Dynamics of Ocean Structures
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Module - 3
Lecture - 2
Response Spectrum

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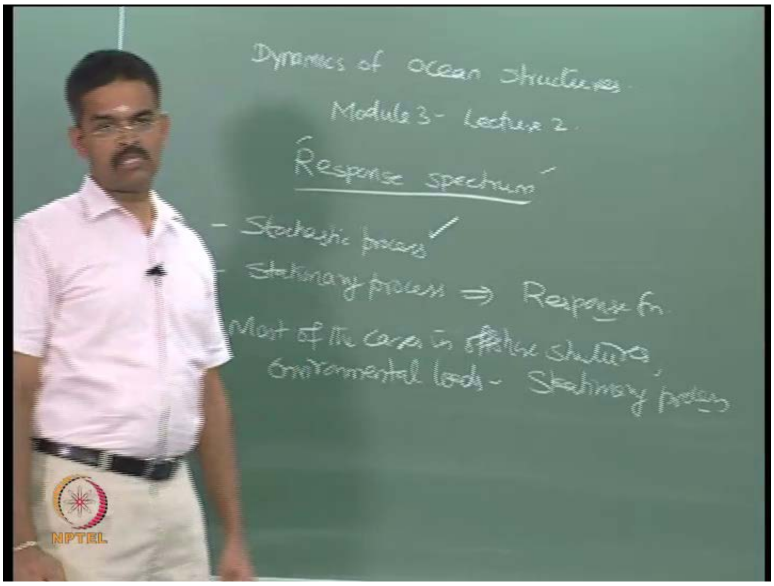

So, this lecture we will focus on the stationary process, we will discuss about the response spectrum. This is lecture 2 on module 3 on dynamics of ocean structures, under the brace of NPTEL, IIT Madras. In the last lecture, we discussed about stochastic process. In this lecture, we will talk about something called stationary process. Then for a stationary process, we will derive the equation for response function or a transfer function. Now, in most of the cases in offshore structures, the environmental loads which act on a system are actually a stationary process.

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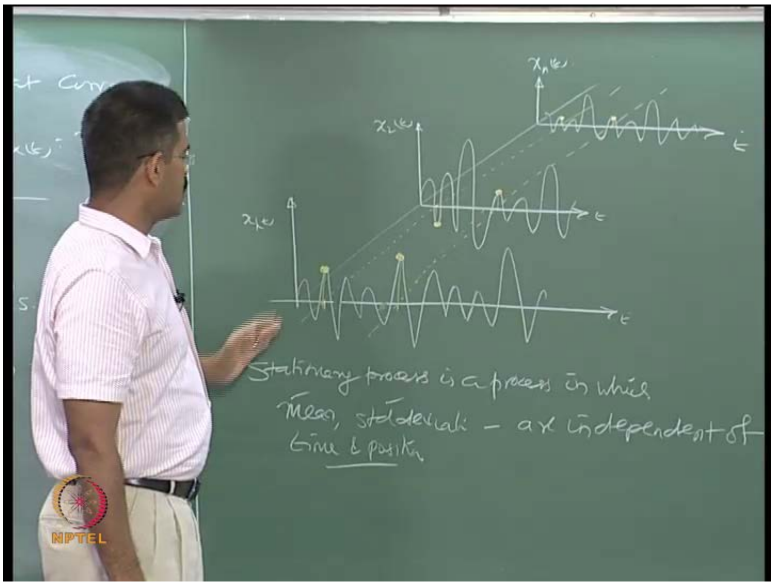

Now, the question comes, what do you understand by a stationary process. So, look at these values, which are x 1 of t x 2 of t and x n of t. So, stationary process is that process where the mean, standard deviation of these variables are independent of time and position.

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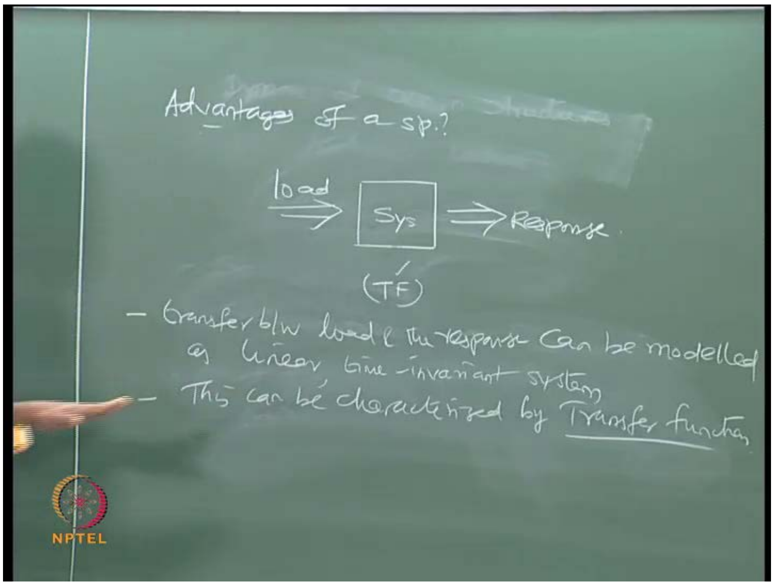

So, literally for a stationary process, if these two conditions are satisfied, then what are the advantages of a stationary process? If we know the advantages, then from the advantages, we will try to derive the transfer function between the responses on the load.

Now, let us see what our objective is. I have a system structure. The system is subjected to some load. Then the system response; now my objective is to obtain a transfer function, which gives me the mapping of for a given load, what is the response. That is my objective actually. Is it not? In the whole analysis, that is my objective. I must have a transfer function, which understands or gives me a mapping of the load onto a structure, which gives me a response for a specific load, right. If the system what we are talking about here is stationary, then the transfer between the load and the response. The transfer between the load and the response can be modeled as linear, time independent or time, let us say invariant. It does not vary with time and system. This can be characterized by a transfer function.

So, our objective is to derive a transfer function, which gives me the connectivity between the load acting on the system and the response received by the system, because of the load acting on it. Fortunately, if the system qualifies to be called as a stationary process, then I can characterize this transfer function as a time invariant problem. Any question here?

