Dynamics of Ocean Structures
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Lecture – 17
Numerical Methods for MDOF Systems

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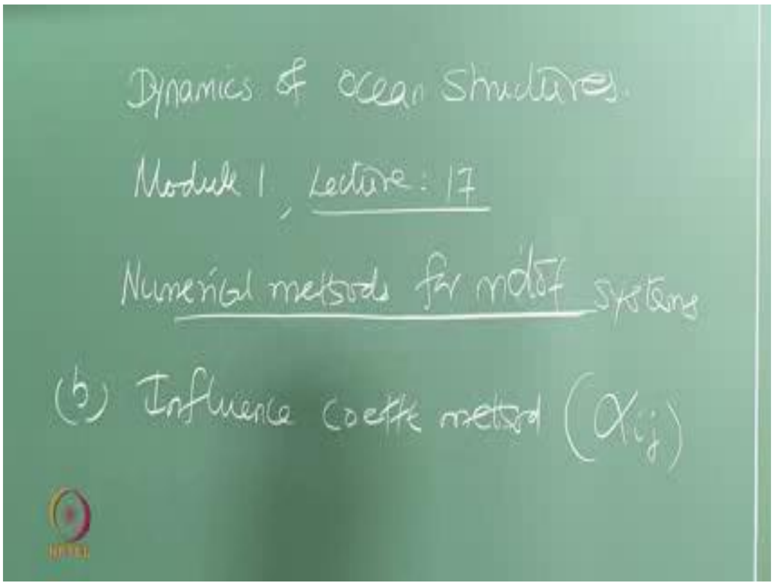

So, in the last lecture we discussed about one of the numerical methods, which can be easily used for multi degree freedom system to find out the fundamental frequency which was given by Dunkerley about 100-150 years back. There is approximate method it is very interesting to know this method converges to the fundamental frequency with respect to the values worked out with the latest numerical method where we can find out all omegas.

Now, there some discrepancies or there are some let us say a unacceptable statements in Dunkerleys theory saying the Dunkerleys theory will give essentially only the fundamentally frequency to an approximate value because it assumes that the systems behaves or is in linear mode of operation, one. Two, he does not give me the mode shapes which are very important because as I told in the beginning also even in classical Eigen solve problem if you do not know the mode shape it is very difficult to converge whether the given mode is fundamental or higher mode, it is difficult. Because you have to always know how many zero crossing the mode shapes has because mode shape is actually the relative displacement of mass points under the given accelerating force or the inertia force of m x double dot. So, looking at the displace portion of the mass at lamped positions one can easily converge and say – yes, the mode is related to the first mode of variation, second mode of variation, and so on.

Therefore since the Dunkerleys principle does not establish anything about the mode shapes it is very difficult to confidently converge that the frequency given by Dunkerleys is addressed in the fundamental frequency. Now, the problem because more serious when you have got closely spaced more there are structures were the first and second frequency will have a very marginal difference, very marginal difference. They will be (Refer Time: 02:23) successfully when you vibrate the system experimentally, you not be able to actually capture, so this system or this solutions can be applicable to systems where you have got distinct band of frequencies where the omega one, omega two, omega 3 are distinctly an explicitly different.

When you got a continuous mode of vibration let say omega one, omega two are very close and therefore, mode shape one and two will be more successfully adopted by the system. It is very difficult to capture the fundamental frequency using Dunkerleys, and is very difficult to establish that the frequency given by Dunkerleys is the fundamental frequency because it does not address anything about the mode shape.

Now, came in operation the numerical methods - out of which the most important and most fundamental method which came in early 70s or let us say late 70s - early 80s is the influence coefficient method which will discuss now. Influence coefficient are indicated by the letter alpha ij and alpha stands for influence coefficient and ij is the classical ith and jth term in the given influence coefficient matrix. So, one should first understand what do you mean by influence coefficient and we will talk about this.

Now, let us get back to the original equation.

