Dynamics of Ocean Structures
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Lecture – 25
Continuous System

In the last class, in this, we will talk about the continuous systems quickly and we will see how the diagonalization will help us to solve the multi-degree freedom system in simple uncoupled single degree freedom system equations.

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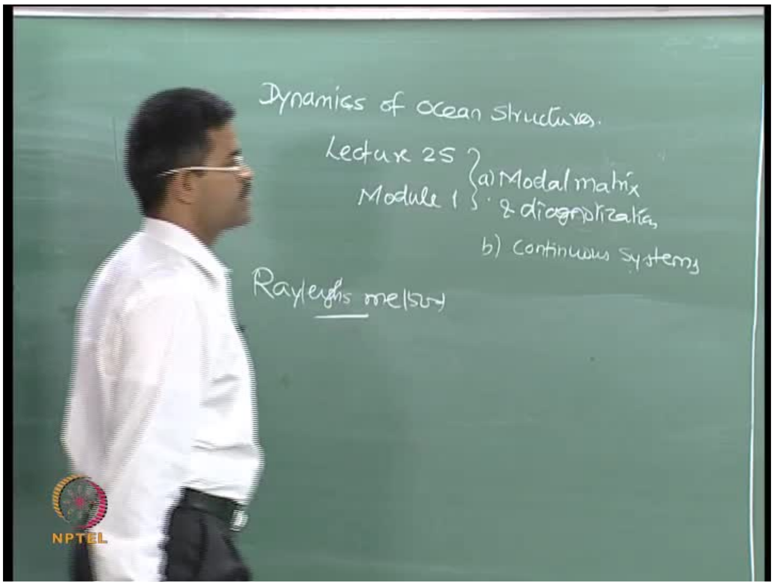

So, I will pick up the Rayleigh's method for solving the continuous system. We will demonstrate this with a simple example.

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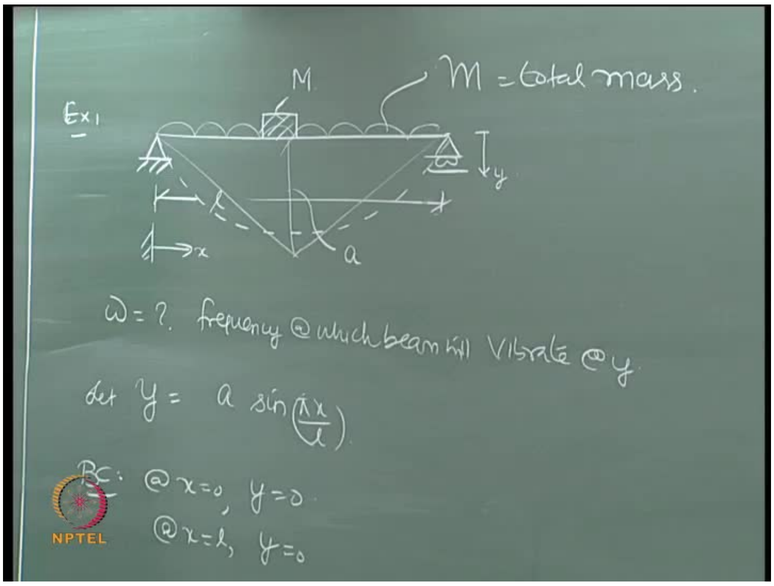

Let us say I have a continuous beam, which is simply supported at the ends. Of course, the self rate of the beam is given as m, which is the total mass of the beam. In addition, it is also subjected to a force at the center. But, for dynamics purpose, I take this as M, which is again a mass, which can be w by g. So, let us say this is the l; and, this becomes the deflected profile for the udl and this becomes the deflected profile for the central point load. And, let us say I have an origin here, where x starts from here. I want to know what is the frequency at which this is vibrating.

So, determine the frequency at which the beam will vibrate in the translational degree, that is, the degree normal to the axis of the member. So, I should say at y; and, this is y. Let us say, first find out the shape function y; let a shape function y be a sin pi x by l – the sinusoidal function let us say. And, I say the maximum deflection, what I will foresee here is a. So, check for the boundary conditions. At x is equal to 0, you will obviously see y tends to 0, because this function will become 0. And, at x is equal to l, again, y will be 0. This will satisfy the boundary condition of this profile.

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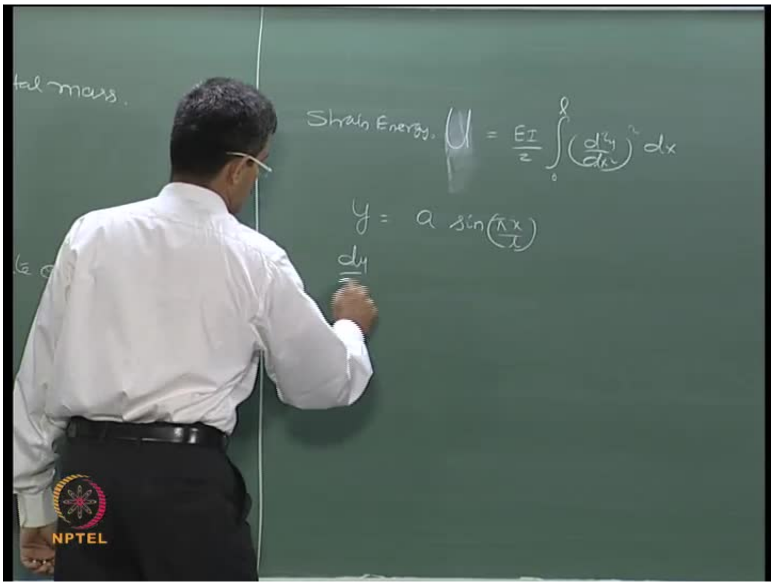

Now, I can find the strain energy capital U, which is EI by 2 0 to l d square y by dx square the whole square dx. So, we already know, the shape function y is a sin pi x by l.

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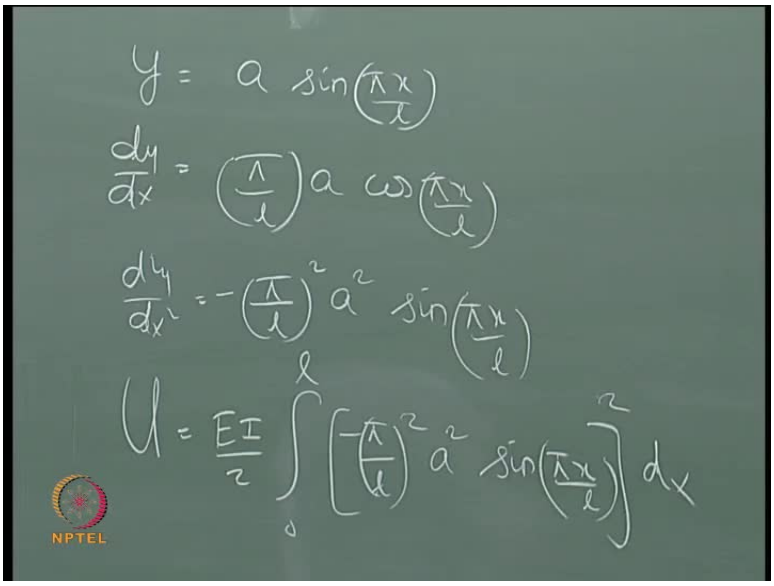

So, dy by dx – the first derivative of this can be pi by l of a of cos pi x by l and d square y by dx square – the second derivative, will be pi by l the whole square minus a square sin pi x by l. So, getting back to U, will be EI by 2 a square pi square by l square 0 to l… Let me put it like this – E i by 2 0 to l minus pi by l the whole square a square sin pi x by l the whole square dx Which will tell me U as E by 2 pi square by l square the whole square a square 0 to l sin square pi x by l dx.