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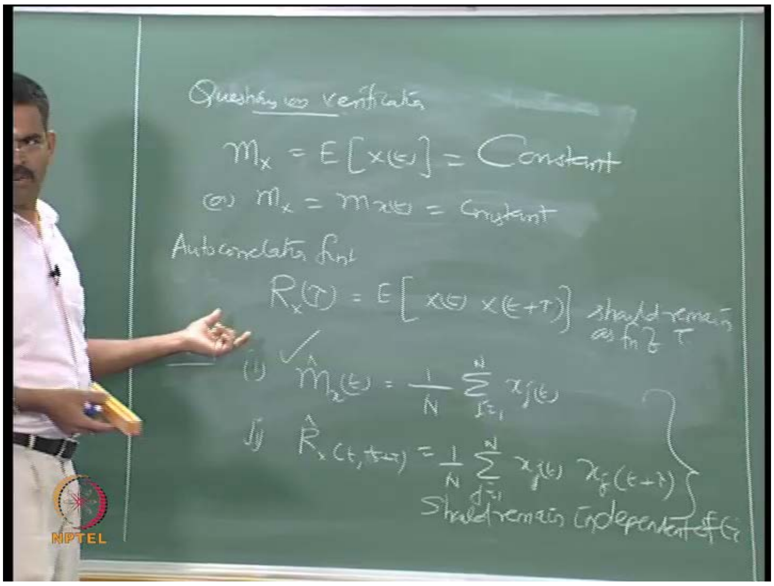

So now, the question which I have got to answer is or verification which I have got to do is the following. What are the verifications we have got to do? The verifications are, the mean value of the variable, which is expected value of x of t should remain as constant. That is, we have got to verify this because then only we can call this process as a stationary process or to be more elaborate, m x is m x of t is constant. I mean time invariant and the autocorrelation function, which is expressed as r x of tau, which is given by expected value of x of t and the product of x of t plus tau should remain as function of tau only and not t.

To be very specific, I have got to verify the following. One is my mean, which is nothing but 1 by n of sum of j is equal to 1 to n of x j of t and my autocorrelation function, which is a function of t and t plus tau, which is given by 1 by n of sum of x j of t x j of t plus tau should remain independent of time. So, for a given process, I must evaluate the mean and autocorrelation function. I must show that they are independent of time. So, for that process, I expand and get a relationship between that loading function and the response function. I call that function as a transfer function. Then leaving to that, we will get into the response spectrum. Any question here? I think I will use this side. It is easy. Is there any problem? Are you able to see from here?

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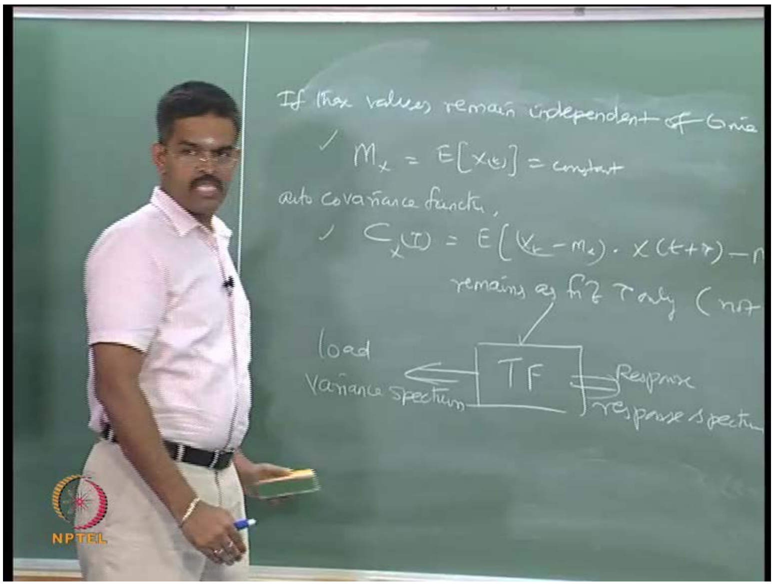

Now, if these values, that is m x and r x tau of t remain independent of time, then we can say that m x expected value of x of t remains as constant and the auto covariance function, which is given by c x of tau, which is expected value of m x minus, sorry, x of t minus m x product of x of t plus tau minus m x remains as function of tau only and not t. So, the advantage is, now for a given load, which varies, I get a variance spectrum. For a given structure, it has got a response; the response varies. I get response spectrum. Connecting these two is my transfer function.

So, I am interested to derive the transfer function, in the process of showing that the process is stationary. So, I must establish these two conditions and show the process is stationary. For that process, I want to connect the variance spectrum of the load to that of the response and I get the transfer function. The moment I get the transfer function, I can find the response spectrum, if I know the load variance spectrum.

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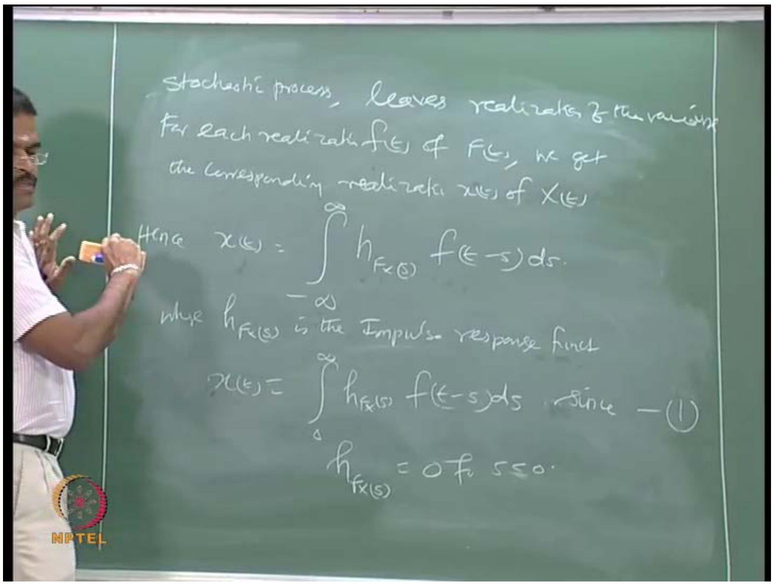

Any question here? Remove this. We have already seen the stochastic process in the last lecture yesterday that, stochastic process leaves a realization of the value of the variable. For example, for each realization small f of t of f of t, small f of t or realization of f of t, capital f of t is a large ensemble, out of which small f of t f 1 f 2 f 3 are all realization values of this f of t. These are all true values present in that set. We get the corresponding realization x of t of x of t. Is that ok? These are all responses. Hence, x of t, which is a realized value, can be given as integral of domains minus to plus infinity h f x s. I will come to that what it is. f t minus s d s, where h f x s is called the impulse response function. What does it mean? For any force, if we know what is the impulse of that force, cost on the response, because these are all realization of the force. f of small f of t these are all realization of the force.