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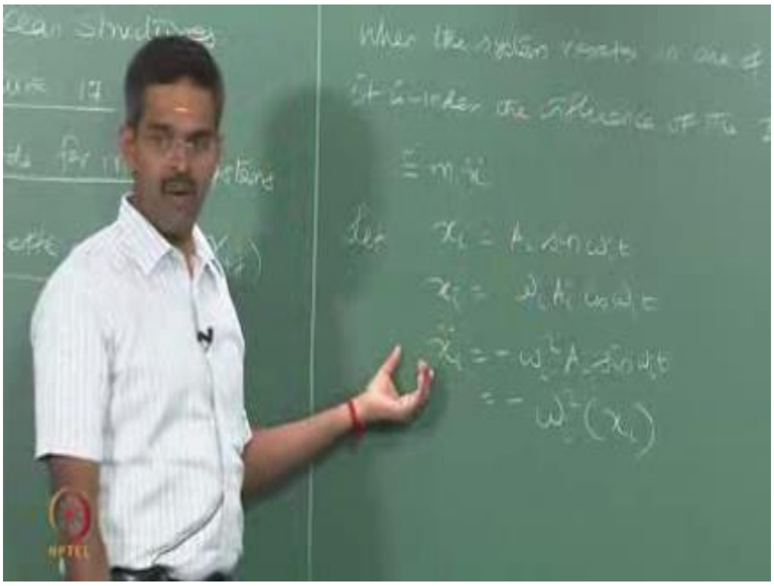

Let us say when the system vibrates in principal mode one of the fundamental modes it is under the influence of the inertia force, quite naturally this inertia force would be m 1 and x 1 double dot, why I am saying m 1 and x 1 double dot because it is understood that the fundamental mode will be converge to an omega 1 and the model mass or model participation factor in the first degree of displacement will be predominantly present in this system. So, approximately it will have influence or the system will be under influence of inertia force. Therefore, the whole method will be now circumscribing under the influence of inertia forces on a given system.

Now, let the displacement of the given be harmonic we can always assume harmonic displacement. So, let x i be a i sign omega t and we know that x i dot which can written as of x. So, the acceleration force which will be mass of these under which influence a system is now vibrating is nothing but a proportional multiplier of displacement itself and then multiplier is nothing but the frequency.

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Having said this, we also know that delta which the flexibility coefficient is inverse of stiffness and omega square is k by m which is now 1 by delta m. Hence, m i x i double dot which is the inertia force under whose influence the system is vibrating can be now considered as minus x i by m delta, I am borrowing this statement from here.

So, therefore, minus x i, m i x i double dot delta because this can be written directly from here. Minus x i can be m i x double dot of delta, I can expand this now for different degrees of freedom minus x 1 minus x 2 minus x 3. Let us say, we will take a 3 degree freedom system model we will expand this statement for different degrees because these are going to be the related displacement which is going to be the first mode shape if omega converges to omega 1 and so on.

So, I will use alphas instead of deltas because there is influence coefficient terminology given in the literature. So, I will use alphas. So, let me write this expression in expanded form - I will say alpha 11 alpha 21 and alpha 31, alpha 12 alpha 22 alpha 32, alpha 13 alpha 23 and alpha 33, m 1 m 1 m 1 why because this belongs to first column. Similarly, m 2 m 2 m 2 similarly m 3 m 3 m 3, simple reason being they belong to the third column m 3, they all belong to second column of the given matrix of alpha therefore second column m 2, they all belong to first column therefore m 1.

And obviously, I am looking for the influence of the inertia force. So, mass of that acceleration, mass of that acceleration, mass of that acceleration. Now it is very carefully written equation, please understand this equation. The common confusion will always start you have a representation of writing m 1 m 2 m 3 no problem. The movement you see alpha two here you will again say x 2 x 3 and x 1 that is a cycle. So, you please remember the equations are return in this form that first you write down the columns of every influence coefficient vector independently and if they belong to 1 indicate m 1 x 1 double dot, they belong to 2 - m 2 x 2 double dot, they belong 3 - m 3 x 3 double dot; why m 3 x 3 double dot or why m 3 m 2 etcetera? We are looking for the influence of the entire system and the inertia forces that is called influence coefficient method.

So, these are nothing, but influence coefficients. Now we can see a very well the displacement of x 1, x 2 and x 3 which is going to the first mode shape if omegas will be the first frequency than they are nothing but proportion of the acceleration components from x 1 double dot, x 2 double dot and x 3 double dot; that is why is called influence coefficient. They gain the influence of every inertia component on the given displacement at every mass point namely x 1, x 2 and x 3 that is why it is called influence coefficient method. So, now, I can write this influence coefficient separately as a matrix just plug it out from here which I called it has an alpha matrix which can be 3 by 3 if you got 3 degrees of freedom.