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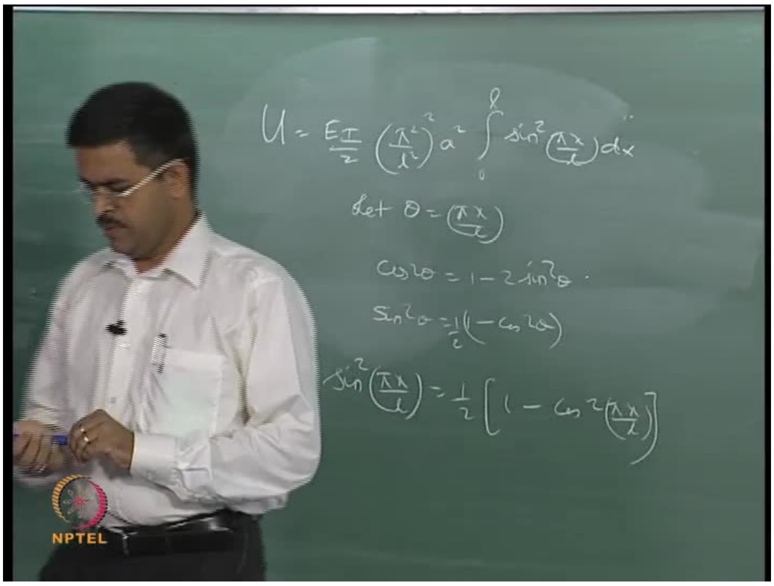

So, we already know… Let us say let theta be pi x by l. Cos 2 theta is 1 minus sin square theta. So, sin square theta is 1 minus cos 2 theta by 2. So, sin square of pi x by l is half of 1 minus cos of 2 pi x by l.

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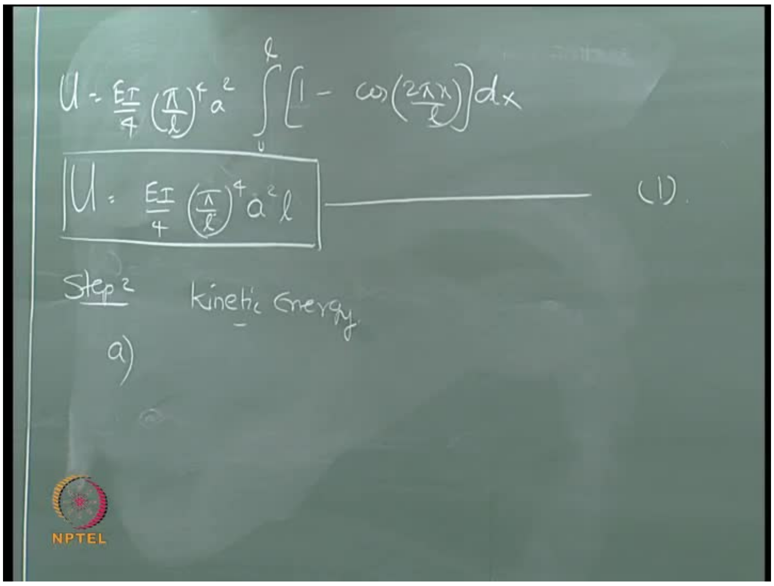

Now, integrating – U is EI by 2 pi by l the whole 4 a square 0 to l; again by 4, because there is 1 by 2 here – 1 minus cos 2pi x by l dx. You integrate and apply the limits and see what happens. So, after integration, you will check that, U will be EI by 4 into pi by l whole to the power 4 a square of l. I get this as U. This is the energy store, which can be also the potential energy; that is, I call this equation as 1. Now, I want to find the kinetic energy. So, this can be again due to two things. One is due to the central concentrator load.

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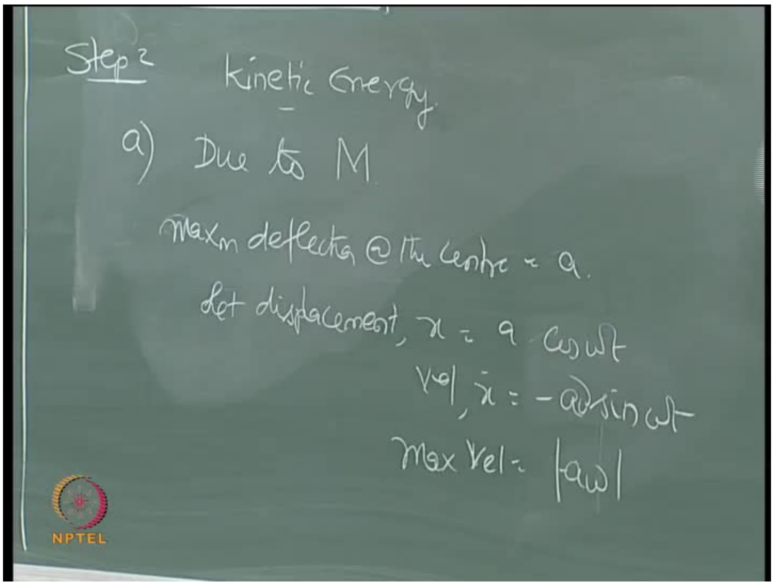

What I should say is due to capital M – the central load. The central load will have a linear deflection. So, the maximum deflection at the center is already known, which is a. Therefore, I say let the displacement be a sin omega t. And, of course, the velocity – let us take it as cos omega t – velocity, which is x dot minus a sin omega t. I am looking for the maximum velocity; I am looking for the maximum kinetic energy. Therefore, the maximum velocity can be determinant of minus a omega.

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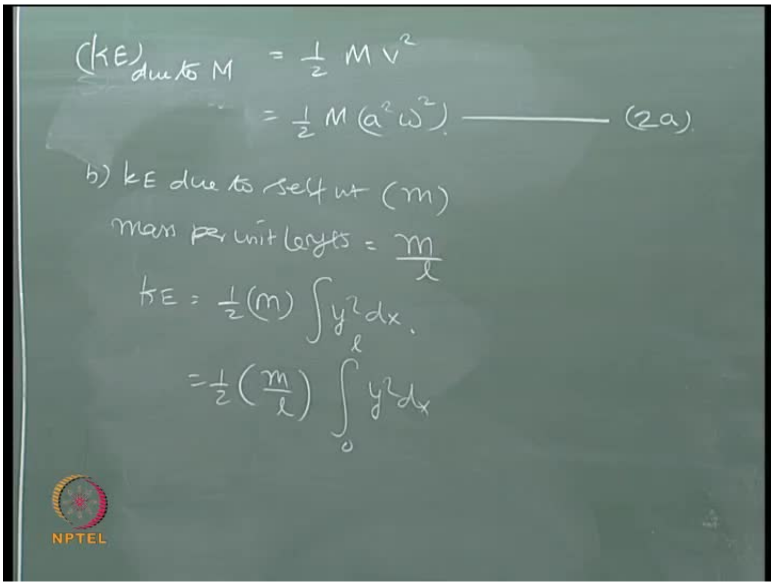

Therefore, the kinetic energy due to this mass M can be simply half M v square; which can be half M a square omega square. I am looking for the maximum velocity; that is, I call this as 2a. Now, kinetic energy due to self weight, that is, small m. So, we know mass per unit length of the member will be simply m by l, because m is the total mass of the whole beam. So, kinetic energy simply can be given as half that m of y square dx; which can be half that m by l of y square dx for the entire length of the member.

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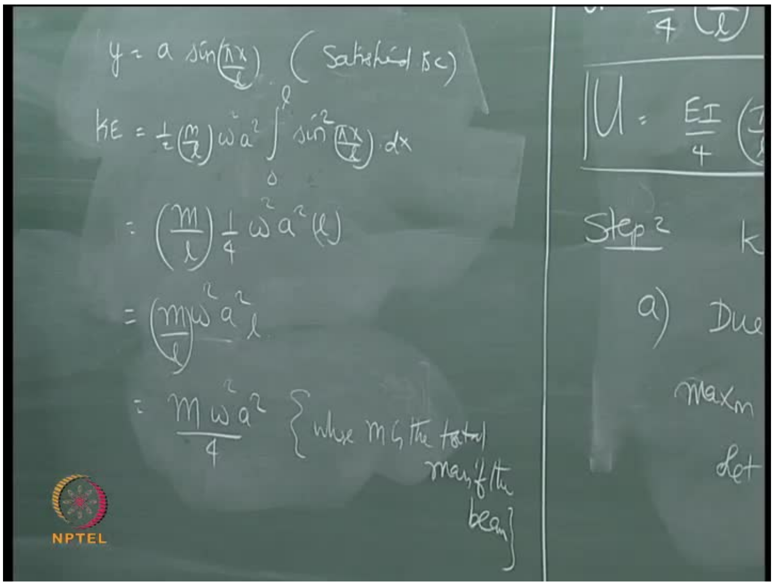

We already know y is a sin pi x by l with a satisfied boundary condition. Therefore, the kinetic energy is going to be half m by l of 0 to l a sin pi x by l of dx; there is a multiplier omega square here, because this and the maximum acceleration are connected by omega square a square. So, I must get omega square a square.