If I know how much realization is being impulsed upon by this factor, I will know the realization of x for a corresponding value of the force multiplied by that response from a domain of minus to plus. You will always agree that impulse response function in a negative domain is always 0, because it is giving a positive impulse only. So, I can rewrite this equation as 0 to infinity h of f x. This is capital x s f of t minus s d s, since h of f x is 0 for s less than or equal to 0. Instead, let us say less than 0. Because, at 0, you have value. You do not know. Let us call this equation number 1. So, this is my x of t. I am getting the response realization for a force realization multiplied by the impulse response function of that. Any questions here? I will remove this.

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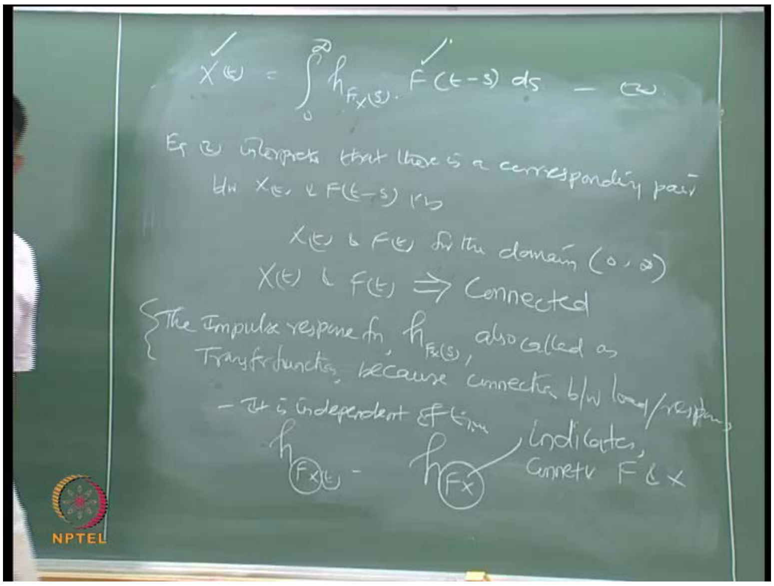

Having said this, we can establish the capital x t values, because we are talking about the x of t value for a whole domain of 0 to infinity. I can establish x of t as also similar to 0 to infinity h of x of s with capital f of t minus s d s. These are all nothing but the space of x and capital f. The earlier expression 1, equation 1 was given you only the realized values within the space. So, I can call this equation number 2. I can say equation 2 interprets that there is a corresponding pair between x of t and f of t minus s or x of t and f of t for the whole domain 0 to infinity. There is a complete connectivity between these two. So, they are connected. So, x of t and f of t are connected. That is the mathematical meaning of this expression. Is it not? So, that connectivity is what we are trying to establish.

Interestingly, the impulse response function, which is h f x of s is also called as transfer function, because it establishes the connection between load and the response. Is it not? This is the load and this is the response. I can always call this as response function also. Please note that the impulse response function is independent of time. There is no time factor here. It is independent of time.

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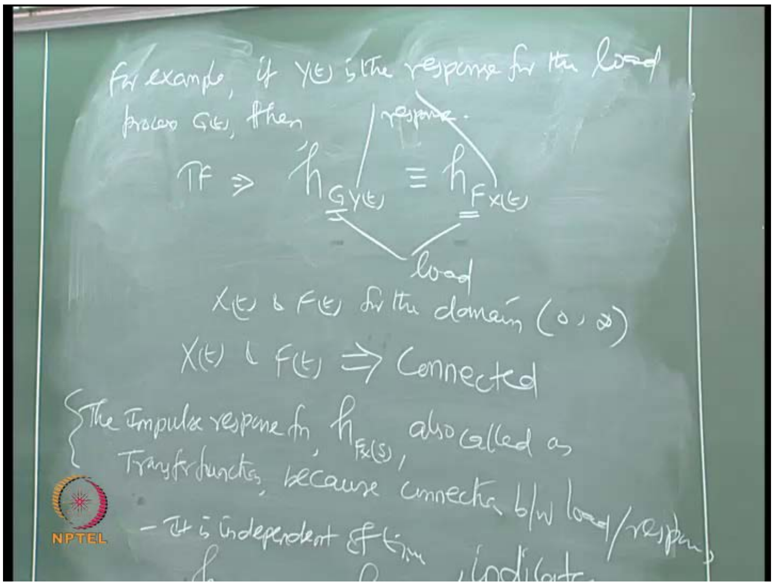

Now, you may wonder that, in h f x of t, s is a variable or t can also be. You can wonder that, yes, since t is there, x of t is there, is it a function of time? It is not. The f x term is indicating, h f x indicates that the connectivity is between f and x. For example, if I have for example, if y of t is the load process and y of t is the response, sorry, is the response for the load process g of t. Then the transfer function can be indicated as; not given by, can be indicated as h of g y of t, which is as same as h of f x of t. So, the first letter here stands for load and the second letter here stands for response. So, our job is to estimate the transfer function now. Because, if we know the transfer function, I can easily find the response, if we know the load spectrum. Can I remove this?

