Dynamics of Ocean Structures
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Module - 1
Lecture - 19
Eigenvalues and Eigenvectors

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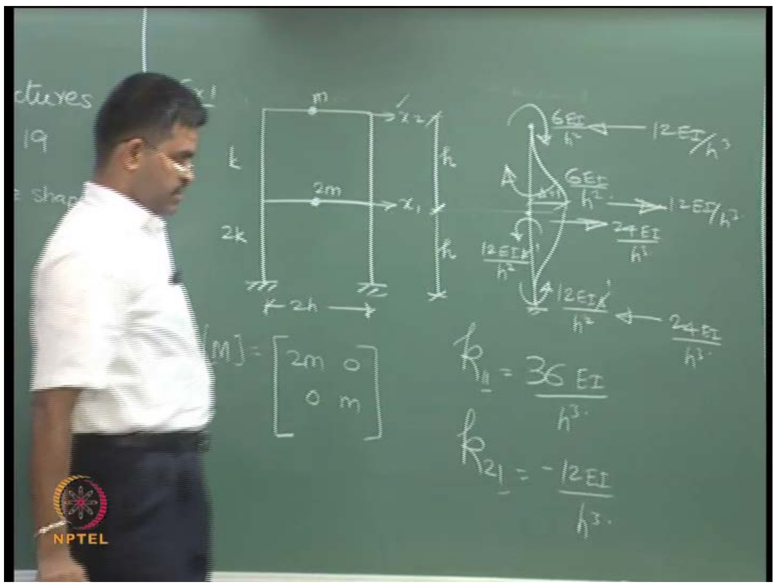

So, we will continue with the discussion on two degree freedom system models, but today we will discuss about natural frequency and mode shapes and some advantages of getting orthogonality of modes. Let us see what the advantages are, and how to obtain an orthogonal mode again. We will take up an example to solve, to find the natural frequencies and mode shapes of a double two degree freedom system model.

Then we will extend this principle in the next class to multi-degree freedom systems using approximate techniques because this method, what we will discuss, will be tedious to find out natural frequency and mode shapes. Then there are approximate methods to find out this for multi-degree freedom systems, that we will see in the next class. So, we will take up an example and take a simple idealized model. So, the first information, what I get from this model is that since my mass are lumped at the points where I am measuring the displacements of degrees of freedom, therefore my mass matrix will be diagonal and off diagonal terms will become 0.

In the last lecture we have clarified this, if you do not measure the displacements at the point where the mass is lumped, and then you will find, that the mass matrix will not be diagonal, off-diagonal elements will also be present. This is what we call as, static decoupling, may be with respect to mass, with respect to stiffness matrices. So, now I can straightaway write the mass matrix for this problem at 2m, m, 0, 0. I am interested to find out the k matrix, that is, the stiffness matrix for this.

So, I will use the first principles again. So, I give unit displacement here and do not give any displacement at any direction or any node, only displacement at the point where we are interested in. So, this will cause a moment at this point and usually when the displacement is delta, then this moment is 6E delta by l square from the first principles. But my stiffness of this column is 2k, therefore this is going to be 12E I delta by h cube, where h is the dimension of this member because it is 2k, therefore it is 12, otherwise it should be 6E delta by l square; similarly, here 12E I delta by h cube.

We already know delta is unity because that is how I get stiffness. Stiffness is nothing but the force responsible to give or to cause unit displacement at any desired location. So, again this is going to be 6E I delta by l square and 6E I delta by h square. Now, there is a confusion here for me, that how to mark this arrow directions. Let us do that very carefully. Here there is a very shortcut to understand how I mark this arrow direction.