Student: (( )) sin pi x by l whole square

Sin pi x by 1 (( )) Sin square of course; so, we again have the same algebra here; no, I removed it. So, I can convert this into the equivalent multiples of cos ratio and try to get the integration of 0 to l and see what happens. So, I will get this as m by l 1 by 4 half; and, again half I will get back again. So, omega square a square. And, this integration will give me l. So, I actually get m omega square a square – m by l now – omega square l. Or, I can say simply m of omega square a square; where, m is the total mass of the beam. So, the total kinetic energy will be the sum of…

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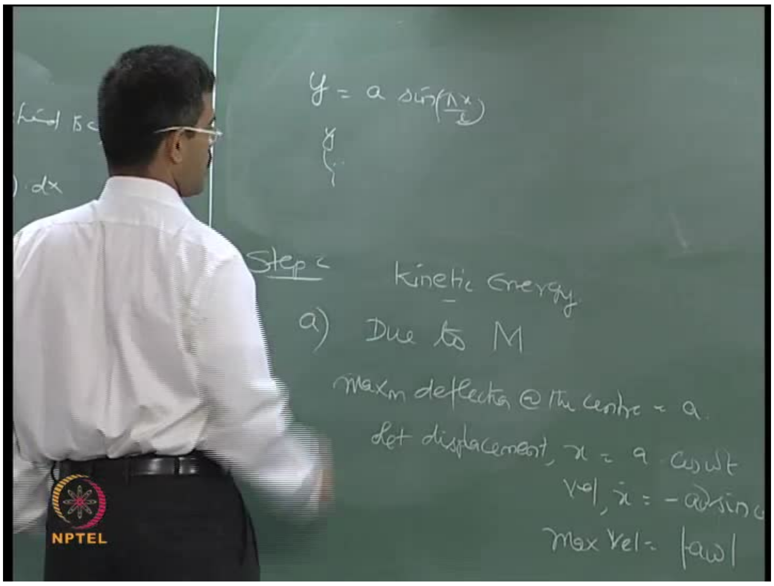

y – a sin pi x by l; y dot – y double dot; and, the maximum magnitude is a square of omega square.

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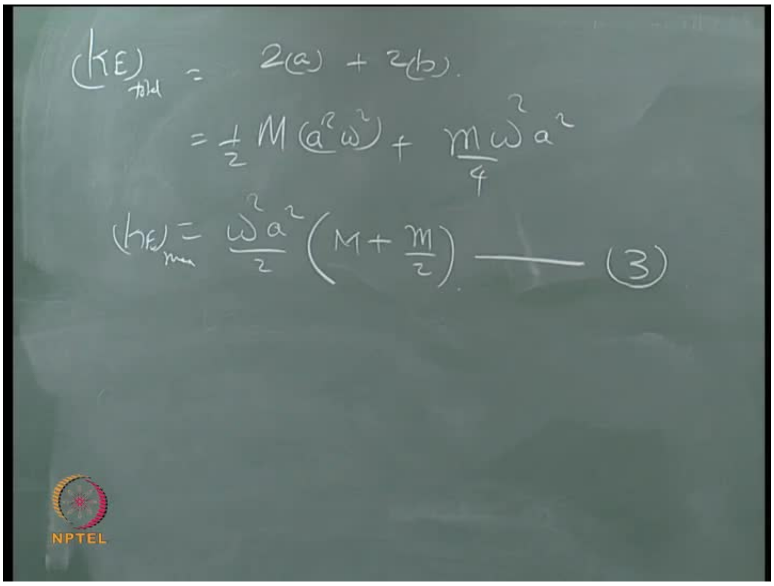

So, the total kinetic energy now will be the sum of – due to the point load M and due to the self weight m. So, it should be 2a plus 2b; I will call this as 2b. So, let us say 2a plus 2b. So, half capital M a square omega square plus m omega square a square by 4; which can be omega square a square by 2 of M plus m by 2. Let me call this as kinetic energy max. This is the equation number let us say 3. So, in a given system, wherever there is a potential energy maximum, kinetic energy is 0; and, for kinetic energy maximum at that position, potential energy is 0.

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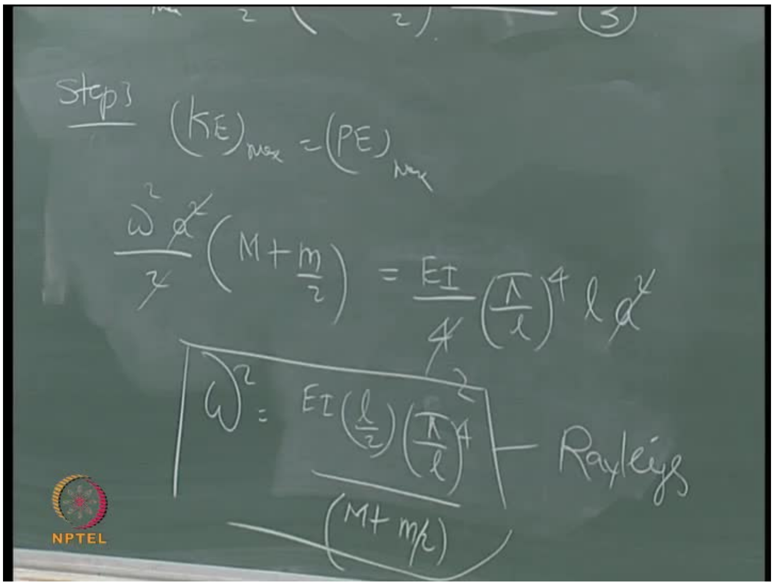

Therefore, I can say KE max should be given to PE max equal for a conservative system. So, equate 3 and 1. So, I should say omega square a square by 2 of M plus m by 2 should be equal to the U value, which is EI by 4 pi by l whole 4 l a square. So, simplify; I get omega; which will be EI l by 2 pi by l the whole power 4 by M plus m by 2. This is what I got from Rayleigh. So, let us see what happens in Dunkerley.

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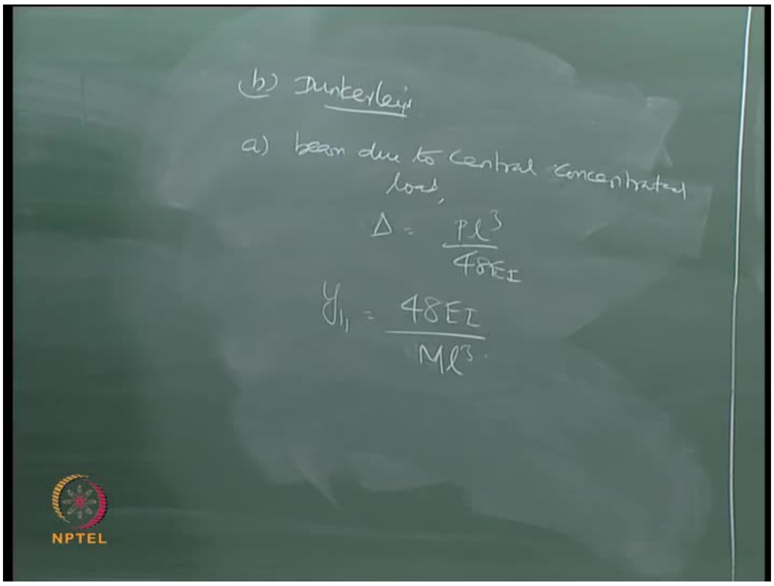

Dunkerley’s one influence coefficient, where I should apply the unit force and get the displacements, which are called otherwise influence coefficients or flexibility coefficients. So, if I am talking about any beam due to central concentrated load; we already know the deflection. It is simply let us say p l cube by 48 EI. So, I must apply unit force to get the deflection. So, I am interested in finding out y 11 as, that is, the influence coefficient, which will be 48 EI – instead of p, I am using M – capital M.