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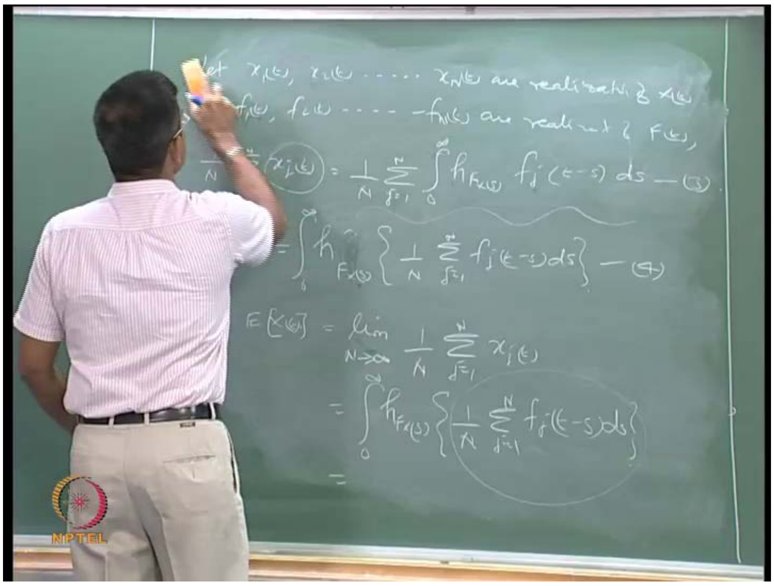

So, let us say x 1 of t x 2 of t x n of t are realizations of x of t and f 1 of t f 2 of t with f n of t are realization of f of t. Then I can say 1 by n of j equals 1 to n of x j of t is 1 by n of integral j equals 1 to n of summation of integral of my transfer function 0 to infinity h of x of s f j of t minus s d s. I call this as equation number 3. This is what I have borrowed from my response. Is it not? We are using the response as a transfer function or impulse response function on the load. I am using all small, because I am talking about realization for entire domain of 0 to infinity, which can be rewritten as, as there is no time component here h of f x s. The integral domain is there for the entire domain and then the summation of 1 by n of j equals 1 to n of f j t minus s d s. Equation 4.

Therefore, what is this value? This is expected value of, what is this value? So, mean of the response function. Is it not? Which is now given as limit n tends to 0, sorry, infinity n tends to infinity 1 by n of summation of j equals to 1 to n x j of t, which I am rewriting from here, which is given as further integral of the transfer function 0 to infinity h of x s summation 1 by n of j 1 to n of f j of t minus s of d s, which can be said as, what is this value? expected value of this variable. Is it not?

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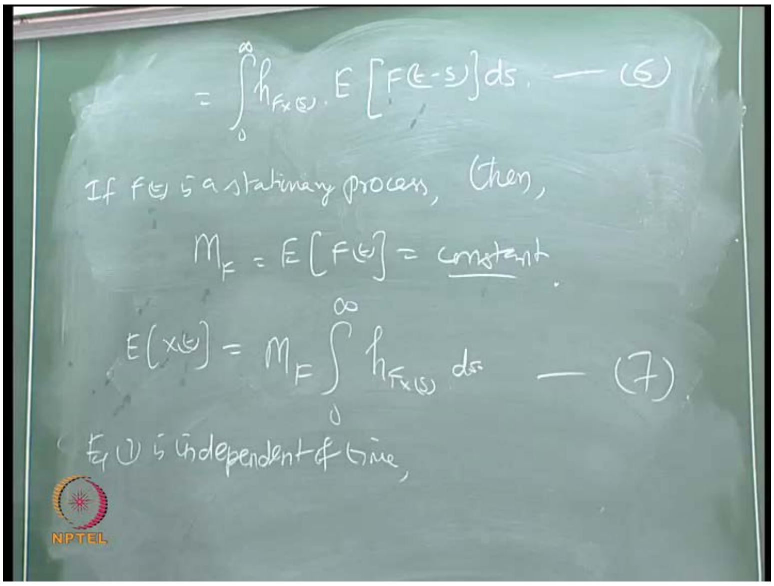

So, I can rewrite this as, I am writing it here. I will remove it. So, I can rewrite this as integral of 0 to infinity h of x s. The expected value of f of t minus s d s. Is it ok? Now, we already know, I will call this as equation number 6. This was fine. Now, we already know that if f of t is a stationary process, which in most of the cases in offshore structures is true, then we have a very important interpretation. Then I will remove this. The mean value of this, which is m of f, I am talking about the load; the mean value of this, which is expected value of f of t, is constant. Is it not? That is the definition for a stationary process.

Therefore, e of x of t, which you have here in the previous equation can be rewritten as, this can be written as a constant. Instead of using a constant as k etcetera, I am using as m f. I am taking this outside the integral. I am simply saying h of s. Is that ok, and d s. Integral is over the domain. I call this equation number 7. Now, interestingly the expected value of the space of x of t is independent of time. So, I can see here equation 7 is independent, which conforms that it is a stationary process. Any question here? Now, my focus is to evaluate this. Any question? I will remove this.

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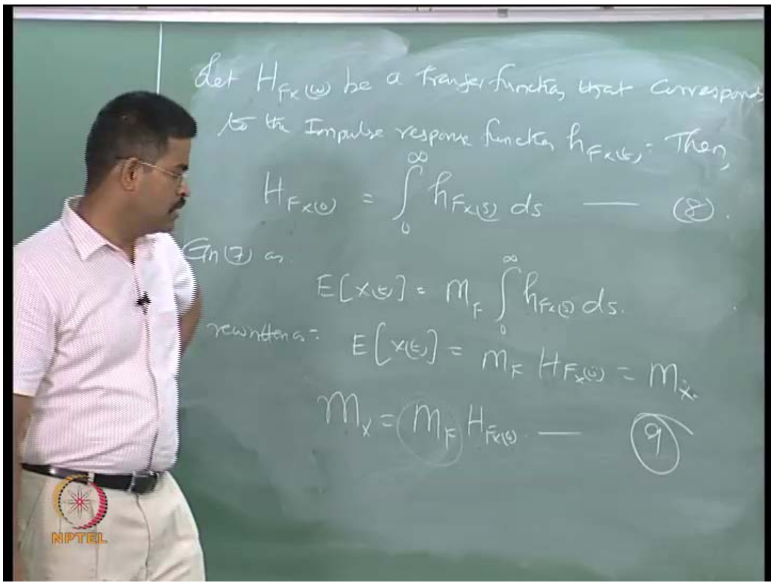

Having said this, now let us say capital h, because we are talking about the transfer function for realized values of f or realized values of x, which are true for the entire domain of 0 to infinity. Now, I am talking about the transfer function of the whole domain as a capital h now. So, capital h f of x omega be a transfer function that corresponds to the impulse response function h of f x of t. Then we can say that h f x of 0 is straight away 0 to omega, sorry, infinity h f x s d s. Equation number 8.