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And of course, the mass matrix for this particular problem can be written again as 3 by 3 and obviously, since the mass matrix may be always generated in such a manner that degrees of freedom will be marked at the same point where the mass is lump is going to be a diagonally dominate case and we can enter the mass matrix.

So, know we will take up this algorithm and update an example and derive the influence coefficient matrix and the mass matrix for a given problem. Now please understand in this case one may wonder, omega is actually a function of k and m - m is available here but why we are not having k. We are trying to find out the influence of the inertia force directly; indirectly the alpha matrix or the influence coefficient matrix is a representation of stiffness. So, k and m are present, but k is present indirectly. Of course, m is directly present here. So, we take up a problem now and try to derive the alpha and the mass matrix for the given problem than we will solve the problem. So, our aim is to find out the Eigen vector and the corresponding Eigen value. On the other hand to get the frequency and the corresponding mode shape for all frequency and all mode shapes for given n degree freedom model system which was not available in Dunkerlys proposition that is the idea. So, we will take up this problem and see how this can be expanded, any questions in algorithm here.

So, one should know how to write this set of equation than I will apply it then you will more comfortable. So, there I have written column wise row first and column next, row first and column next, second column, third column and so on so forth. And all will be the inertia forces which on whose influence the system is vibrating at x 1 and x 2 and x 3 displacements for a given omega. So, all this or this vector now will be actually an Eigen vector if omega corresponds to Eigen value. So, now, in this equation omega is not appearing, but omega is there are indirectly I will just place it here you come to know how we get extracting omega from this. So, let us take example and try to derive alpha and m. So, the example is like this.

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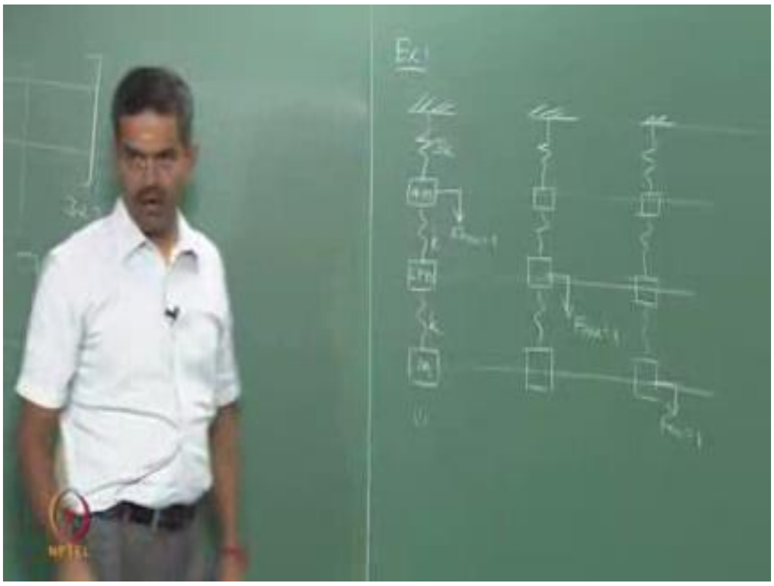

So, as usual I want to derive the influence coefficient which is nothing but the inverse of the stiffness matrix. We must give unit force at this location this is x 1, the second case could be give unit force at this location, force is unity and the third could be give unit force at this location and check what could be the restoring forces.

So, look at the specific case of sub number one, when I give a unit force here the restoring force will be obviously equal to - we now the stiffness is force per diffraction I have given unit force here. So, I must find out the diffraction which will be the stiffness of this.

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So, in this case it is going to be alpha 11 let us say, why 11? I am having unit force at the first degree and finding the response at the first degree itself, alpha 11 which is going to be 1 by 3 k because 3 k is stiffness of that spring. And the remaining two mass points, 2 m and m will subside automatically with the same amount therefore; I can say alpha 21 and alpha 31 will be also equal to 1 by 3 k because they will be pulled on with the same amount.

When I talk about the second sub division here, giving unit force here now we know that both these springs are in series I can find equivalence stiffness of this therefore, alpha 12, alpha 22 and alpha 32 need to be found out. So, now, I can understand that the springs 3 k and k these two. On the other hand these two will be in series. So, one can work out an equivalence stiffness of this. So, 1 by k equivalent 1 by 3 k minus 1 by k, 4 by 3 k or k equivalent is 3 k by 4. So, alpha 22 will be 4 by 3 k alpha 12 will also be 4 by 3 k because they are in series alpha 32 sorry the next one - alpha 32 and alpha 22 will be in the same amount. So, they will before by 3 k, an alpha 12 will be as same as alpha 21 by reciprocal theorem which already we have.

