Dynamics of Ocean Structures
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Module - 1
Lecture - 7
Duhamel's Integrals

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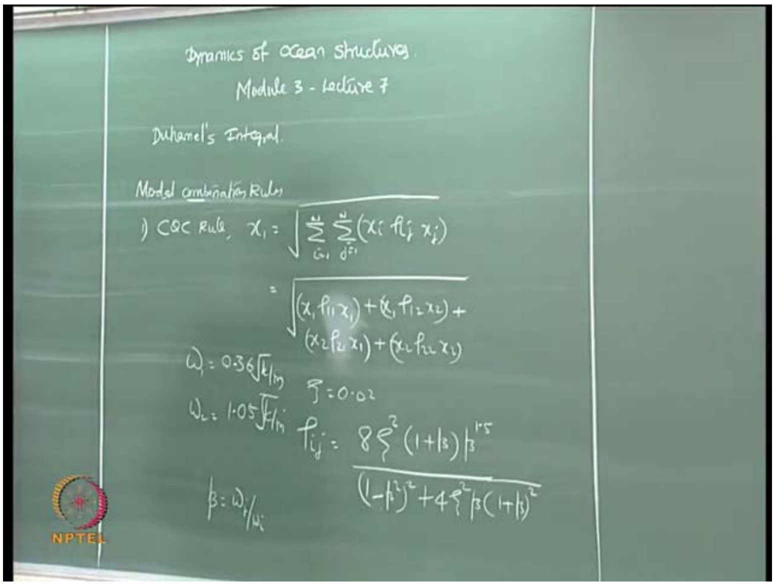

So, in the last lecturer we discussed about combinational rules, we spoke about the truncation of modes. We also spoke about the static correction we want to avoid the higher modes. Now, we take up an example and show that how the modal combinational rule can be employed, for different categories of problems very quickly by simple example. So, we have taken this example where I have got four degrees of freedom problem, omega 1, omega 2 are given to me.

Though in the example showed in the last lecture we have clearly said that the modal mass participated from the first mode is closely to 90 percent, we need not have to look for the contribution from the higher modes. But still just for understanding, to operate this rule, see how it can be d1 let say for example, I want to know the maximum response in the first level of the mass concentration may be the first floor in the given problem.

So, if I know the responses in different floors independently how can I combine them to get the maximum response because one is interest in getting the maximum value by some combinational rules. So, picked up an example of 1 and 2 let us say for example, mode 1 and 2.

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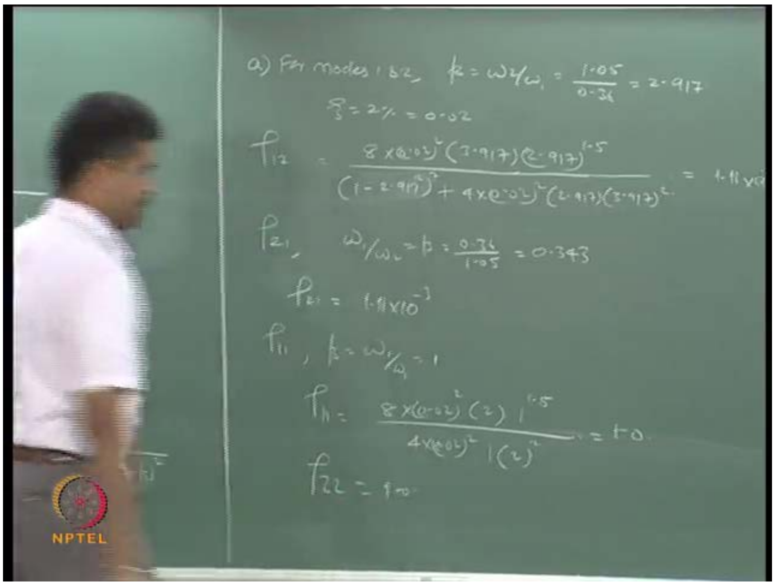

I have the values omega 1 and omega 2 here, so let us compute beta very quickly. So, beta in my case is going to be omega 2 by omega 1, which is going to be 1.05 by 0.36 which becomes 2.917. And zeta already we know it is 2 percent which is 0.02. So, in this given equation of computing x 1 max what I want it know is the cross modal co-efficient 1 1 1 2 2 2 1 and 2 2 because assuming that I already know this responses independently.

So, let us compute 1 2 so i j stands for 1 and j stands for 2, I should say 8 into 0.02 square that zeta of 1 plus beta. So, 3.917 and 2.917 to the power 1.5 divided by 1 minus beta square. So, 1.917 the whole square of plus 4 0.02 square 2.917 and 3.917 the whole square. How much is this, this comes to 1.11 10 power minus 3. So, the cross co-efficient the cross modal co-efficient for 1 2 and 2 1 should be same.