Now, I have the equation 7. I am rewriting equation 7 as, the original equation 7 was this, e of expected value of x of t was given as the mean value of the force of 0 to infinity h of x of s d s. Is that ok? This was the original equation 7 we had. I rewrite this as, rewritten as the expected value of x of t as m f. Instead of this, I am using this. So, h f x 0. I call this as m x. So, I can say, m x is nothing but m f of h of f x of t. Equation number 9. So, if I know the mean value of the forcing function or the process, we can always find the mean value of the response function, when I know the transfer function. In stochastic process, we will not get the absolute values of x. We will get only the statistical parameters of x, like mean, standard deviation and coefficient of variation. So, I can get this, if I know this. So, my objective is, for a given problem, how to get this value. Agreed? There is no time dependency here. Therefore, the process is completely stationary.

Now, we will take up a single degree freedom as an example. Try to connect what is the connectivity this function has with the problem. So, I will remove this now. Any question here? So, what we picked up, to be very brief, you can turn back and see that we started with the stationary process explanation and we wanted to emphasis that stationary process is that process, where the mean and standard deviation are independent of time and space or position. So, we said two values have got to be qualified. One is, the mean should be constant and second is, the auto correlation function should remain independent of time. We picked up that. Then if a process is said to be stationary, the load acting on the system, which is the process, is said to be stationary, then there exist term connectivity between the load acting on the system with the response given by the system on to the load.

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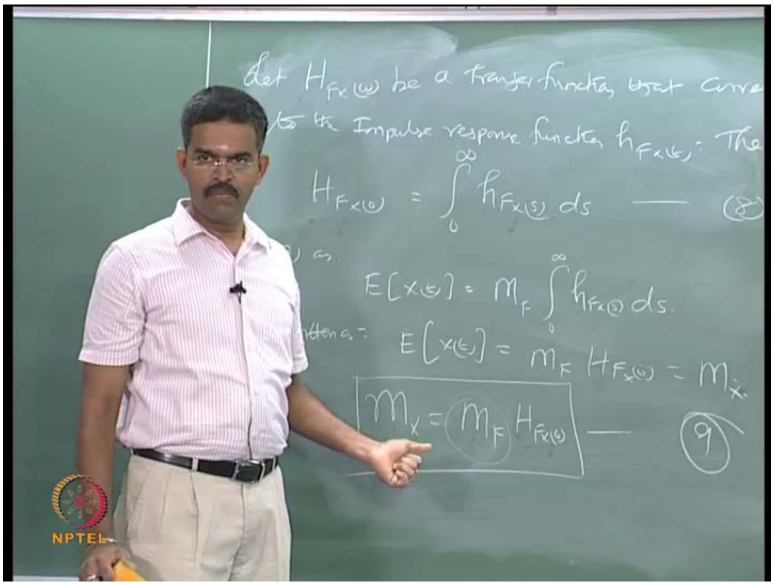

We can easily find out this mapping using a function what I call impulse response function or a transfer function. So, for a stationary process, this should remain independent of time again. So, we showed that small h of x of s is independent of time, right. We picked up that. Now, we connected the mean of response to that of mean of forcing function with this value. Now, we will apply this concept back again in a simple problem and derive the transfer function for that problem. Let us pick up a single degree freedom system problem quickly with a damped system and see how I can derive the transfer function for that particular problem. Then from that problem, how do I get the response spectrum.

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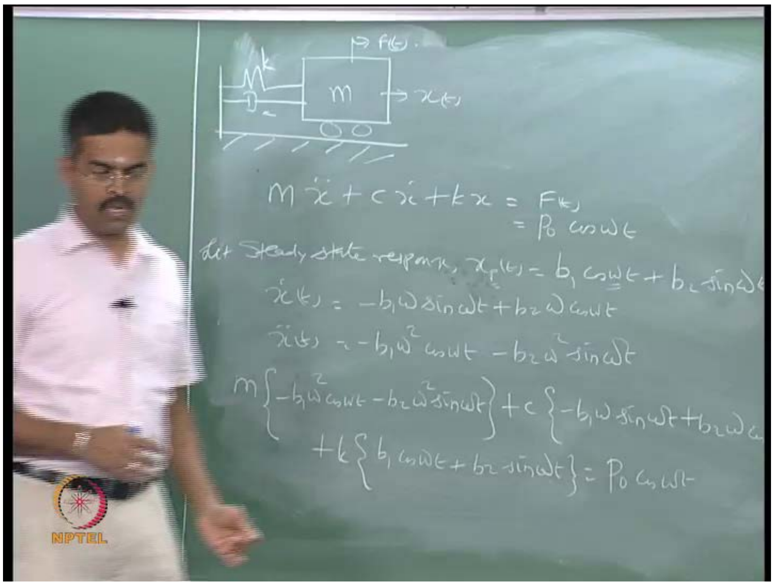

So, I have a single degree freedom system problem, m k c x of t f of t. So, we already know m x double dot plus c x dot plus k x is f of t. In my case, I take this value as p 0, some harmonic function. So, the steady state response, referring back to the lectures on module 1, I am using p here to indicate it is a particular integral. There are two solutions for this differential equation. One is the complementary function and another is the particular integral. Complementary function is closely related to my transient vibration. I am dying down and take only the steady state response, which is assumed as, let b 1 cos omega t plus b 2 sin omega t.

We have already known that the response of my steady state will have a functional component of the forcing function. So, can we quickly find out x dot and x double dot and substitute these values in this original equation, can you get me b 1 and b 2 quickly. So, you have to simply differentiate it once and twice and substitute these values back and get the values of b 1 and b 2, quick, So, minus b 1 omega sin omega t plus b 2 omega cos omega t minus b 1 omega square cos omega t minus b 2 omega square sin omega t. Substituting back, m of minus b 1 omega square of omega t b 2 omega square sin omega t plus c of minus b 1 omega sin omega t plus b 2 omega cos omega t plus k of b 1 cos omega t plus b 2 sin omega t is p naught cos omega t. Now, I pick up the terms associated to cos and associated to sin separately; two equations. Get b’s, your b 1 and b 2 quickly.