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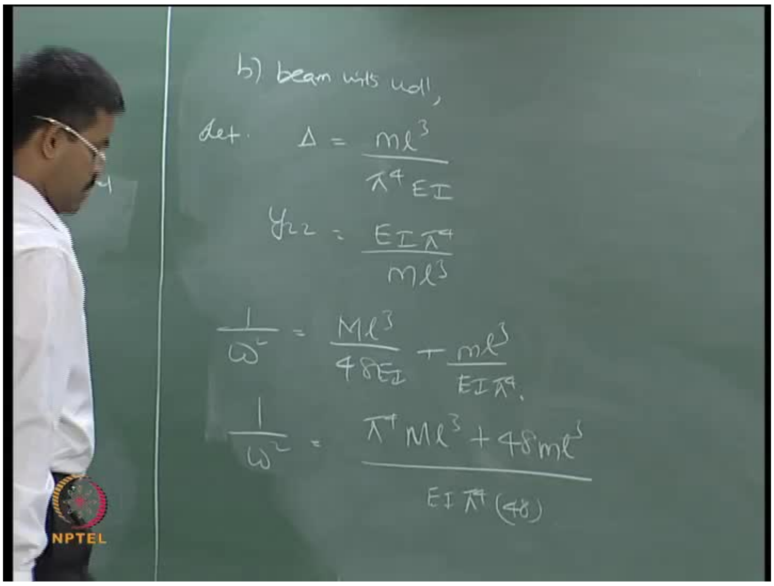

Similarly, for a beam with udl under (( )) we already know deflection is given by phi w l power 4 by 384 EI. But, here I will use deflection as m l cube; instead of l power 4, this M is already m l divided by pi power 4 EI – let. It is an approximate function actually; we do not know the value. Therefore, y 22, which is required for Dunkerley will be EI pi power 4 by m l cube. Dunkerley already says 1 by omega square is equal to m i of y i's. So, I should say M l cube by… That is the force here; that is the force here – m l cube by 48 EI. I must use M and forces. So, M l cube by 48 EI plus m l cube by EI pi power 4. And, omega square is now simplified as pi power 4 M l cube plus 48 m l cube by EI pi power 4 of 48. That is 1 by omega square.

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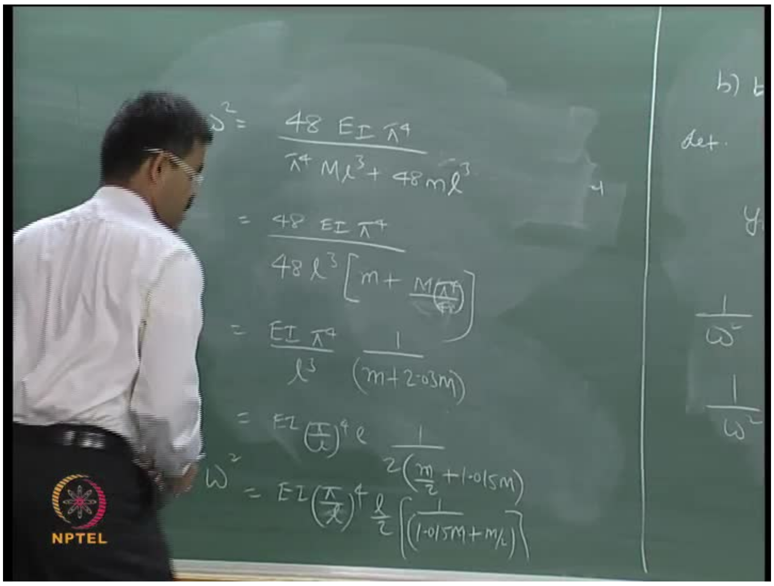

So, omega square can be rewritten as 48 EI pi power 4 by pi power 4 M l cube plus 48 m l cube. So, let us say 48 EI pi power 4 by 48 l cube pi power 4 M by 48… 48 l cube into m plus M pi power 4 by 48. So, this will turn out to be EI pi power 4 by l cube of 1 by… This ratio will become 2.02… This is 2.03 only. So, m plus 2.03 of capital M. So, I can rewrite this again as EI pi by l the whole 4 of l 1 by twice of m by 2 plus 1.015 of m. I am just taking it out. So, I can say this as EI pi by l the whole 4 l by 2 into 1 by 1.015 M plus m by 2. That is omega square.

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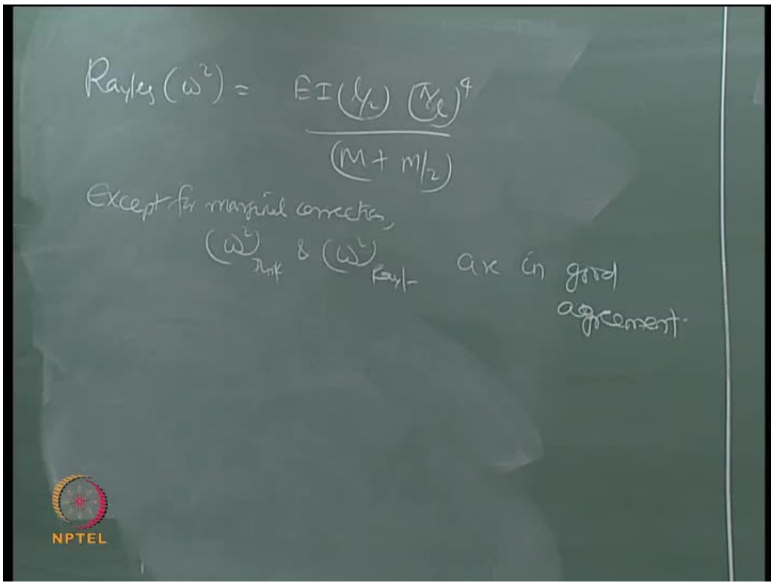

Let us compare this omega square with what I got from Rayleigh. Rayleigh's omega square is EI l by 2 pi by l whole 4 by M plus m by 2. Was it right? So, I am getting the same equation back again, except there is a marginal error in l. So, Rayleigh's method as well as Dunkerley is closely matching even for a continuous system. So, except for marginal correction, omega square obtained from Dunkerley and omega square obtained from Rayleigh are in good agreement. But, of course, Dunkerley can be used only for discrete systems. But, for comparison, we have shown here that, they are matching. So, this (( )) a discussion on the first module, except only one small portion, which I want to discuss; this is very important for dynamic as well as multi-degree freedom systems.

Now, do you have any difficulty, any questions so far what we have covered in first module? So, we have only one important aspect to be discussed; which I will do now quickly and show you the advantage of normalizing the modes, because somebody asked that, how orthogonality will help us in dynamic analysis; I will show you that here now by an example; simple. Two-degree freedom system example I will pick up and I will show you that, how the normalization of weighted modes will help you in solving the equations of motion for a multi-degree freedom system. Single degree – there is no problem; we already know the solution; so, no difficulty. For multi degree, there is a difficulty; let us say how we can use the procedure of normalization or orthogonality principle in using this for solving Mdof systems comfortably. That is our focus.

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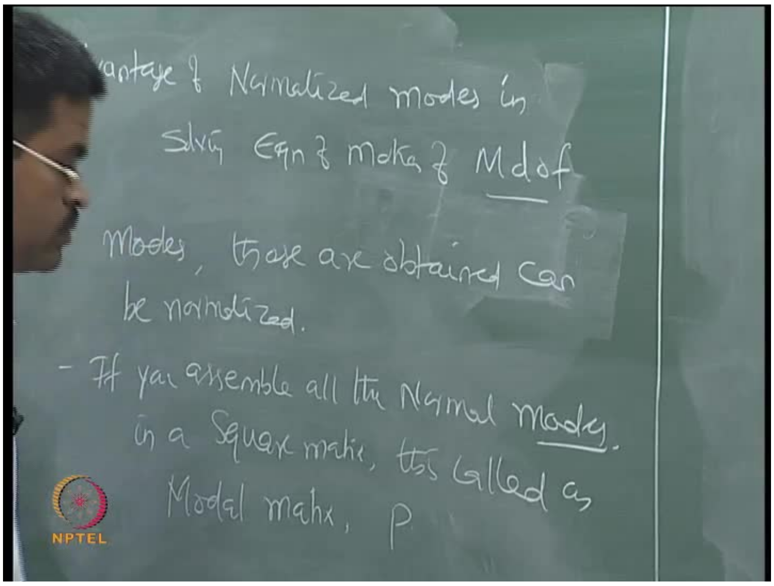

Let us say advantage of normalized modes in solving equation of motion of Mdof – multi degree freedom systems. We already know the modes, which are obtained can be normalized. We have shown a procedure. If you assemble all the normal modes in a square matrix, this is called modal matrix denoted by p.