Let us check that so to compute rho 2 1 I must compute the ratio beta which will be omega 1 by omega 2, which will be the inverse of this which is 0.343. So, can you find out rho 2 1 in the same manner. So, let us compute rho 2 1 in the same fashion as we did for the previous case, you will check that it will also become the same value of 1 2. Let us compute rho 1 1 the cross co-efficient with the same mode so for computing rho 1 1, I require beta this is omega 1 by omega 1 which is become unity.

So, then rho 1 1 can be computed as 8 in to 0.02 square of twice of 1 ratio to the power of 1.5. So, this term goes away 4 0.02 square 1 and 2 square, which will come to 1 actually. Any way the answer is 1.11 fine yeah, let us write it as 1 minus 2.917 the whole square. So, this comes to be unity. Similarly, we will check rho 2 2 will also be unity. So, one can easily find because I have the cross modal co-efficient of all the four value, have the independent responses I can get the maximum response in any floor I want by this equation. This combination rule what we call as complete quadratic combination rule, which is c q c rule this general rule which can be applied to find out the maximum response. Now, Alternatively if the modes are not closely spaced, any questions here?

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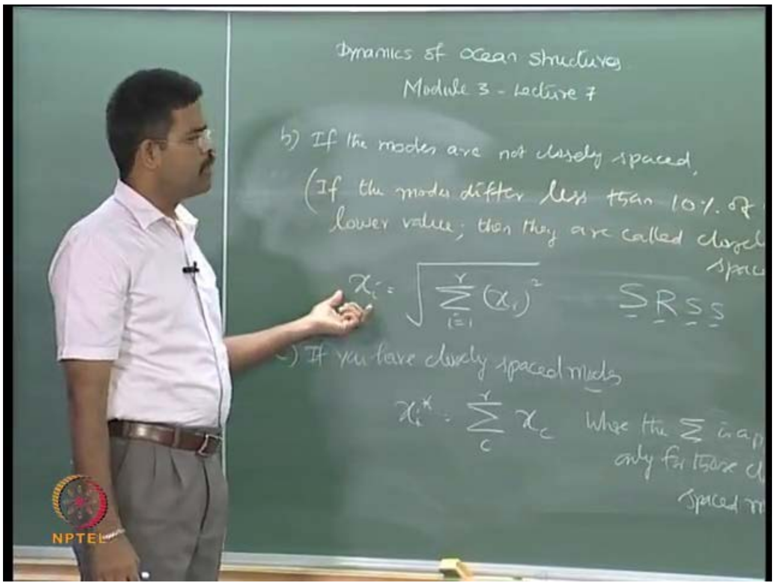

If the modes are not closely spaced, what do you understand by not closely spaced? The modes are said to be not closely spaced, if the modes differ less than ten percent of the lower value, then they are called closely spaced. Suppose if you have modes or if you have frequencies, if you frequencies whose values differ successfully less than ten percent of the lower value, then we call them as closely spaced frequencies. Suppose in that example if you do not have a closely spaced frequency, then the response max can be worked out using this rule, which is which is a simple statistical rule which we call as square root of sum of squares.

So, I made a square sum them up and took a root, so it is a square root of sum of squares it is a simple statistical combination, which can be employed if the modes are not closely spaced.

Suppose, if you find or if you have closely spaced modes the moment I say modes I am always talking about frequency because you cannot actually, compare the modes to be this closeness. I am talking about the frequencies, if you have closely spaced modes then again you can find the maximum response in any floor i th floor, using a simple combination which is c to r c, where this is apply where the summation is apply only for those closely spaced modes. Only for those modes you can apply this and take the values simply sum them up.

So, these are the three different combinational rules which are available in the literature with the help of which you can find the maximum response, for a multi degree freedom system models. if you know the mode shapes and frequencies and if you know the damping ratio as a percentage of a statistical damping. Having said this now there is 1 more level of complexity involved in computing the solutions for dynamic systems prescribed dynamic loading, we already said what a dynamic loading is.

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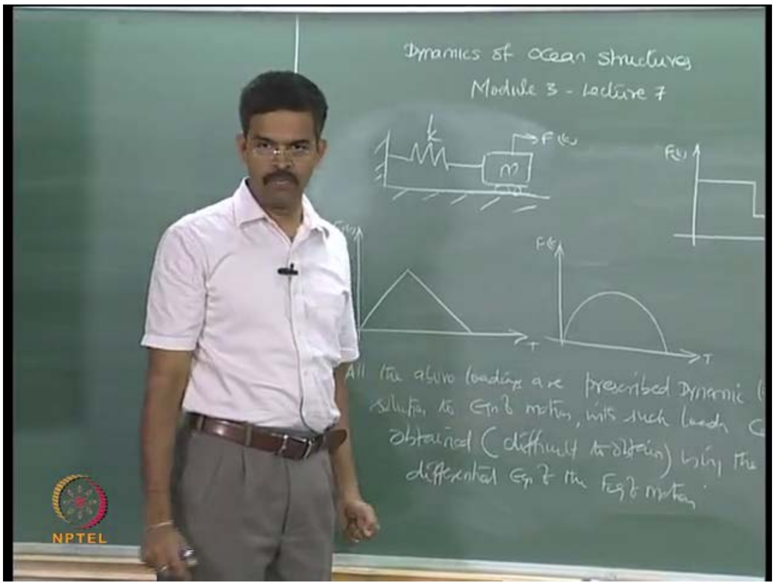