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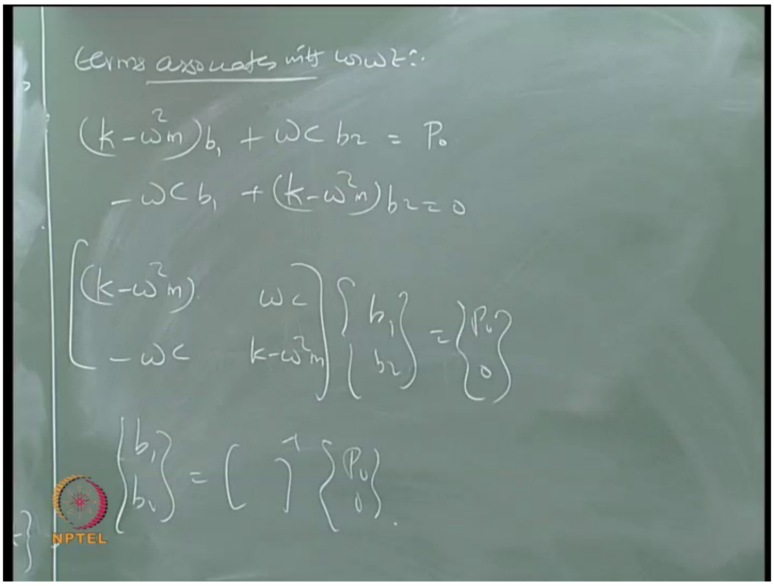

If we look at the cos terms associated with cos omega t, both l h s and r h s, so I can say k minus omega square m of b 1 plus omega c of b 2 is p naught minus omega c of b 1 plus k minus omega square m of b 2 is 0. Is that ok? So, I write in matrix form. Invert it and get b 1 and b 2. I can write this as k minus omega square m omega c minus omega c k minus omega square m of b 1 b 2 is p naught 0 So, b 1 b 2 is nothing but inverse of this matrix of p 0.

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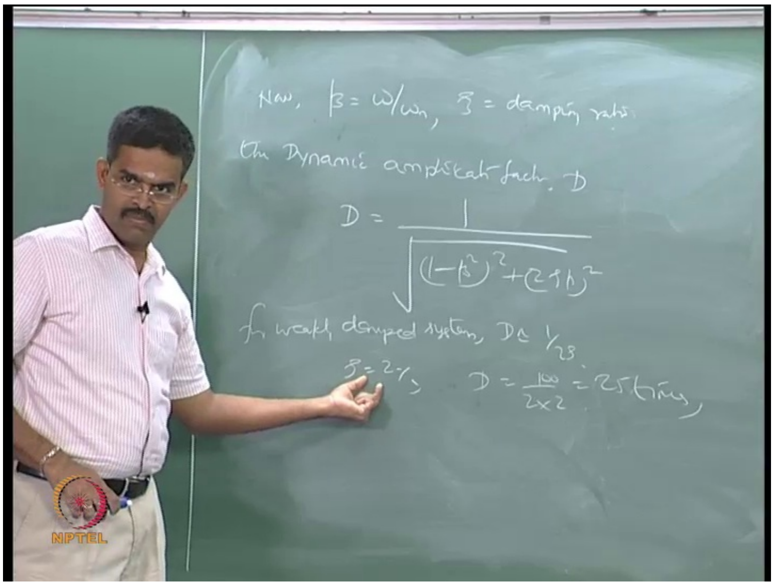

Now, for beta to be a ratio of omega by omega n, it is a frequency ratio and zeta be the damping ratio. We already know the dynamic amplification factor d for this problem is given by, can you give me what is that equation for d, if you remember. I will write down here. I have the equation here. Can you, if you have the notes earlier, can you just tell me what is the equation of d?

Good. So, 1 upon square root of 1 minus beta square the whole square plus 2 zeta beta the whole square. Is that ok? What is the upper of this for a weakly damped system? The upper limit? So, I should say, let us say for a weakly damped system d is approximately given by 1 by 2 zeta. What does it mean? If I have zeta as 2 percent, the magnification will be 50 times, 25. So, if you have a zeta of just 2 percent in a given system, the maximum amplification what I will get for the system will be about 25 times. Is it not? That is the meaning physically. What does it mean? Even for a small oscillation, the magnification is very high. The damping 2 percent means small oscillations. Is it not? It is not damping at all. It is not dying down.

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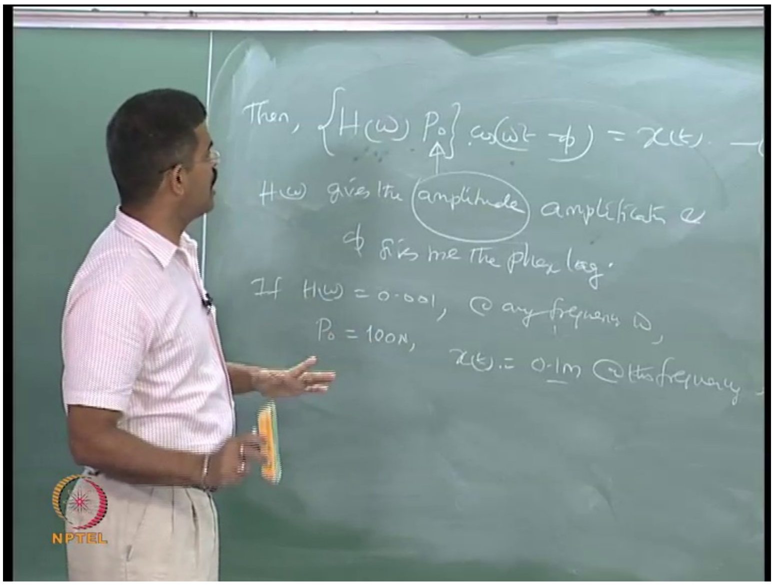