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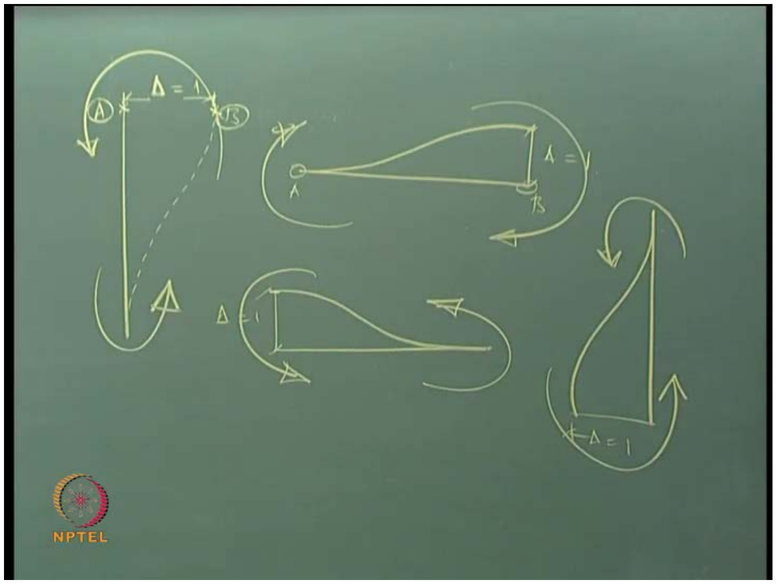

Let us say I have a column member, this way I give a unit displacement here, so obviously, this should be my deflected position. So, when I have displaced a point from A to B, this is delta from A to B, then try to bring back B to A and similarly, the same way, you rotate this. For example, if I have a member, which is horizontal, you want to give displacement here, which is delta and call this as A and this as B and this is going to be displaced position. Since A displaced B from the normal position here, bring back this and so on. If you have a column or if you have a member, I want to give displacement here and this is going to be a displaced profile, so bring this back and so on. For example, if I have a column member I want to give displacement here. So, this is going be a displaced profile, so bring this back to my normal portion. That is how I marked the arrow directions; that is how these arrow directions are marked.

So, now this is going to be a total moment of 24 ABHQ h square, please make this correction, this is h square 24 ABH square. So, now I will have a couple, which is 24 EI by h cube and 24 EI by h cube. Similarly, I will have a couple again here, which is 12 EI by h cube and 12 EI by h cube. So, therefore, I am interested in finding out the force or the stiffness at this point because of unit displacement given here and looking for the force in this direction. So, therefore, I should say k 11, from my derivation, will be equal to these forces, which is 36 EI by h cube, which is positive because this force in the same direction as that of the displacement. I talk about k 21, stiffness matrix is always derived column-wise, I am looking for the second one, this, here displacement is towards the right, force is towards the left, so minus 12 EI by h cube.

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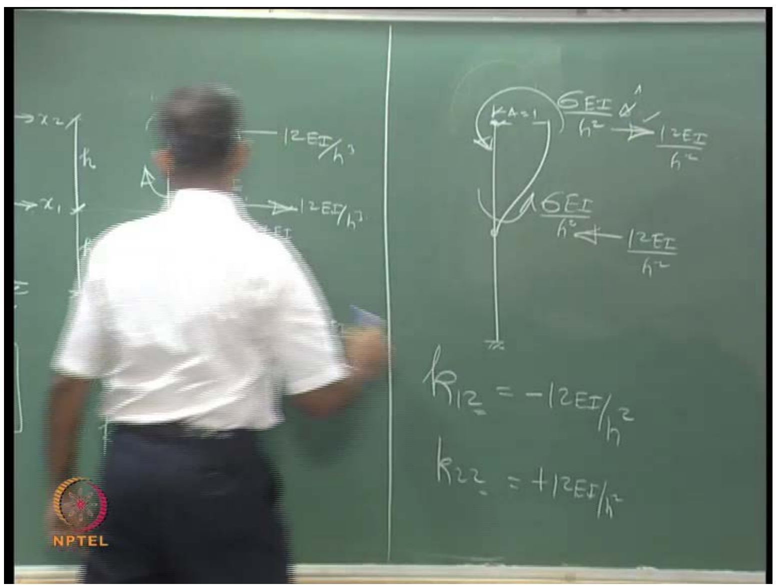

Let me give delta here, so this is going to cause a moment, which is 6 EI by h square. Of course, delta is unity, I am putting here I because this is only k, the stiffness is only single, whereas this is 12 because this was 2k, so this is an anticlockwise moment of 12. So, there is going to be a couple, which is 12 EI by h square. So, I am going to derive now k 12 and k 22.