Let say I have a simple system now f of t can be any 1 of the following. Many examples of this order all the above loading are prescribed dynamic loading, you already know what is a prescribed dynamic loading because the time history of the loading is completely known. But solution to equation of motion with such loads cannot be attained or I should say difficult to attain using. the solution of differential equation of the equation of motion. Simply using the classical equation of motion, which is second order differential equation by solving that equation we will not be able to get solution, for f of t which becomes a prescribed loading of this order. In such cases what do we do? (No Audio: 13:37 to 13:51)

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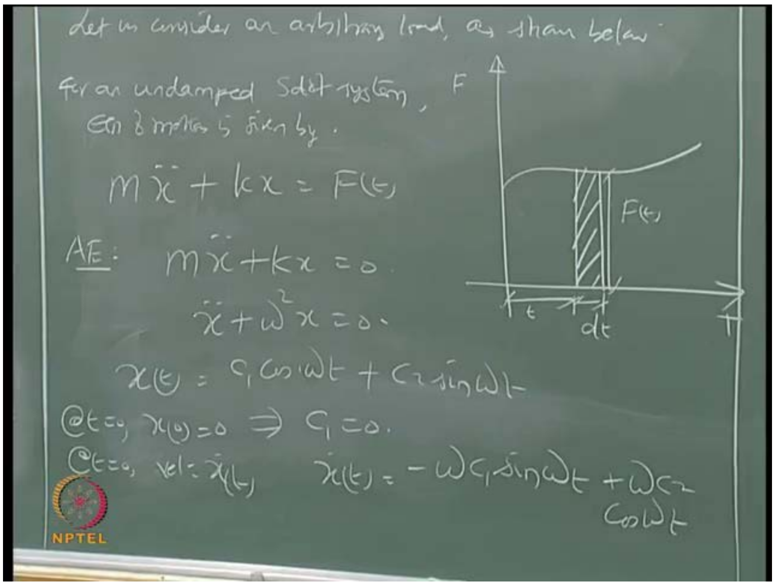

We employ a special type of integral called Duhamel's integral. Let us take an arbiter loading as shown here, let us consider an arbiter load as shown below the arbiter load is here this is my T some load I pick up this load, for a small segment and I call this as T and this as of course, d t and this is my f of t for this problem. Of course, the vertical axis is showing the load, for an un damped single degree freedom system we already know the equation of motion is given by m x double dot plus k x is f of t.

The auxiliary equation can be written for pre vibration, we already know that which is m x double dot plus k x is 0, which is x double dot plus omega square x is 0 which gives me x of t as c 1 cos omega t plus c 2 sine omega t at t equal 0 x 0 is 0 initially no displacement which implies c 1 become 0 at t equal 0 x velocity is x dot of t. So, let us say x dot of t from this equation is minus omega c 1 sine omega t plus omega c 2 cos omega t.

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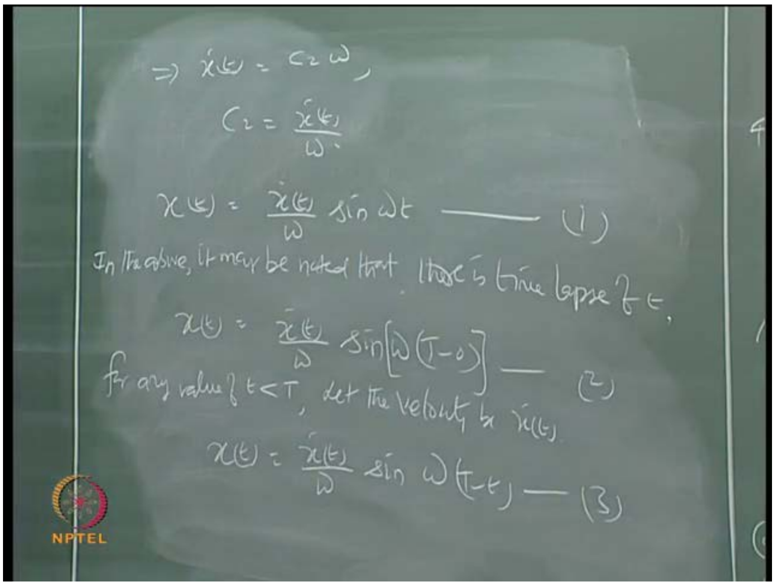

We already know c 1 is 0, which implies x dot of t is c 2 omega says c 2 omega is x dot of t by omega. Therefore, x of t is x dot of t by omega sine omega t this becomes my solution, for the pre evaluation problem. Now, in the above it may be noted that there is a time lapse of t, when you are measuring the response. Therefore, x of t can be re written as x dot of t by omega sine omega, let us say t minus 0, when there is no lapse let us say t minus 0, this becomes my equation.