So then I can say, remove this, then h of omega multiplied by p naught of cos omega t minus 5 can be set as x of t. Is that ok? h omega is nothing but the function on p naught. What does it mean? It is nothing but the amplification of p naught on the system. So, h omega, this is equation number 12, I mean my numbers are continuous. I am getting equation number 12 here. 11was somewhere down the line. d was 11, I think. So, h omega gives me the amplitude amplification. What is meant by amplitude? p 0 is my amplitude. You see the forcing function, this p 0 cos omega t. So, p 0 is the amplitude of the forcing function. Amplification is this, content magnification and phi gives me a phase lag. Now, for h omega as 0.001, the amplification is so small, 0.1 percent. 0.001 at the specific frequency, at any frequency, omega and p naught, let us say it is about 100 Newton. I can straight away say, my response at that specific frequency of time is about 0.1 meter at this frequency. Is it not? Now, let us compare this equation with my standard equation what I had in a single degree freedom system model. Any questions? Any doubts here till?

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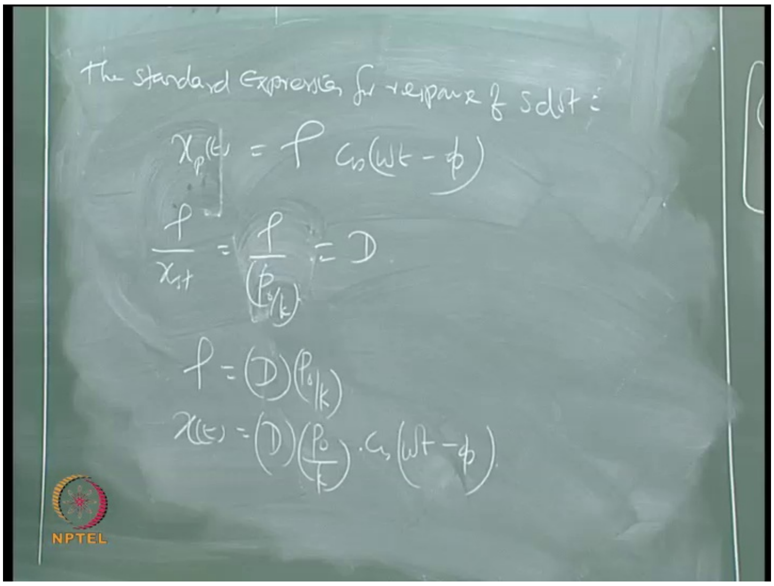

So, all of you agree how did we get this equation, right, from my sample example what we are trying to do. Now, I am just comparing this equation, retain this equation as it is and I will compare this equation with a standard expression is given by u. Sorry, I am using p only to show it is a steady state response, particular integral. Otherwise, it is only x of t. This is equal to nothing but some value rho of cos omega t minus phi. Is that right? We had this equation earlier somewhere in the first module and we already said the rho by x static is nothing but p naught by, sorry, rho by p naught k, which is the amplification factor. Is it not? So, I can straight away say, therefore, let me put it back here. x of t is nothing but d of p naught by k of cos omega t minus phi and what is d. I will rub it here.

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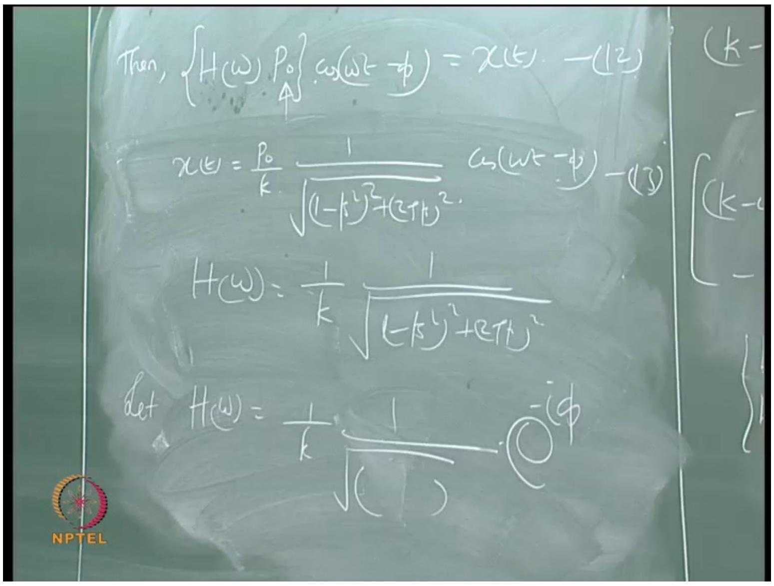

Yeah, dynamic amplification factor. I will rub this. I will rewrite it. So, x of t is p naught by k of root of 1 minus beta square plus 2 zeta beta square of cos omega t minus phi. Compare these two equations now. This is my equation number 13. Can we compare 12 and 13 and can we write h omega as 1 by k of root of 1 minus beta square plus 2 zeta beta square. Is it ok? This p 0 by k p 0 k cancels and 1 by k. Now, interestingly the transfer function h omega should also have all the components of d. Is it not? It should have the components of the dynamic amplification factor of what d has in the time domain. Is it not? It should qualify me, because I am now going to connect h omega with the x and f omega. Now, the question is, I will get the maximum amplitude, but I will not be able to capture the phase lag. So, I must modify x omega as, I am writing this as, I mean, e to the power of i phi. Now, one may wonder that why I am using an exponential function here. I can use cos or I can use sin. Why I am using exponential?