I am deriving the second column because I have given unit displacement in the second degree that is why I am deriving the second column. k 12 is the point of the force where I am looking at here, this is opposite to my displacement direction, therefore it is going to be 12 EI by h square where this is positive, because this is positive, because this is in the same direction as that of the displacement, so let me write down the k matrix.

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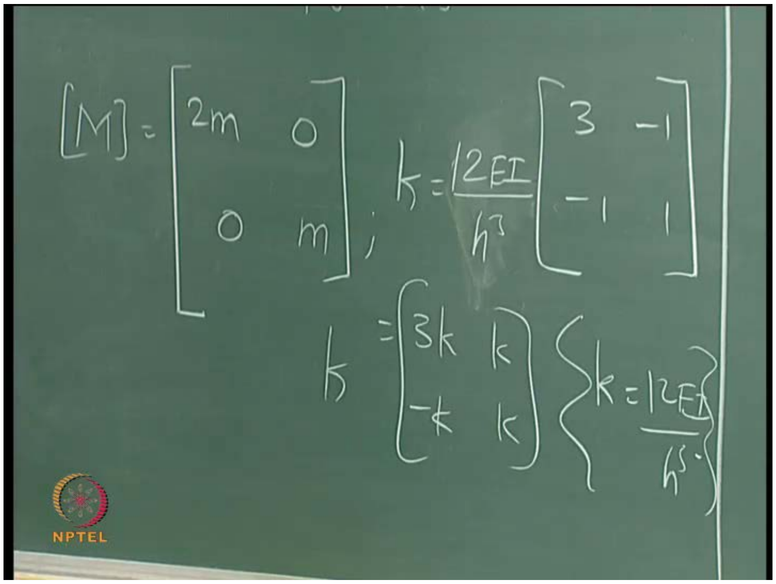

So, I will write down the M matrix here, 2m, 0, 0, m and k matrix let me put, 12 EI by h cube common 3, minus 1, minus 1, 1. You can also say this as 3k, k, minus k, k as my k matrix, where k is 12 EI by h cube, any questions here? Fundamental mechanics, where I derived the k matrix and mass matrix, mass matrix will become diagonal because we already know, that the displacements are measured in the same direction where or the same point where the mass is being lumped. I will remove this.

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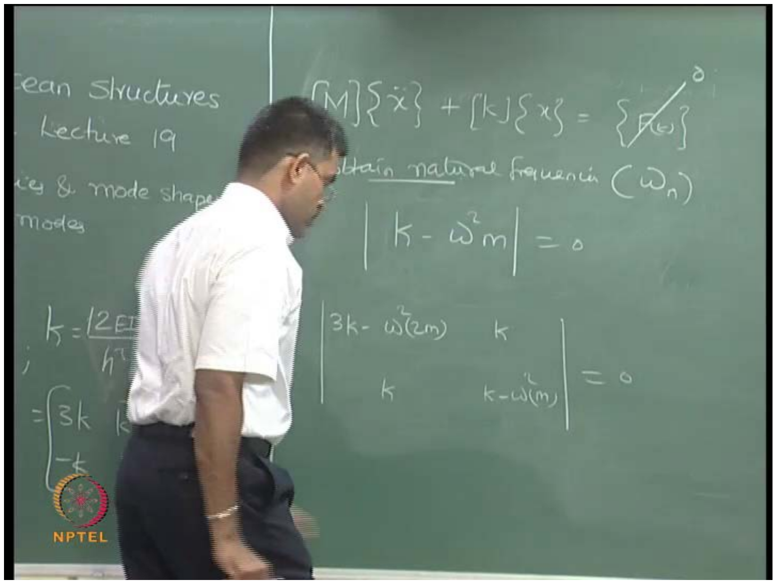