For any value of t less than capital T, let the velocity be x dot of t therefore, x of t now becomes x dot of t by omega sine omega t minus t because I am implementing the velocity lapse also, is that clear? Because there is a lapse of small t, which is lower than the capital T is the period of the loading. So, for that x of t velocity is replaced with the lapse of t minus t here. Now, out of this x of t we pick up a small segment.

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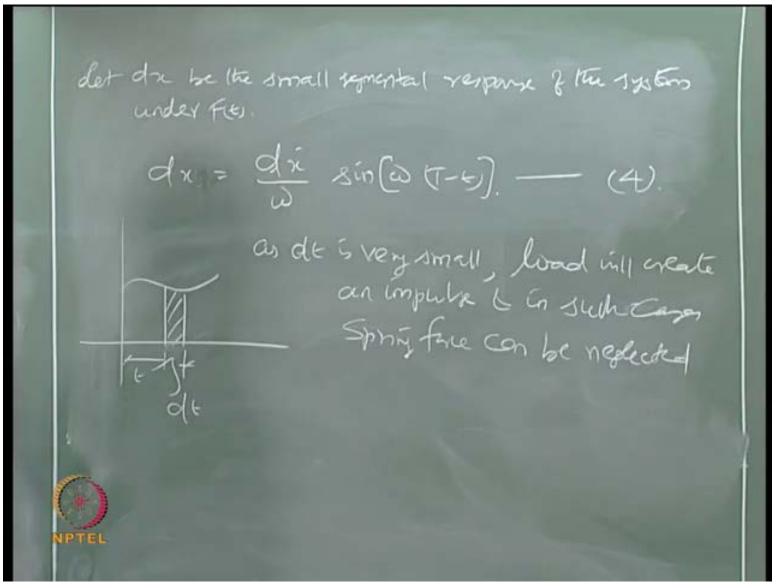

Let d x be the small segmental response of the system under f of t. So, I can write d x as d x dot by omega sine omega t minus t, equation four. Now, this is my loading criteria as d t is very small as d t is very small the load will create an impulse and in such cases the spring force can be neglected, can you tell me why? Spring force is a restoring force that is an impulse force this will have a very high amplitude, for a very short duration for that duration the restoring force can be neglected is it not you can always practically see any structure any system excited to an impulse force for a very short duration, the restoration will only follow that. So, for that duration I can neglect the spring force.

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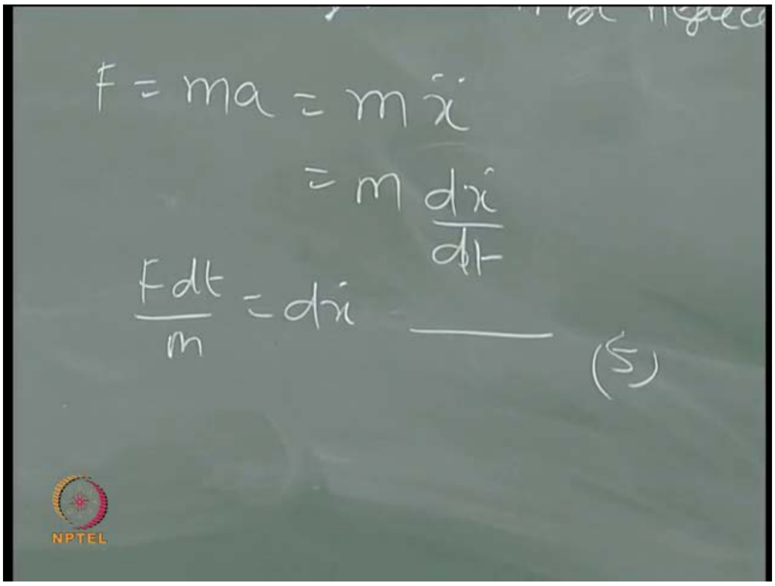

Therefore, f should be equal to m a, which is m x double dot which is m d x dot by d t. So, f d t by m is the x dot substitute d x dot in equation four, I call this equation five.

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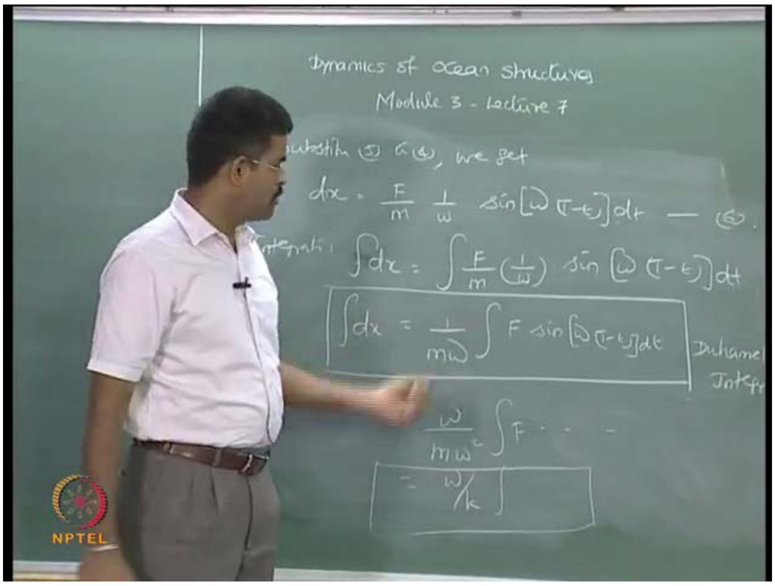

