with an example in the next lecture. Is there any confusion here?

So, we have answered couple of questions now here - one, why do we truncate a mode, while you truncate a mode actually we are not truncating the mass at all, mass we are not anyway compromising we are only truncating the modal participation of the mass from an r factor we already know to model and participation factor how many mode numbers I must include I have a clue. But however, when you neglect higher modes what would be the contribution of that mass in the original response need to be corrected and that amounts to be significantly will do a static correction, because the dynamic component in that equation is minimizing for higher frequencies and that static correction component is addressed here. So, I must try to get an equation and then apply a problem and show that this static correction is compromising for the neglected modes sufficiently - that is what we have learnt here.