You can also write the equation of motion for this from Newton's law. I will simply write this as, in this case it is 0, whereas M corresponds to the mass matrix, k corresponds to my stiffness matrix and of course, the vectors x and x double dot corresponds to displacements and acceleration respectively in the appropriate degrees of freedom. Now, I am interested to find out the natural frequencies of vibration of this model, which I call as omega n. So, simply said, determinant of k minus omega square m equal to 0, so let me do that. So, k is going to be k matrix of 3k, k, minus k and k and this is 2m. So, I keep on appropriately using them and substituting it in this and get a determinant of this. So, 3k minus omega square of 2m, k, k, k minus omega square of m, set determinant of this to 0.

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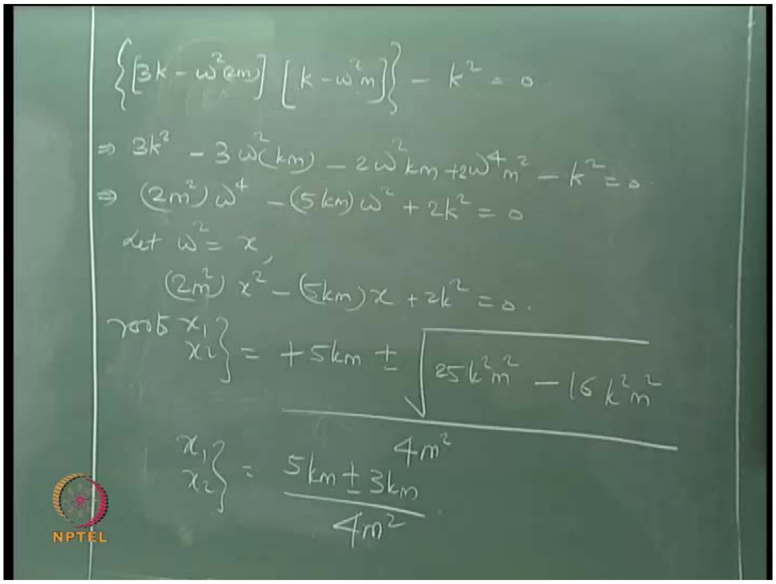

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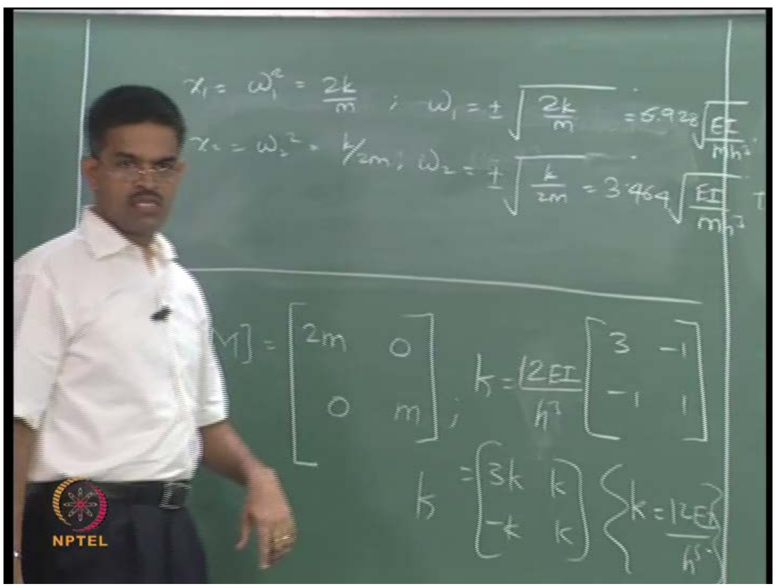