Substituting five in four (No Audio: 21:46 to 22:15) is that okay? d t f d t equation six. Actually we interested to find x because d x is a segmental value of my response. So, integrate, (No Audio: 22:43 to 23:23) this my, this classical integral which is called as Duhamel integral. There is a another form of this I can multiply numerator and denominator by omega, what I get is omega by m omega square of integral, which will become omega by k is it not of this. This is another form of integral you can use any one of the form for your calculations.

Let us quickly apply Duhamel integral for a prescribed dynamic loading and try to find the dynamic amplification factor for a given problem. For example, we take a single degree problem we will apply this and see how I can use conveniently, the Duhamel's integral for such cases which has got a prescribed dynamic loading. Any questions here, so we should be able to agree and understand how the Duhamel integral can be derived for an impulse function of this order which I showed for a prescribed loading. I will remove this, we will do 1 quick example. And apply Duhamel's integral for this problem.

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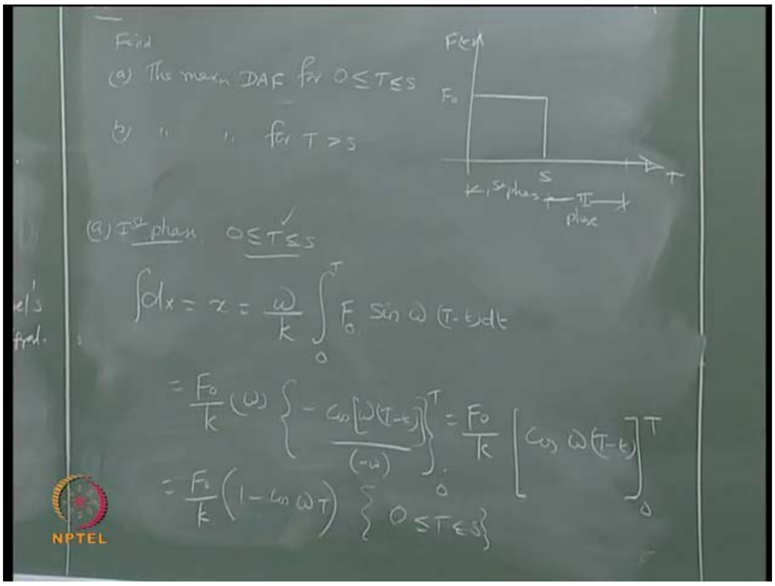

A rectangular pulse load is shown in the figure, the figure is this, the duration of load is s. And for s the maximum value is f naught, what is asked is a find a. The maximum dynamic amplification factor, for 0 less than T, less than s, I call this as first phase, this my first phase. And of course, this my second phase, and also so let us say a first phase, 0 less than T less than s, we already know d x integral which gives me x is omega by k, I am using this form integral f sine omega t minus t d t.

And this is varying from 0 to t and this applied only for the phase, where is between 0 and s because after that f naught is actually 0. So, I should say during the phase of 0 to s this should be f naught. So, can you integrate and get me this value. So, I will get f naught by k omega minus 1 cos omega t minus t by minus omega, which will give me simply f 0 by k. Of course this applied from 0 to t cos of omega t minus t 0 to t, which will give me f 0 by k, 1 minus cos omega t is it, this will be for a range 0 less than t less than s because after that there is no f naught it is becoming 0. I will remove this, we already know f 0 by k is x static.

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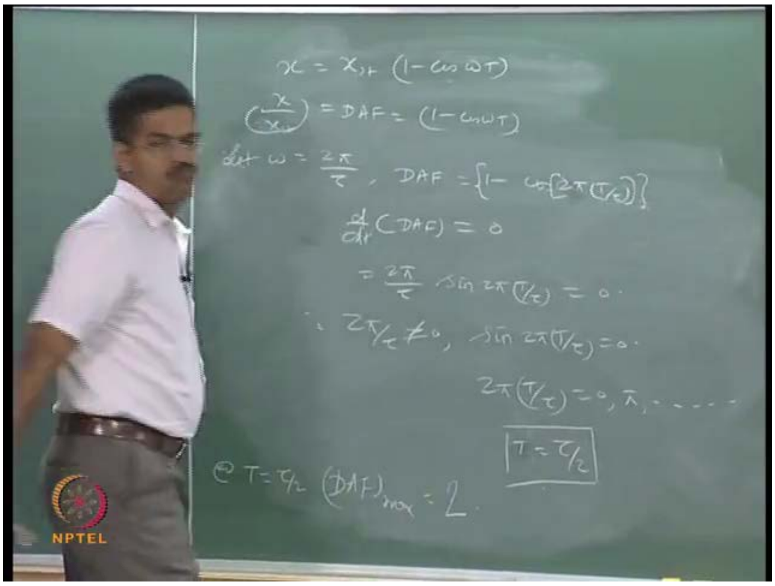