Similarly, for the third case the springs 3k, k and k will be n series.

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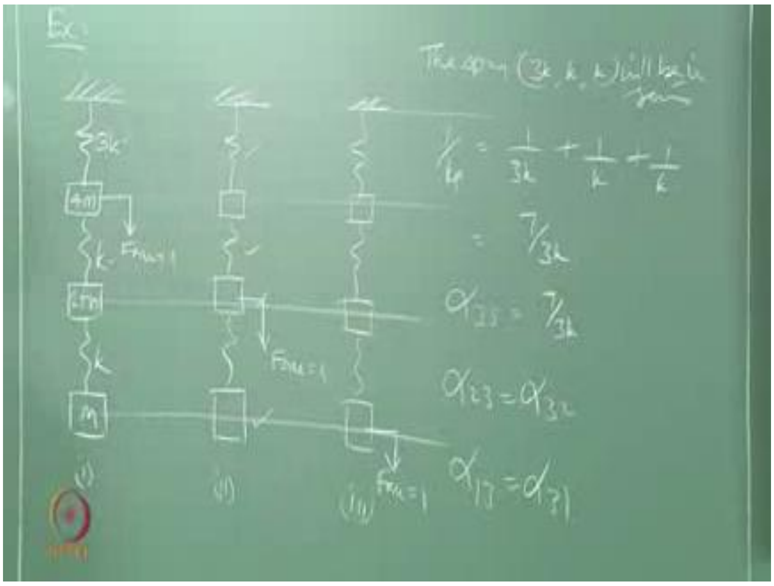

Hence 1 by k equivalent can be 1 by 3 k plus 1 by k plus 1 by k 7 by 3 k therefore, alpha 33 will be 7 by 3 k and alpha 23 will be as same as alpha 32 and alpha 13 will be as same as alpha 31 which we already have here. So, let us enter the alpha matrix here which we want to have which will be a common denominator of 1 by 3 k, so 1 1 1, 1 4 4, 1 4 and 7.

So, it is interesting that this matrix is symmetric, diagonally dominate whose inverse will now exist, whose inverse will be this stiffness matrix. So, alpha matrix is now generated which will be used as a multiplier here for the given algorithm. Mass matrix of course, can be generated quickly it is going to be this is the first degree is x 1, x 2 and x 3 therefore, 4 m, 2 m and m and we all know why the (Refer Time: 19:27) diagonal limits are 0 because they are taken in the same point where the mass lumped.

The influence coefficient which is as same as the delta matrix, I have the linear diagonal values; I have the mass values we can easily find the Dunkerleys frequency for this. Can you find out Dunkerleys frequencies for this?

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It says 1 by omega square m 1 let us say alpha 11, m 2 alpha 22, m 3 alpha 33 we have all the values with us now, please find out and tell me what is value; what is the value of omega? Yes, that is 1 by omega square that is why we get k by m.

Let me apply this algorithm now for influence coefficient method, I will write down the alpha matrix here because I may require it.

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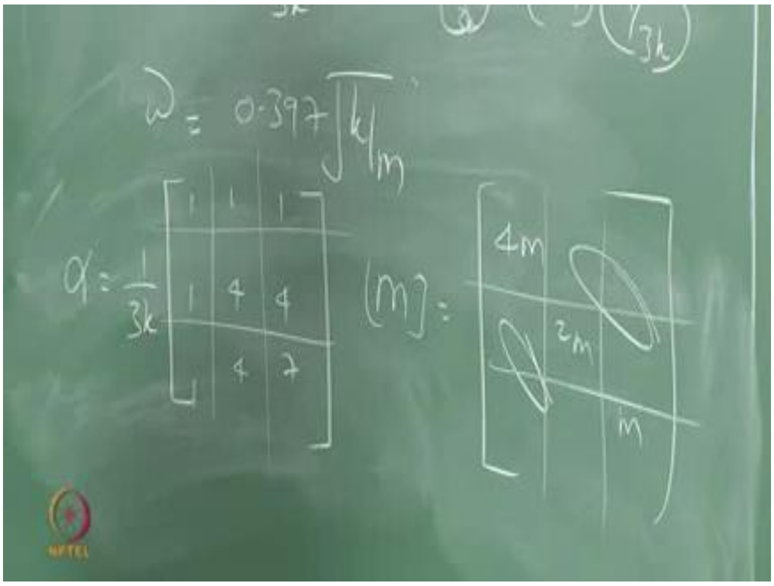

I may require the mass matrix also, so minus x 1 minus x 2 minus x 3 alpha 11 21 31.