So, we have found the roots of this quadratic equation and we have got the natural frequencies, which are called Eigen solver method. It is a classical Eigen solver technique where I have found out the natural frequencies omega 1 and omega 2. Now, I really do not know which is the lower one of course the number gives me a value if this number is not indicated here I really do not know which is the first fundamental frequency therefore do not classify the frequency as omega one omega two by arriving at the roots of this Eigen solver so let us try to qualify this from the mode shapes let's see what is the mode shape. I will remove this.

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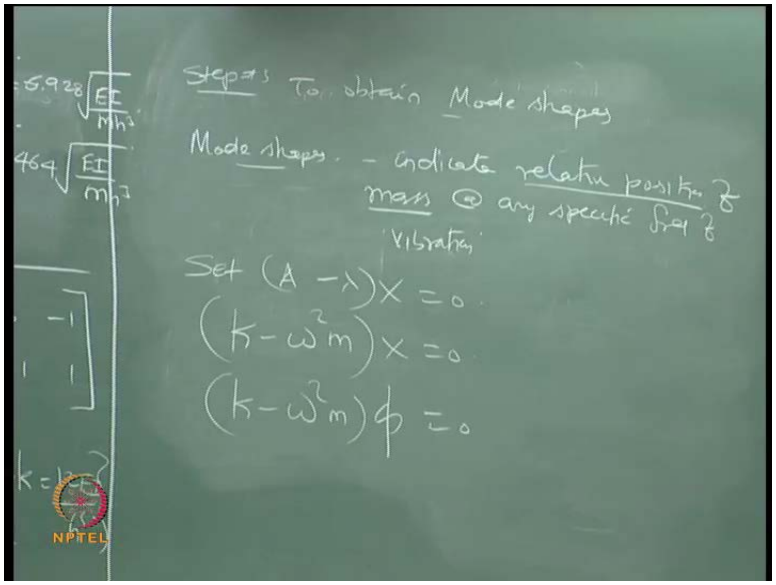

So, step number three, to obtain mode shapes. Now, what are mode shapes? Mode shapes actually indicate the relative position of mass at any specific frequency of vibration. For example, if the structural system is vibrating at a specific frequency of 6.928 EI m h cube root, then what would be the relative position of the mass points in the given system? If I try to plot that graphically, that is what my mode shape is, so I want to find the mode shape now.

So, classical equation for Eigen solver is AI minus lambda x is 0, set A minus lambda x is equal to 0, set A minus lambda of x is 0. In our case, it is going to be k minus omega square m of lambda set to 0. So, in my case I am going to say, k minus omega square m of phi is 0 or phi is my mode shape, so I will remove this. So, for every specific frequency of vibration, there will exist a mode shape. So, frequencies and mode shapes are couples, they are pairs.

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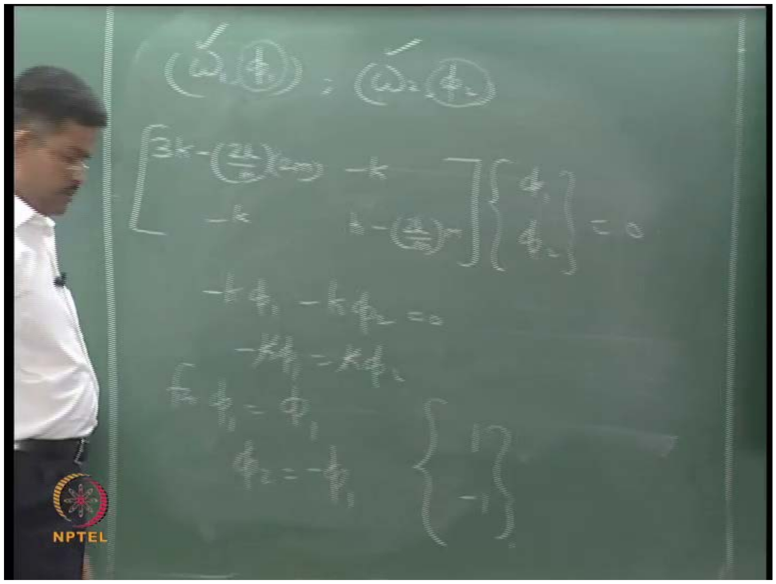