So, x is x static of 1 minus cos omega t and x by x static is nothing but my dynamic amplification factor, which is 1 minus cos omega t. I am interested in finding the maximum value of this. So, let omega be 2 phi by tau I cannot use t because t is the period of the loading, I am using tau. So, then dynamic amplification factor becomes 1 minus cos omega sorry 2 phi t by tau. Let us take d by d t of this and set it to 0 for d a f to be maximum, if we do that I will get 2 phi by tau that is a multiplier there I get sine 2 phi t by tau set it to 0 because I want d a f to be maximum.

Obviously 2 phi by tau cannot be 0 because I have a frequency in this problem. So, obviously since 2 phi by tau cannot be 0 sine 2 phi t by tau is set to 0. Therefore, I should say 2 phi t by tau should be either 0 or phi etcetera, taking the value of phi I will see that t will be equal to tau by 2 is it not. Substituting this as phi I get t as tau by 2, for t as tau by 2 my DAF maximum will be at t as tau by 2 is given by this equation as 1 minus cos omega t substitute it here, omega you now its 2 phi by tau and t you already know. So, what is the value 2, let us say solution for the a part of the problem.

I want to know the maximum DAF for this problem, you will appreciate that we have not solved the equation of motion of this problem, using the conventional equation of motion and differential equation. We have used an integral because the loading is prescribed it is applied only for a specific duration I use Duhamel integral to find out the x. Now, the second problem in the second phase is you will see that at s t becomes s or t greater than s f naught is 0 there is no load, but the free vibration of the spring system at response of t equals s will be there because the spring the system is going to respond here.

So, we already have a general equation for x of t we must find out x of t at s and x dot of t at s that will be my solution for my second part of the problem. Alternatively if I have another loading here, which is f 0 by 2 we will have two component on part b, one component is because of the f 0 by 2 load. On the duration s plus s plus 2 whatever may be plus the vibration because of initial conditions of first phase in this case you are not doing that we are removing this. Let us remove this any questions here any doubt will remove this.

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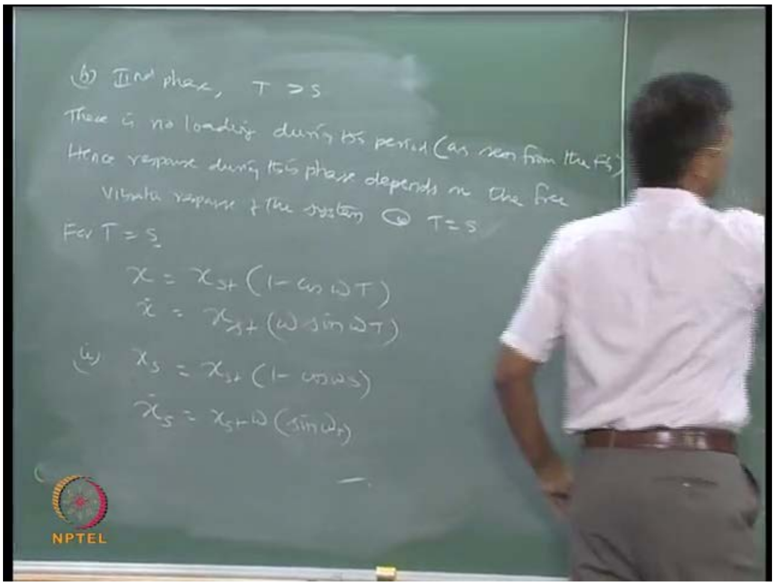

Let us say I want to find out the second phase where t is greater than s. Now, we already know there is no loading during this period, you can see as seen from the figure. I am not drawing the figure here; you have the figure there with you, so as seen from the loading diagram you can that there is no loading here. Therefore, the response during this phase depends on the free vibration response of the system at T equals s. Now, for T equals s, x is given by x static of 1 minus cos omega T and x dot is x static of omega sine omega t. It means x s that is T is equal to s and x dot s or x s t 1 minus cos omega s x s t omega sine omega s. For free vibration we already know x of t. is given by a general equation.

Can I write this, so substitute back for x s and x dot s from this equation and get x s. So, if I write that substituting for x s and x naught s, x will become x static 1 minus cos omega s cos omega t. I should rewrite this as cos omega because I am looking for a value, which is more than s there is a lapse of s here plus. So, x s t ok x s t by omega sine omega s of sine omega t minus s. So, this goes away I get x s t of 1 minus cos omega s of cos omega t minus s plus sine omega s sine omega t minus s. So, x by x s t is dynamic amplification factor, which gives me this value as it is.