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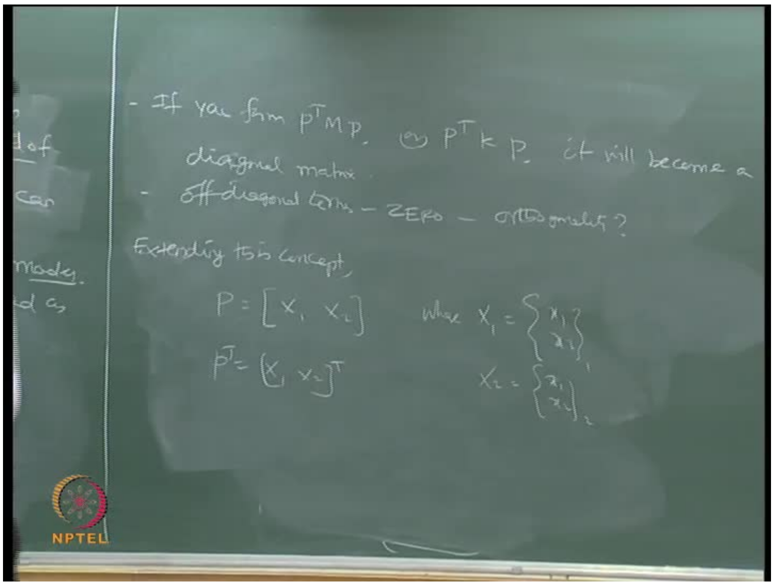

Now, if you form p transpose M p or p transpose k p, where m and k are the mass and stiffness matrices respectively; it will become a diagonal matrix. Now, diagonal matrices are generally easy for solving them. Why? Because off-diagonal terms will lead to 0 due to orthogonality. You may wonder where orthogonality is coming into play. I already said p matrix is of normalized mode. So, interestingly, extending this concept; let us say p is x 1, x 2; where, x 1 is x 1, x 2; and, x 2 is x 1, x 2 of the first and second mode respectively. Let us say p is a vector; I mean x are all vectors, but p is a square matrix now. And, of course, p transpose will obviously, x 1, x 2 of transpose.

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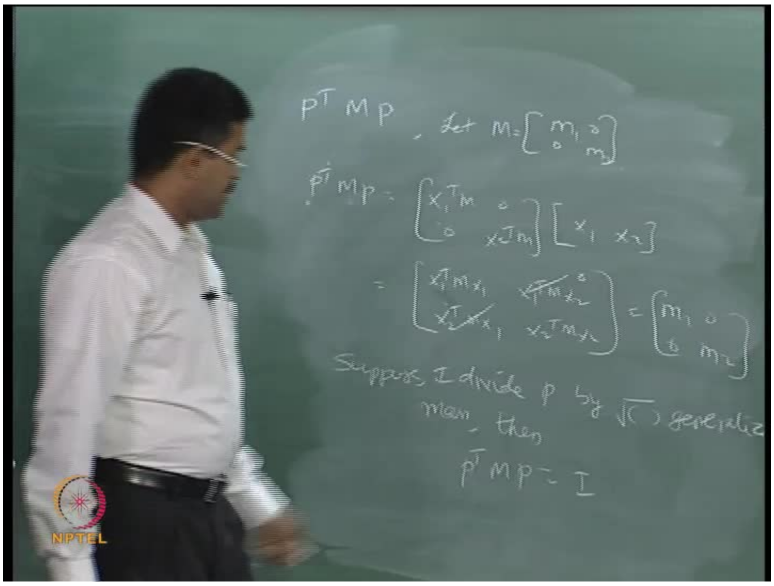

If I say p transpose M p; will give me… Let M be m, 0, 0, m – the diagonal matrix, where the coordinates of measurements of x 1, x 2 are taken at the point where mass is lumped. So, p transpose M p now can be x 1 transpose M; I multiply p transpose M separately and then multiply with p – x 1 transpose M – 0, 0, x 2 transpose M with x 1, x 2. These are all vectors remember; they are not two; they are vectors – column vectors as I show here. So, obviously, if I complete this multiplication, I will get this as x 1 transpose M x 1, x 1 transpose M x 2, x 2 transpose M x 1, x 2 transpose M x 2. These two will become 0, because x 1, x 2 are orthogonal, which will ultimately lead to m 1, 0, 0, m 2 if these are m 1 and m 2. Suppose I divide the p matrix by square root of the generalized mass matrix; obviously then, p transpose M p will become an identity matrix. In one example, we have already shown this. Let us see what happens – p transpose k p.

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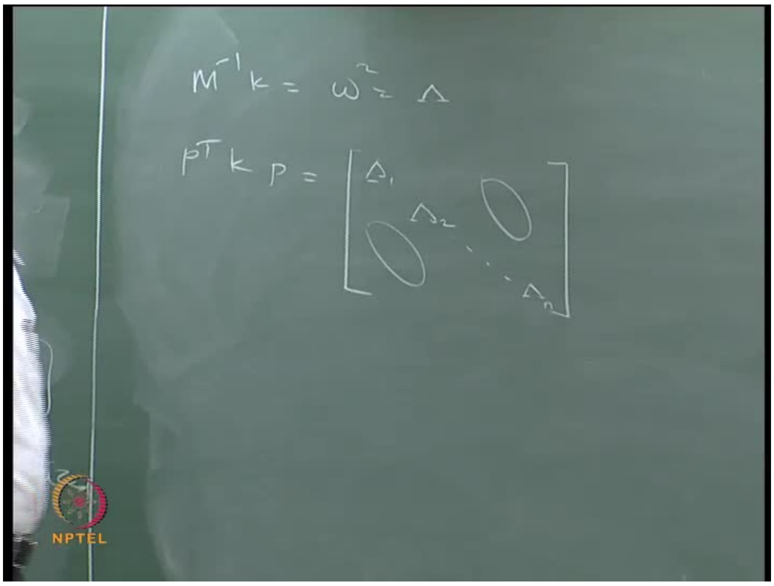

We already know M inverse k is let us say omega square, which is I am calling as lambda. Therefore, p transpose k p will interestingly given me all eigenvalues; remaining all will be 0. So, using these two concepts, let us see how I will use the weighted model mass matrix for solving a multi-degree freedom system by decoupling the equations of motion. That is an advantage, because all this off-diagonal terms get 0; I can decouple the equations of motion; I will show you how. I will take a very simple example. So, quickly we can demonstrate that in few minutes and show how this can be done. Any questions here in the principle of orthogonality applied to mass and k matrices.

All these are matrices. Therefore, they are multi-degree freedom system models. For single degree, we have no problem. Any doubt here? Any questions how we are applying the principle of orthogonality for mass matrix and k matrix? I will explain you how we obtain the weighted model mass matrix by dividing the square root of the generalized mass value. I will show you that just now in an example; I demonstrate this. We have already done this for one example; but, now again I will do it, so that I will decouple the equation of motion and show you how this method can be very powerfully operated for multi-degrees. I will take an example of 2 by 2, but you can apply this for n by n also. Any question? We will remove this.

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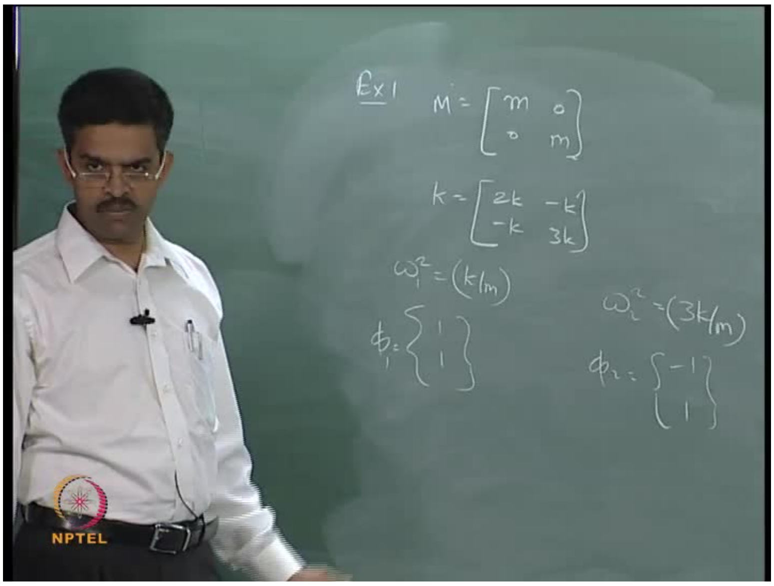

Let us say I have an example, where mass matrix, k matrix, omega and phi are computed; I already know them. So, I will take up an example, where the mass matrix is simply m, 0, 0, m; where, the k matrix is 2k, minus k, minus k and 3k. And, omega 1 square is simply k by m. And, the first mode phi 1 is 1 and 1. Omega 2 square is 3k by m; and, phi 2 is minus 1 and 1. These are obtained standard procedure; we already know this. So, the first step is I want to find their generalized mass matrix. We already know the modal matrix; if they are normal, I can simply form a capital phi matrix or capital P matrix, which will be nothing but 1, 1 and minus 1 and 1. But, I want to normalize them. So, I must divide them by a square root of a generalized mass matrix. First, let me find the generalized mass matrix.