So, for every frequency there is a corresponding mode shape; for every frequency there is a corresponding mode shape. I know these two values, I do not know them, I am interested to find out this. So, if I really wanted to know phi 1, I must substitute here. Instead of omega I should say omega 1 and get phi 1; if I substitute omega 2, I will get phi 2. So, corresponding values, this cannot be pair decoupled, they are pairs and this combination is unique.

There can be infinite number of frequency and mode shapes, but for any given frequency there will be only one mode shape. There is a strong coupling between these two, this cannot be mismatched. Let us say, phi 1 cannot go with omega 2 or phi 2 cannot go with omega 1 except that very rare possibilities. We will discuss that later. So, here instead of omega, I substitute omega 1, which is my omega 1 square is 2k by m, substitute here and say phi 1 and phi 2. Let me get this mode shapes, so try to substitute this for… So, 3 k minus omega 1 is let us say 2k by m of m, minus k, minus k and minus k, minus k and k minus omega square of phi 1 and phi 2. So, simplify this and get phi 1 and phi 2. So, if I look at the first equation, 2k minus 3 k minus 2k, so k of phi 1 minus k of phi 2 is 0.

This is 2 m, is it? There is a change here, it is 2 m here, so this becomes minus k. There is a 2m here, the mass matrix, first value is 2m. So, this implies, minus k phi 1 is equal to k phi 2. If I say, for phi 1 equals phi 1, phi 2 is equal to minus of phi 1, is that ok? I am trying to find out the relative value, so the mode shape is going to be 1 and minus 1; ok, 1 and minus 1. Similarly, substitute for omega 2 and try to find the second mode shape. I will remove this.

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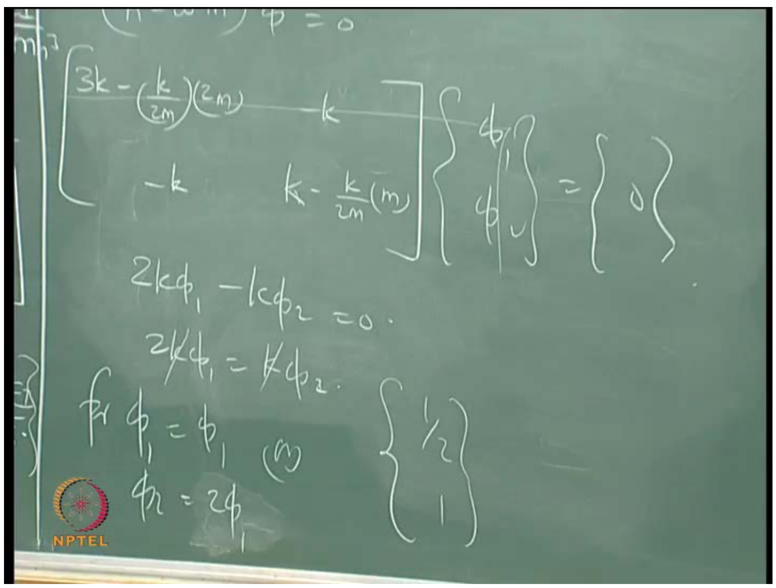

So, for omega 2 square as k by 2m, from here find out again k m minus omega square m. k minus omega square m of phi is 0, so 3k minus k by 2m of 2m, minus k, minus k, k minus k by 2 m of m phi 1, phi 2 said to be 0. So, if we look at the first equation again, 2k phi 1 minus k phi 2 is 0. 2k phi 1 is k phi 2, for phi 1 is phi 1, phi 2 will be half of phi 1, is that twice of phi or mode shape can be half and 1. Let me remove this here.