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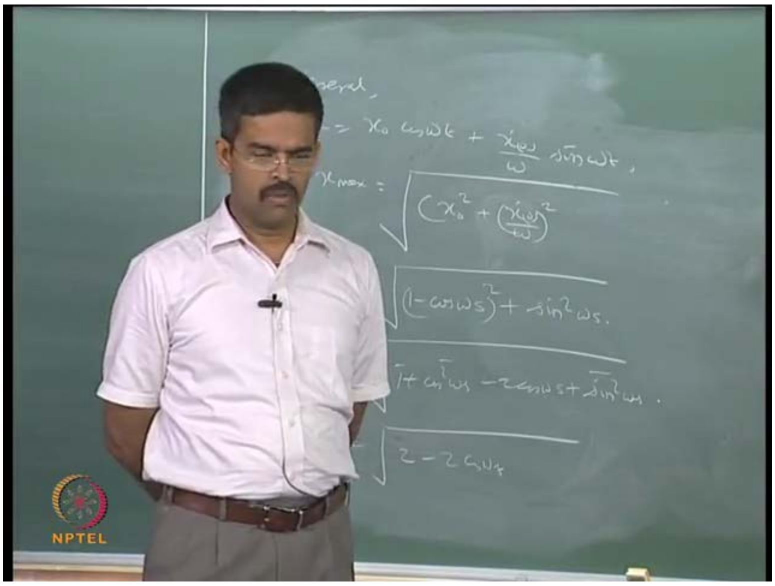

Now, in general if x is x naught cos omega t plus x 0 by omega sine omega t x max is simply given by the squares of these coefficient's. That is a general equation, which is nothing but x 0 square plus x naught 0 by omega square, the co-efficient squares. Applying the same algorithm by comparing this equation with what I have here, I must pick up the coefficients of cos omega t and sine omega t that is nothing but this value and this value, square them up take a root and find the x max is it not.

So, I should say now the dynamic amplification factor max is simply the square root of squares of 1 minus cos omega s the whole square plus sine square omega s, is it okay? Expand and simplify and see what happens square root of 1 plus cos square omega s minus 2 cos omega s plus sine square omega s, I get this as 1 1 1 2 minus 2 cos omega s.

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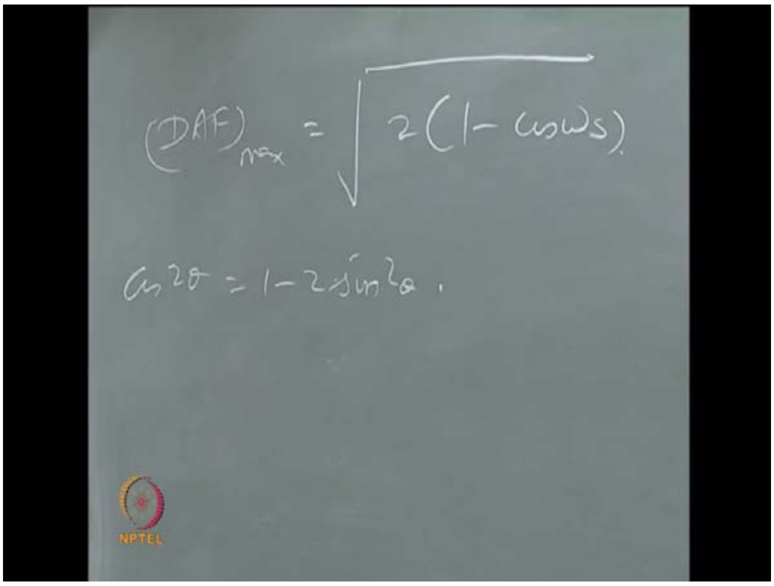

So, use this equation substitute back here and see what is the DAF max, you will see DAF max becomes twice of sine omega s by 2, is it okay? This for the second phase, so we have used Duhamel integral, where ever the prescribed loading available we have used Duhamel integral. And evaluated x and of course, in this problem we never wanted to stop at x the question wanted us to work out the dynamic amplification factor, and maximize the value so we found out that.

So, when you got a prescribed loading it is not advisable to use the solution procedure for the classical differential equation, you must use a short cut method as you seen in Duhamel integral case. For all prescribed loading you should use this and this is a very common application in astro structures.

The loading is not always continuously applied except that we have a basic loading is continuously applied, where as a instantaneous loads like seismic excitation, like wind forces, should be consider only as a prescribed loading. They are dynamic because they vary with time and they are prescribed, you have the full formulation of the load available to you, but it is rather difficult to use that in your general formulation for finding out the response of the system.

So, people use Duhamel's integral as a tool to solve such problems. Though we have taken a very simple problem, the same concept can be applied to multi degree freedom system as well. The only difference could be now the m and k matrices as we have seen here, this is what I wanted to explain there is one more question which is generally left in mind is that, when we do any iteration scheme how does the iteration scheme land up in the lowest possible frequency, how you get omega 1 as the omega n the natural frequency the lowest value how?