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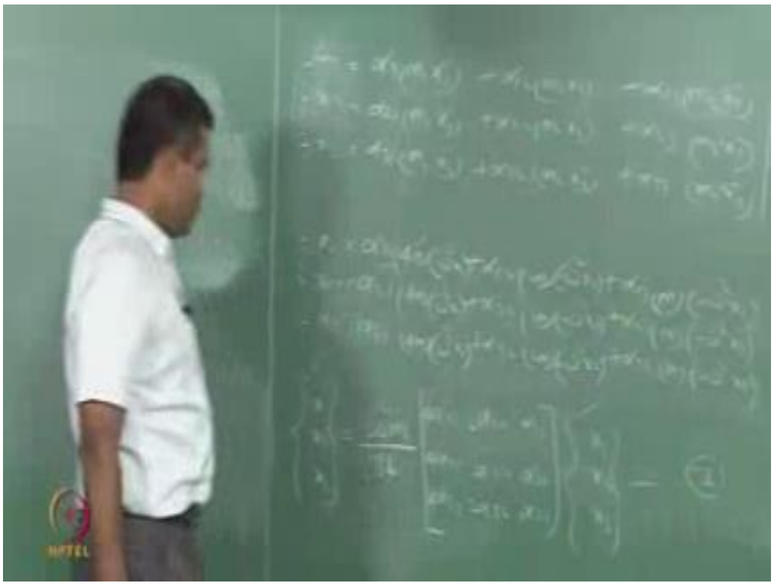

The common mistake what people make is they will write m 1 m 2 m 3 here that is I am saying please remember the first column represents the first mass, the second column represents the second mass, all other terms will be zero because mass matrix is diagonal.

Let us substitute it here I will call this equation set as 1, the full set. So, minus x 1 minus x 2 minus x 3, alpha 11 I already have we have written it as it is alpha 11 alpha 21 alpha 31, alpha 12 alpha 22 alpha 32, alpha 13 alpha 23 alpha 33; m 1 is 4 m, m 2 is 2 m, m is as m 3.

I (Refer Time: 23:27) x 1 double dot, x 2 double dot and x 3 double dot here. We already know x 1 double dot can be connected to x 1 by a relationship which is minus omega square x 1. So, I can write here minus omega square x 1 instead of x 1 double dot assuming harmonic excitation, so minus omega square. Similarly minus omega square x 2, minus omega square x 2, minus omega square x 2, minus omega square x 3, minus omega square x 3, minus omega square x 3. The whole expression the negative term goes off I can rewrite this equation set in a matrix form like this I will, write it here; x 1 x 2 x 3 which is a vector which will be omega square m by 3 k. I am getting m from here, I am getting omega square from here, denominator of 3 k is alpha coefficient where I have it here - which is now going to be alpha 11 21 31, alpha 12 22 32, alpha 13 23 33 with a multiplier of 4 4 and 4, 2 2 by 2 and 1 1 1; multiplied by x 1 x 2 x 3.

Now, just read one line of this equation, x 1 will be equal to omega square m by 3 k, 4alpha 11 x 1, 2 alpha 12 x 2, alpha 1 3 x 3 which is as same as here - x 1 is equal to omega square m by 3 k that is a denominator here of alpha 11 x 1 plus 2 alpha 22 alpha 12 x 2 and alpha 1 3 x 3 is available here. So, I have transformed the equation set into a matrix form like this.

Now you may look at this equation, this equation has got both sides a vector which is not known to me. What you know this equation is the matrix and of course, the k value and the m value. I do not know because k value is a known factor here, this stiffness spring is known to me k and m are known to me, but I neither know omega nor I know the vector. Once you have any algebraic equation which is LHS and RHS of the same unknown the best procedure to solve this iteration, you assume a scheme, assume a vector and keep on substituting the vector for the left hand side and multiply and get a new vector, compare the new vector with the assumed vector and keep on doing this exercise until the new vector and the assumed vector converges. That is a basic scheme to do any numeric integration like this, let us try to do that numerically and see who we can solve this.