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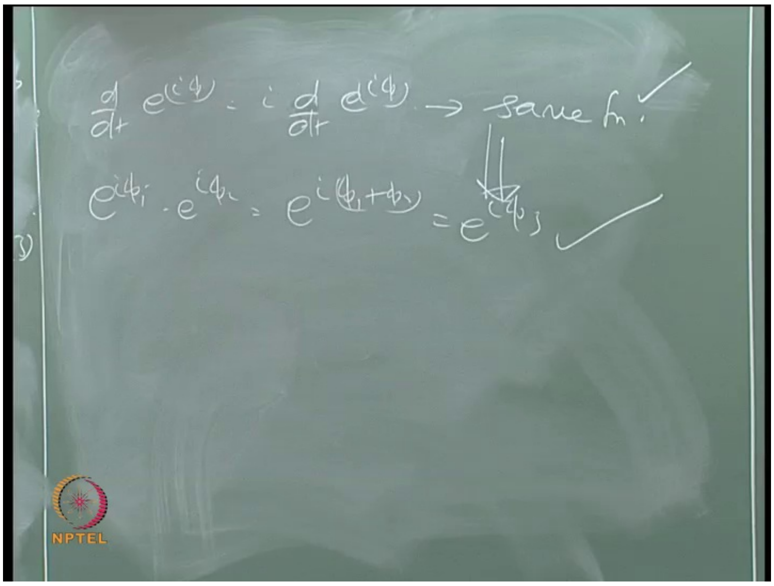

There are two explicit advantages of using this. Now, d by d phi of e i phi will be i e i phi. Is it not? Let us say time domain. I am talking about the time domain. Let us say d by d t of e i phi is i d by d t of e i phi, same. It means, I am getting back the same function. Suppose, I have two combinations, let us say i phi 1 and i phi 2. I will again get back something of phi 1 plus phi 2, which I can say, e i phi 3.

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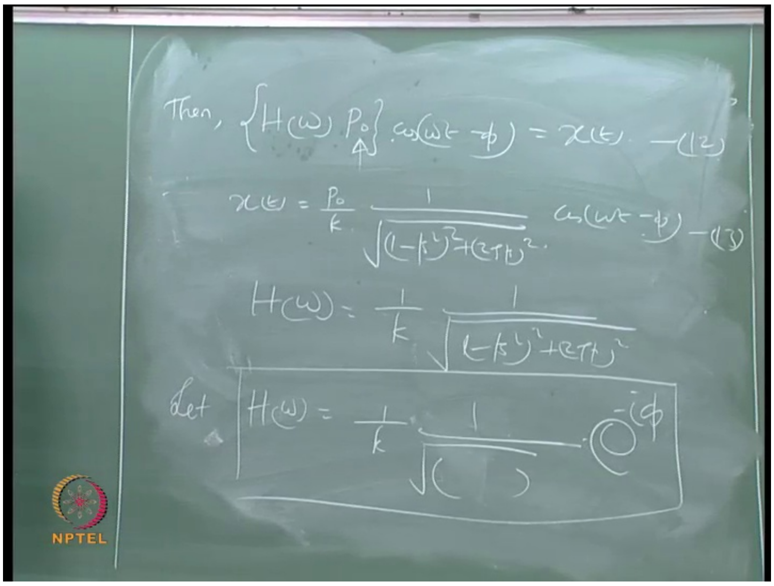

Again, I am getting a similar function back again. Is it not? So, there is an advantage that the multiplier here will not qualitatively vary your h omega. So, this is my transfer function.

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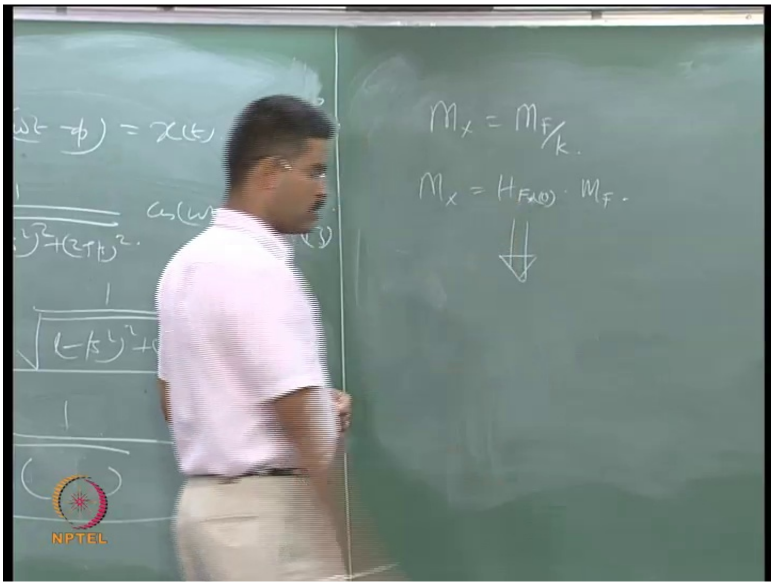

Now, what is the advantage after getting this? very simple. I will rub this. If I know h omega, we can now straight away say, one advantage is m of x is mean of this by k and m of x is also equal to m f. So, I know the mean of a forcing function using the transfer function, I can easily find the mean of my response. Now, taking this forward in the next lecture, we will talk about how to get the response spectrum plots. So, any question here? Any doubts? Any question?

So, we started with the stationary process and we understood what a stationary process is. Then we picked up the mean and the auto correlation function and shown that they are independent of time. So, the process remains stationary. We applied this concept on a single degree damped system for a forcing function of f 0 sin cos omega t and found out the link between the response and the forcing function. We compared that with a standard expression of dynamic amplification factor. We understood that h omega contains all details of that of d, but the phase lag on the whole expression of x of t is not captured in realization of x of t in this domain. So, I have used a power exponential power here, which will have a phase lag phi, which will capture the quality of that content of realization of x of t on my domain.

So, it amounts to a very simple expression. If I have the mean of the forcing function, which is a stationary process, which is true for wave loading, I can easily find the realization of the response of the system from the load using the transfer function, which is independent of time. There is no time domain. There is not time component here at all. Is it not? Phase component and of course, zeta component and of course, the ratio of omega and omega t. That of course anyway will come in d, right. So, if I got a variance in the load, I call load spectrum. If we multiply the variance of the load spectrum with my transfer function, I will get response spectrum, which I will discuss in the next class.

So, if you know the response spectrum for any given system, you can always characterize the system for any load. Therefore, we can design the system. That is what the catch is. Somebody gives you a response spectrum and you wanted to check the system, if is it safe for a specific band width of omega and the load spectrum is given to you. You can always examine the characterization of the system using only the transfer function and see what has happened to the amplification of the amplitude of the response or of the force. So, it is simple. So, we stop here for this lecture.