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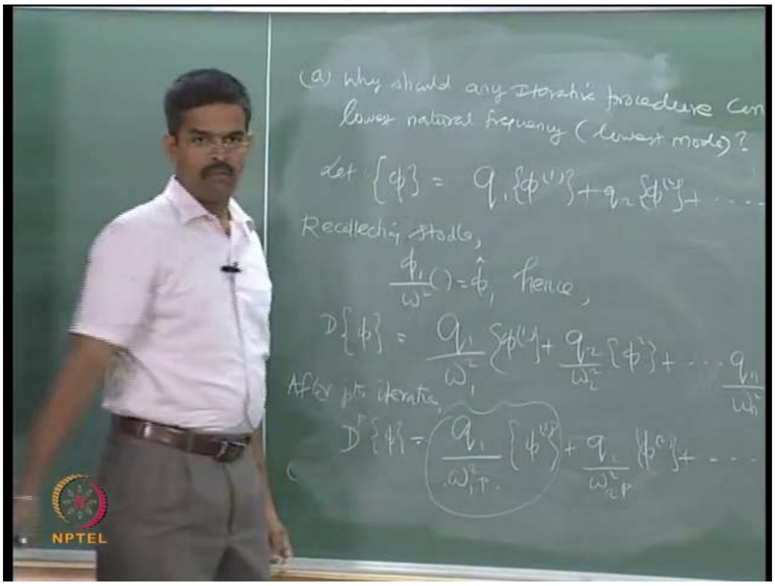

So, the question is why should any iterative procedure, why should any iterative procedure converge to the lowest natural frequency or I should say the lowest mode because it gives me the mode shape also. For example, stored law, why? We can mathematically prove this, let say any arbiter mode shape achieve, in iteration most of the iterative cases you have seen the mode shape is it not stored law Rayleigh procedure, you assume the mode shape actually is it not gets converged. You are not assuming the frequency right assume the mode shape, which you say is a sum of different modes together, which we say sum of phi 1 that is the first mode plus sum proportion of phi 2 plus sum proportion of phi is it not, that is what we are assuming.

If you recollect stored law, if you have the modes I want you to see that if you recollect stored law you will see that we already said phi 1 by omega square is phi 1 hat, which is the new value phi 1 omega square with some multiplier will be our phi 1 hat. You look back your stored law you will see that there is a division here. This multiplier can be any value as you see from this above equation. So, using the same concept here I say iterative scheme of any mode shape can be rewritten as q 1 by omega 1 square of phi 1 plus q 2 of omega 2 square of phi 2 plus q n of omega n square of phi.

I should say hence, I am writing this I keep on iterating this vector v number of times. So, after p th iteration I may get this as p th iteration of phi is q 1 of you will see the multiplies always taken out, we have no assigned value for this multiplier, we use it later. So, summation is it not as omega 1 square p i am saying p th iteration this is not the multiplier this an iteration of phi 1 and so on q 2 omega 2 square of p of phi 2 and so on.

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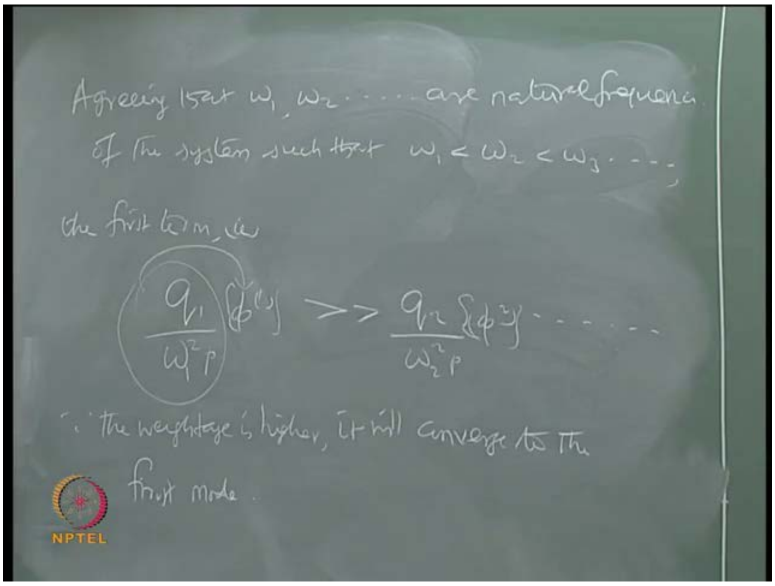

Agreeing that omega 1, omega 2 are natural frequencies of the system such that omega 1 is lesser than omega 2 lesser than omega 3. Where the number comes from an ascending order, if you agree this then you will see that the first value will be higher than the second value. The first term that is q 1 by omega 1 square p of phi 1 will be much higher than q 2 by omega 2 square p of phi 2 and so on, what is it mean? The proportional weightage of this value on the first mode is higher, it will converge to the first mode automatically. Since, the weightage is higher, it will converge to the first mode, so that is why in stored law, we say that iteration scheme will converge to the first mode that is a lowest frequency.