So, this may scheme available to be now, which is equation 3 in a matrix form I must assume the vector. Now the questions is how will I assume this vector, one can start all as 1, one can start 1 0 1 so on. Now the question is what will be the relevant value of assuming this vector. Now this problem is actually more focused on the type of mass concentration on every degree of freedom. But one thing is very certain; I am first attempting to look at the fundamental frequency of the problem. The moment I say I attempting to look at the fundamental frequency of the problem, I must always know that the vector will have all positive number or all negative number.

So, there is no re zero crossing in the given vector, the vector should not have any zero crossing therefore, 1 0 1 will not work out may be plot 1 0 1 you will always have a vector crossing at zero. So, I do not want any of the number plus and minus mixture because I want a zero crossing here, why? I am attempting look in the fundamental frequency which is a capturing point of the problem. Therefore, I must take all positive to start with we can say 1 1 1, keep on iterating. The good thing about this algorithm is whatever a vector you start with provided you are logically correct it will converge, I do not have time to demonstrate so I will take up 1 2 and 4.

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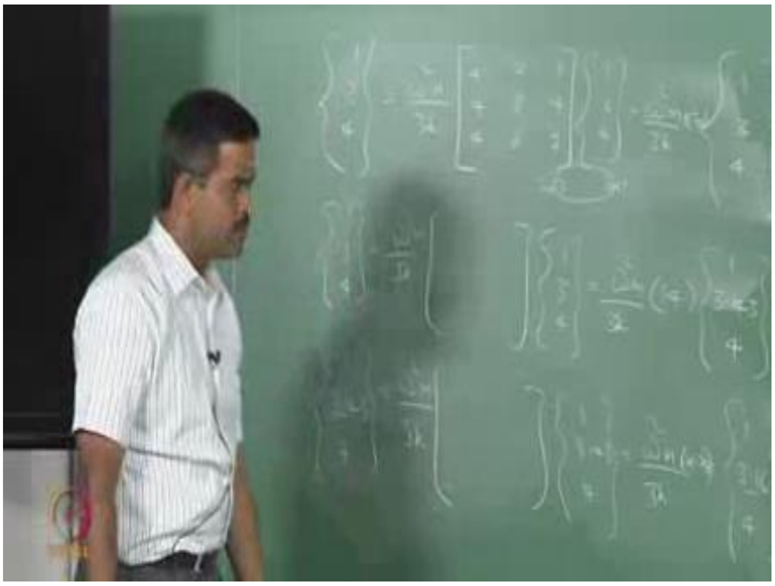

Please do not ask me how did we get this number of 1 2 and 4, I have taken it exactly from here 1 2 and 4, I have assumed it like this. We can also start with 1 1 and check, it is going to converge automatically with the same value do me ask sir 1 2 and 4 vector is not algebraically equal to 1 1 1 vector. Please understand mode shape is the relative displacement of the mass point. I have always take a multiplier out and then I give the vector which is going to be the same meaning right, let us see how it is that. As long as all vector points are positive, as long as the vector denotes the first frequency our answers are all right. So, 1 2 4 I have started with omega square m by 3 k will be the multiplier, even though I have a value of m and k may be m is 100 kg, k is 100 Newton per meter do not have to multiple at this stage here. Live it as an unknown value which can be substituted later because do not get complicated by multiplying this and having a fraction on the left hand side of this equation. It is not required, let us have this equation of 4 alpha 11 is what, what is the value.

So, can I write here simply as 4 4 4 alpha 1. Similarly the next one is alpha 12 twice of alpha 12. So, I should say 2 8 and 8, the third column will be 1 4 and 7 multiplying with 1 2 4 that is what I start. So, now, I will get a value when I multiple, you see here this is 3 by 3, this is 3 by 1. So, I will get a vector now which will be a new vector, this vector if fortunately becomes same as this then I converge already achieved, but let us see what is the vector I am getting multiple this and tell me what is the vector I am getting. So, any way I have a multiplier omega square m by 3 k here I take 12 out because it is 4 plus 4 8 plus 4 12, I take 12 out therefore, I get 1 here - be happy because I have 1 here and 1here also.