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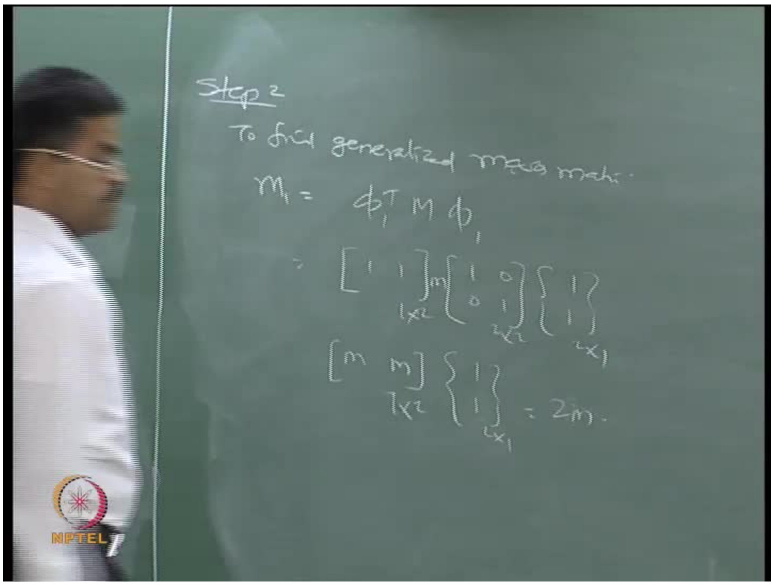

So, let us say, to find the generalized mass matrix, let us say m 1 with respect to the first degree.

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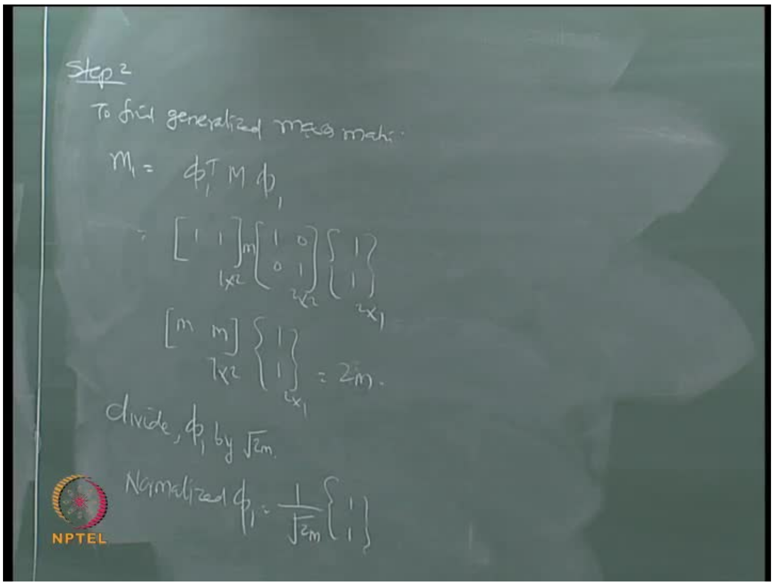

So, I should say phi 1 transpose M phi 1. So, let us try to do that as 1, 1 m of 1, 0, 0, 1 of 1, 1. This is 1 by 2; this is 2 by 2; this is 2 by 1. I get 1 by 2 first, which will be m, m; and, again multiply by this. This is 1 by 2; this is 2 by 1. I will get ultimately a single value, which will be 2m. So, divide the first vector – the first vector phi 1 by root 2m.

So, divide phi 1 by root 2m. So, I must say normalized phi 1 is 1 by root 2m of 1, 1. I will retain this; I will move here.

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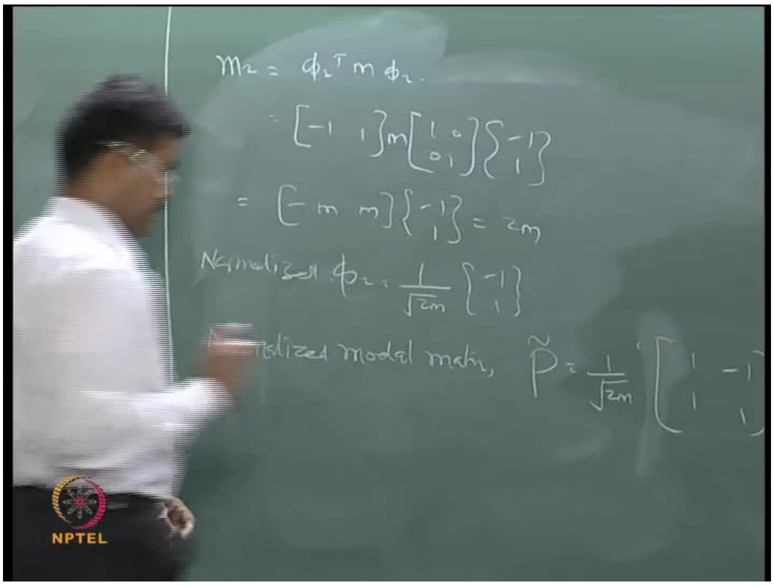

So, let us say, to find m 2, I should say phi 2 m phi 2. This is minus 1, 1 of m of 1, 0, 0, 1 of minus 1, 1. So, can you get me the generalized value of this? What is the value? So, minus m, m of minus 1, 1; again I think you get 2m. So, the normalized phi 2 will be again 1 by root 2m of minus 1, 1. So, let me write the normalized modal matrix; matrix, not a vector – p as 1 by root 2m of 1, 1, minus 1. I call this matrix as just for understanding, p tilde. This p was a normalized matrix.

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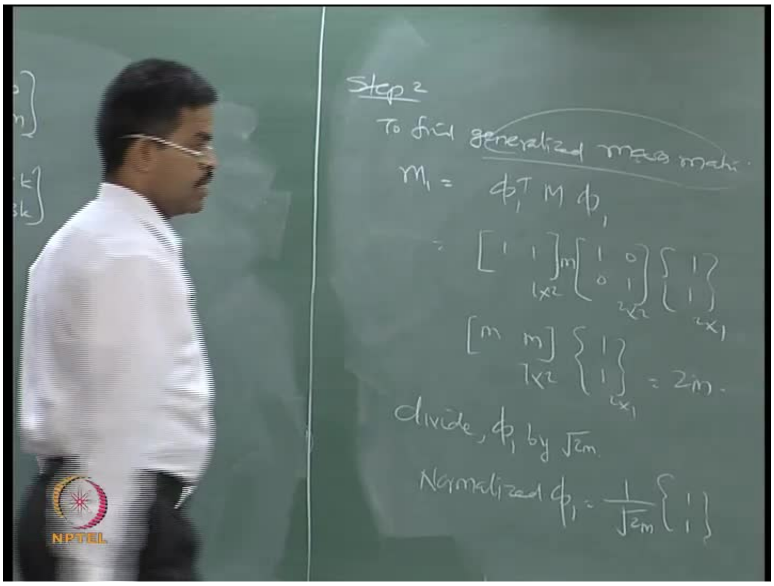

Now, I divided this by generalized mass matrix. If I now multiply this with mass, I will get identity. If I multiply this with k, I must get lambda squares or omega squares. That is the advantage of this matrix. But, I am not going to use that now. I am going to use this advantage for solving the equation of motion. What do you understand by solving equation of motion? I want to find x 1 and x 2; that is the main reason. My interest of omega and phi is not the solution of equation of motion; that is the first characteristic of the system. But, I am interested in ultimately find the displacement value, that is, x 1 and x 2 in terms of time history. That is our aim.