Let us see what happens to the other numbers. So, this is going to be 4 plus 4 8 plus 4 (Refer Time: 30:48) 4 plus 16 - 20 plus 1, 36. So, this becomes 3 this going to be 4 plus 16 - 20 plus 28 - 48 it means 4. So, I have started with 1 2 4 landing in 1 3 4. So, it is not right therefore, let me start again with 1 3 4. Now there is a very important statement here for example, you started with 12.1 and you are getting 2.2 here let us say, do not try to go back in between because the convergence is continuous, if you jump the step you never get converged. So, you have got compulsory borrow the same vector and do it.

So, 1 3 4 omega square m by 3 k of the same matrix of 1 3 4 give me the new vector quickly omega square m by 3 k of some multiplier of; this multiplier is going to be 14.

So, 1 and 4 are matching, but the number was not matching therefore, let us do 1 3.143 of 4 is omega square m by 3 k of the same matrix of 1 3.143 of 4 and I get omega square m by 3 k multiplier of some vector – 14.28, 1 3.16. So, let us check again with 3.16 because 3.1 we have converged let us go for the next digit also. So, let us try with 1 3.16 in left hand side and see whether it is converging. So, you must check whether I am getting the same vector back.

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Here the multiple will change because it is.

Student: 14.32.

Of.

Student: (Refer Time: 33:14) 16.

So, we have converged. Now I will tell you where are the mistakes, what people generally commit the first. Mistake what you will commit is when you are writing the governing equation to form this matrix you make an error, I told you very deliberately it should alpha 11 21 and 31, the first column means m 1 x 1 double dot, second column means m 2 x 2 double dot, third column means m 3 x 3 double dot therefore, that error should not be their once you practice number of time you will not do that error.

The very common and very psychiatrist mistake what people make is they take this multiplier and put it here. Please understand I am not bother about this multiplier at all, I am only bother about the relative displacement of the mass. Once you have this point in your mind you will only pick up this and get this back again. So, this number will be keep on changing.

The third issue is if it is progressively increasing it will increase, if it is progressively decreasing it will decrease, if it increase and decrease you are missing a mode shape in between somewhere, so your assumption is wrong. So, this method is having some self check also, you see this progressively increasing means it will increase - no minus and plus will be changing, right. So, you have to be very carefully in that. And the forth issue is how to get this vector in the start, had your start with 1 1 1 what would have happened. So, I think you should experience that you know what would have happened; you land up in the same vector with this.

Now, the question is how do you get me omega from this. So, that is read this equation the left hand side equated right hand side can I say - algebraically 1 equals omega square m by 3 k of 14.32 of 1 can I say this? From this I can easily find omega. So, what is omega? Some value of k by m point.

Student: 0.457.

0.457 k by m, whereas if you compare this quick value with the Dunkerleys, is close to 0.4; 0.397 is close to 0.4 So, after doing so many mathematics you got a number where as Dunkerleys came close to this number in minutes. So, that was idea, that was a efficiency of the method proposed with Dunkerleys earlier.

Now, I am not interested in going to stop at omega 1 and phi 1, my job is to find omega 2 phi 2, omega 3 phi 3 also. I already told you that once you start finding out the higher modes it is always better to find the orthogonal modes automatically because then you get an advantage. In the previous example what we did in Eigen solver is you found the modes in vectors, you found to be they are non orthogonal you diagonalize (Refer Time: 36:19) orthogonalize them by a multiple vector mass matrix. Now in this case I will automatically set an equation to derive the orthogonal vectors automatically. So, whatever vectors I will get they will become automatically orthogonal. So, we will see that in a next lecture.

So, there is a small take home assignment for you, the take home assignment is very simple. For this problem try to write down the equation of motion, try write down the equation of motion, try to find the mass matrix which you already have with me, try to write the stiffness matrix which you already generate and take an inverse of this and check is this inverse same as at delta matrix what you got. So, that is an assignment for you. And the second assignment could be start with any positive vector and see is it converging or not, are you getting the same proportional displacement like this. Then one may ask very interesting questions here, the question is very clear. If the displacement is not exactly proportion to this number this multiplier will keep on changing, I will get 14.32, you may get 15.55, and she may get 14.86.

On the other hand for different people one should have different omegas, the answer is not there you will get the same multiplier with the same fraction whatever you assume here - that is you please check and tell me whether coming out or not. So, the omega and corresponding phi will not change it will not change.