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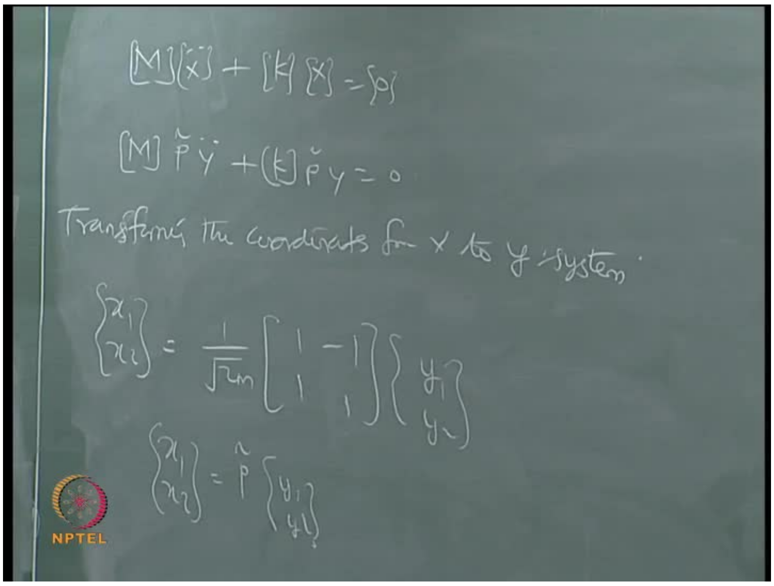

I will use this property now for solving the equations of motion. The original equation is M x double dot plus k x – all are vectors and matrices, etcetera, is 0. What I am going to write here is M p tilde y double dot plus k p tilde y is 0. I am transforming. the coordinates from x to y system. So, what does it mean? x 1, x 2 will be nothing but 1 by root 2 m of 1, 1, minus 1, 1 of y 1, y 2. Simply I multiply p tilde to get the new one. Or, I can say simply, x 1 vector is nothing but p tilde of y 1. What I am interested to find out? The x 1, x 2. But, what I will find is y 1, y 2. Then, I will transform it back to x 1, x 2. So, now, let us see what happens to p tilde M p. Can you quickly find out what is p tilde M p.

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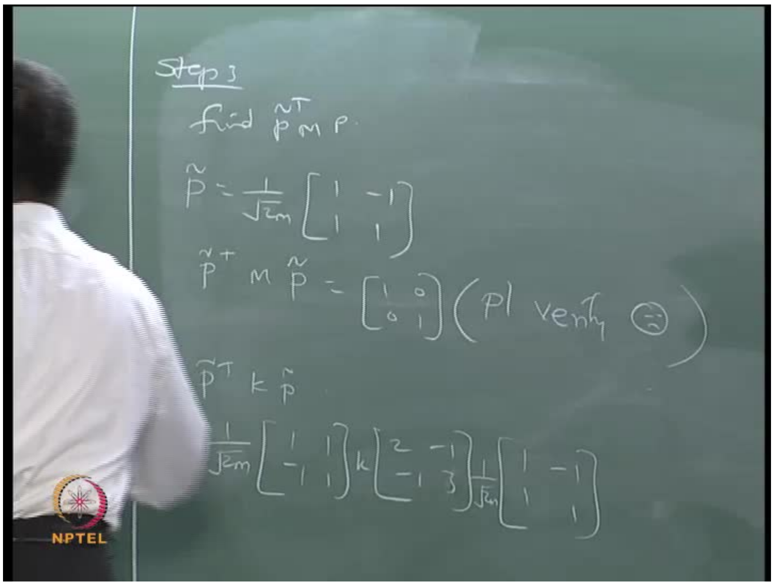

Step number 3 – find p tilde M p, because I need it here – p tilde transpose M p quickly. So, p matrix already we have; let me write that. So, p tilde matrix is nothing but 1 by root 2m of 1, 1, minus 1, 1. Is that right? And, M matrix already you have it here. So, do this transformation and tell me what is p tilde M matrix p; quick, quick. What you should get otherwise? Identity matrix. Please check immediately; quick, quick. If you do not get i, there is a problem. Are you getting?

Student: Yes.

Good. So, I am not showing it here; will be 1, 0, 0, 1. I write here please verify. I am leaving it for you. But, verify it. Can you also find p tilde k p? So, 1 by root 2m of 1, minus 1, 1, 1 of k, which is 2, minus 1, minus 1, 3 of 1 by root 2m of 1, 1, minus 1, 1.

That is what I want. I will remove this. What is that? Hurry up. It is simple 2 by 2 multiplication; you should be able to do it very fast. Please. What is the answer?

Student: (( )) 3, 1, 1 (( ))

Wrong answer. Try it again; quick, quick; it is a simple multiplication. Am I writing the matrices correctly?

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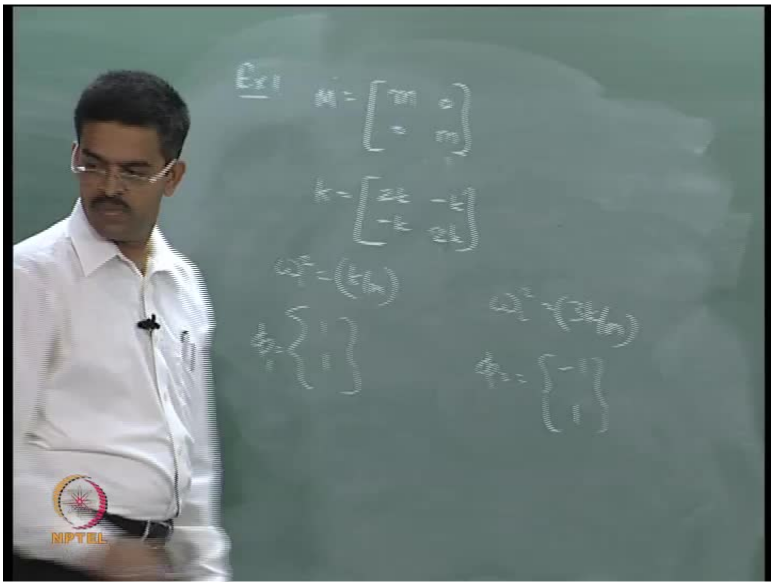

This is 2. This is 2. Please change this. This is 2k, not 3k. You are using it only for the first time here. So, there is no botheration, but still. Now, your answering may be corrected; please correct it. 2k, minus k, k and 2k yeah. What is the multiplier? k by m or nothing, no multiplier? It is a multiplier. Then, what you are getting inside?

Student: 1, 0, 0, 1.

1, 0, 0, 1. Take the multiplier as k by m; take the multiplier as k by m and give me the value, because I want omega square (( ))

Student: 1, 0, 1, 0 (( ))

Good.

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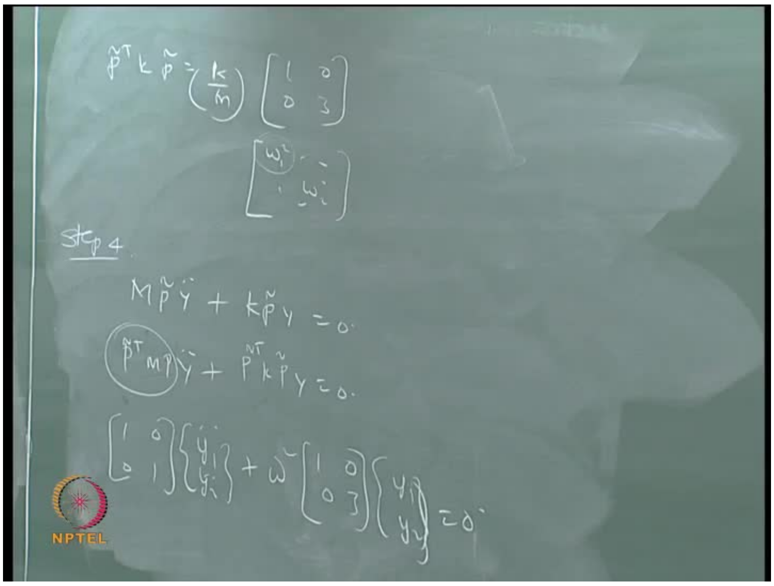

So, I get this value as p tilde k p will be k by m of 1, 0, 0, 3. As I already said, this will give me omega square 1 and omega square 2 and so on. I already said this. So, omega square 1 was actually k by m and omega 2 square was actually 3k by m. You must have got this. So, those who are not getting it, please check; do not try to copy this. So, step number 4 – we already said M p tilde y double dot plus k p tilde y is 0. I have multiplied this side and transformed the equation from x to y's. Let me pre-multiply with p tilde transpose. So, p tilde transpose M p of y double dot plus p tilde transpose k p tilde of y should be 0. I already know this value just now, which is equal to 1, 0, 0, 1. And, I already know this value as 1, 0, 0, 3. Now, I can write 1, 0, 0, 1 of let us say y 1 double dot, y 2 double dot plus omega square of 1, 0, 0, 3 of y 1, y 2, is 0; I can remove this now.