So, if you are not able to understand this you must go to the algorithm first how did you derive this algorithm and try to practice yourself to write the equations. I would argue to write the equation same are as you wrote black board, do not write row wise - write step by step like this that is how this equation can be done without any error. This is one of the easiest possible computers methods to find all Eigen values and Eigen vectors where many softwares follow this method only. The only difficult in this method is to start with a vector; generally softwares have a program where the vector will be multipliers of m ratios. If all m’s are almost equal in a given multi story building or a given platform then you will start; obviously, with 1 1 1 and you will always start only with the positive mode.

The another advantage of method is depending upon your choice, you can always get either the first frequency or the highest frequency also because it all depends on how do you assume the vector whereas, Dunkerleys will always give you only the fundamental frequency. So, since I assume a vector all to be positive I am claiming that this will be the natural frequency I will verify this only after I get all the frequencies and mode shapes then will conform. Now we presume that it should be an actually frequency, but all you carry may suspect that so the next mode can also be all positive we do not know actually let us try - let us try, examine, write all the vectors and then will conform yes gives me only positive vector therefore, this is my fundamental frequency. If it is accepted then you should appreciate for finding this as a fundamental frequency you are suppose to do all the omega and phi's whereas, Dunkerley did not mean any such exercise is simple said - pick up the value, get the value it will be only the fundamental frequency.

Now, Dunkerleys proposal in 1859 was not supported by algebraic definitions that it will converge a fundamental frequency therefore, they are not popular. I will show you mathematical proof subsequently some lecture where always it will converge to only fundamental frequency, mathematically I can proof this, an algebraic equation set I can proof. Then Dunkerleys value would have been more appreciated. So, this method is also sometimes called as matrix inversion method, in literature they calls matrix inversion method some authors call them as matrix method. Essentially the original name given to this method not by me, the authors are influence coefficient method available in the literature because the whole system is under the influence of inertia forces that is why the whole equation is circumscribed under the influence of m 1 or m i x i double dot that is the reason.

And this equation sets principally assume that the structures undergo in harmonic linear excitation that is an assumption in this system. And this can be applied to any n degree of freedom models provided you are able to multiple the matrixes easily. The most advantages procedure in this method is this method will not demand inverse of a matrix at all. Inverse of a matrix the most difficult task in matrix algebra, it is not only true with me it is true to all of us, inverse always dangerous. This method will never ask you to invert is asked you only to multiple, if you are problem multiplying also you should sit on Dunkerley only, we cannot help. So, that is why inversion required. And people generally are confusion in deriving flexibility matrix, what they generally do is they very good in stiffness matrix they derive the stiffness matrix and take an inversion, when they take an inverse they make mistakes.

So, this method indirectly deliveries your flexibility matrix for you, which will verify as a take home assignment and see, it is true or not that is why it is called this method as got lot of advantages easily programmable, so I would argue that at least few if you or at least all of you should write the computer program and try to run this for n degree of freedom, parallelly check for Dunkerley etcetera then at the end of the lectures you should have a coding which input in takes the mass and stiffness value in n degrees of freedom and instantaneously you must get omega and phi by at least 5 methods. I can compare and give a percentile error immediately in minutes, which software can do after you do a mathematical modeling of this in the software, so which you can do.

So, this is a very handy tool for research people we really want to know what are may lower band and higher band frequency given geometric form which is non standard form this method is very helpful. So, need not have to invert a matrix you do not actually form a matrix at all, except that you have to only understand the fundamental system of equation and simple multiply and within a minutes you will be able to get the omegas and phi's all omegas and all phi's.

The most advantages method in this case is it will give you omega and phi sequentially, it will give omega 1 then omega 2 and omega 3 that is the beauty of this method. So, this method is appreciable by lot of an; even today this method is valuable because if you want really resolve omegas and phi's by hand there is one of the easiest tool you have. There are many other method, every method will have merits and demerits. There are demerits in the method also, I will come back to that at the end; there are demerits in this method we will see that later. As of now this method is very advantages for us and let me tell you how to find omegas and phi in the higher order in the next class, any question here.

So, try to do this take home assignment and try to verify whether is converge or 1 1 1 and see what is a difficulty where and start writing coding here in MATLAB or in C++ any language coding you know try to start doing the coding now so that you at least have the solution ready. The moment problem is posted on the board you will have the answer instantaneously; we can check the answer in the black board parallelly with the solution given by the program.

Thank you.