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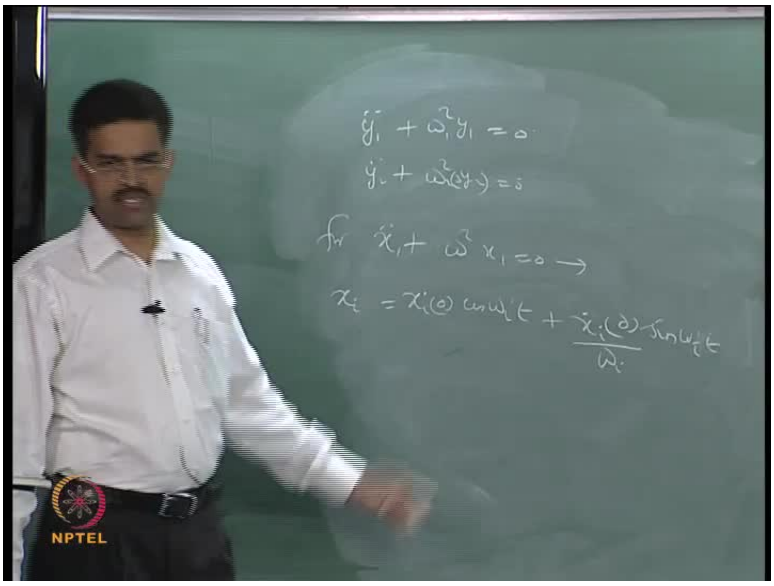

So, what does it become now? y 1 double dot plus omega square y 1 is 0; y 2 double dot plus omega y 2 is 0. Is that right? So, I have decoupled actually the equation of motion into n number of single degree freedom systems. Of course, there is a multiplier here 3. I am just expanding it and writing. So, what I get from this equation will be omega 2; what I solve from this equation will be omega 1. So, initially, I had a coupled equations of motion; I have decoupled them now. So, for a single degree of let us say x 1 double dot omega square x 1 set to 0; we already know the solution. What is the solution? I should say x i cos omega i t; I am just making it general – plus x i dot sin omega i t by omega i. Is that right? So, I can write this in y's now. What I will get is y 1 and y 2; y 1 and y 2 I will get. How to transform it back to x? Because what answers I am getting are all will be in y's; I do not want them in y's; I want in x. How do I transform it back? How do I transform it back? Quick.

Student: Inverse of p (( ))

Inverse of p tilde, yes.

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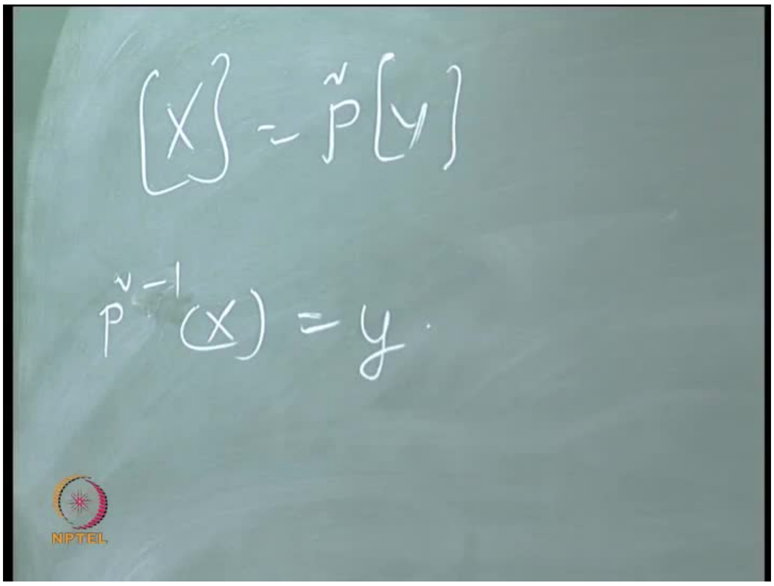

So, I already said X is p tilde of Y. So, I can always find the values of Y first as p tilde inverse of X. Why we are using p tilde inverse of X? Any idea?

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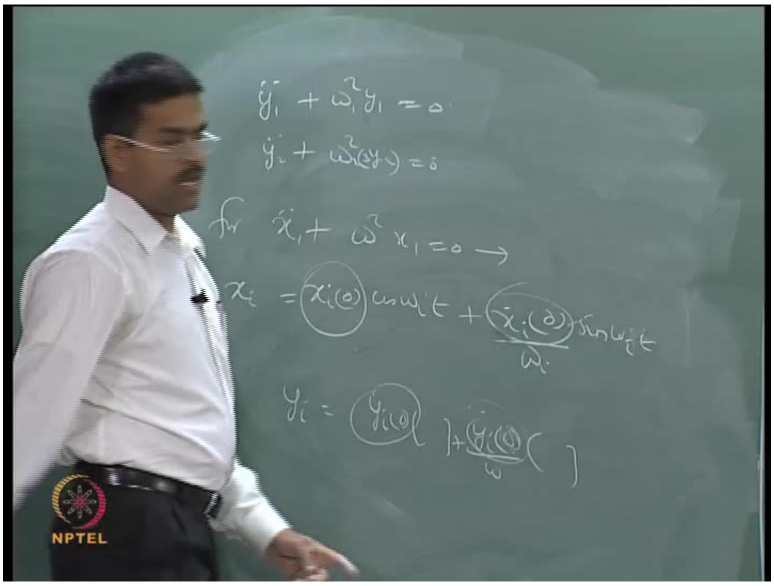

To find y's, because I want the initial conditions; I am having initial conditions only on x, not on y's. To write an equation on y, I need initial conditions of this and this. So, I do not have these with me; I have to transform it and get this. Substitute back; you will get the values of y's. Substitute back again in this equation; get the values in x. So, I have decoupled the equation of motion using the same algorithm of orthogonality principle.

So, n number of degrees of freedom or n numbers of equations of motion of an Mdof system can be decoupled as simple n numbers of single degree freedom systems if we use a weighted modal matrix, which is p tilde. So, what we are going to do is very simple; write the equations of motion; find k matrix, m matrix. Solve by any of the methods what you already know – eigensolver, Stodla, Rayleigh's, Dunkerley, any values; get phi 1's – phi's and omega's related to this. You have got the phi matrix now. For every degree of freedom, find the generalized mass matrix – m 1, m 2, m 3 and so on and divide that square root value by phi's. We have got weighted modal matrix now.

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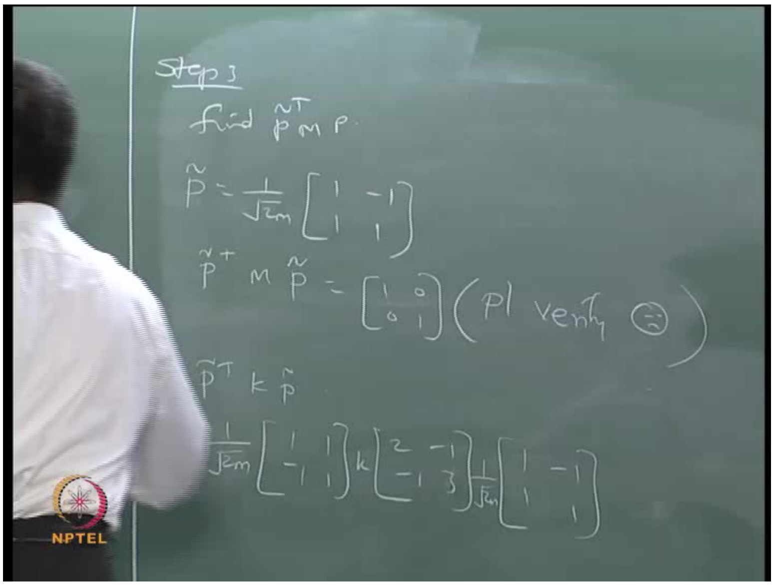

Use this relationship, which we wrote – this and this; simplify the equation; decouple them and solve them in a different coordinate system now. And, this transformation is required, because I have all the initial conditions on x coordinates, not on y. So, I get those conditions back on y first; solve for y 1's and y 2's using the same algorithm. Once you get y 1 and y 2, solve back, substitute back again and get x 1 and x 2, which we wanted actually. My original equation is not in y; it is in x. So, this ends the discussion of single degree.