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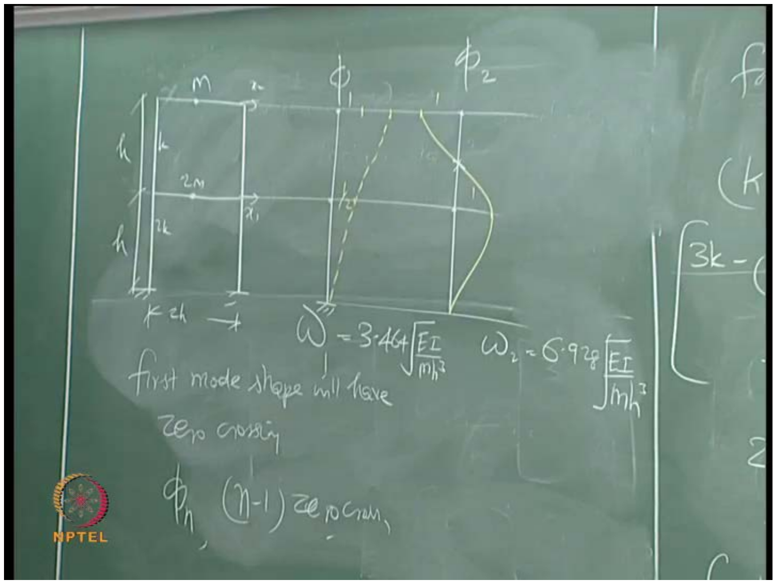

This is my given structure, 2 m and m; this is x 1, x 2, k, 2k, h, h and 2h. So, my first mode shape is my stick model, these are my mass positions, this is fixed. So, the first mode shape says, if phi 1 moves by half, phi 2 moves by double of this. That is what my first mode shape is… I will come to that, why I am plotting this. First, this one mode shape, this is half, this is 1, there is one more shape I have, which, this is my mass position. If the mass m 1 moves to the right by one value, mass m 2 moves to the left by one value. This is my second. The first mode shape will have 0 crossing. What does that mean is all the value will be either positive or negative. It will not cross in the given frame, whereas here it crosses at one point.

So, it means, if the mode shape is phi n, it will have n minus 1 0 crossing. So, for the second mode shape there will be one crossing, for the first mode shape there is no crossing. Since this is my first mode shape, I will call the corresponding frequency as a fundamental frequency, which is 3.464 root of EI by mh cube and this is my phi 1 and this is my phi 2 and therefore, this is my omega 2, which is 6.928 EI by mh. It is a classical Eigen solver theory by which we have found out the natural frequency of mode shapes of a two-degree freedom system.

The same principle can be extended for multi-degree provided you can see the complexities of solving these equations for the classical Eigen solver theory. So, therefore, when you resolve to the multi-degree freedom systems, people are looking for some approximate techniques by which the frequency of mode shapes can be obtained, which we will discuss in the next class, that is tomorrow.

So, one is interested to find out the natural frequency and mode shape, you can use one of the method I have demonstrated here. It is a classical Eigen solver theory where omegas are called Eigen values and phi's are called Eigen vectors, whereas I call them as frequency and mode shape omegas are Eigen values and phi's are Eigen vectors, that is why we call this as Eigen solver.

So, you can always write a conventional form of equation of motion, try to find the determinant of k minus omega square m matrix, set it to 0 and find out omegas and phi's, that is a classical Eigen solver theory, where every program, every inbuilt software has this facility of finding out the roots of the characteristic equation, which we call as Eigen values and the corresponding vectors, which we call them as mode shapes.

Now, there is a very important property associated with these mode shapes, let us see what this is? How it is beneficial to us. So, now is there any question here? Any doubt here in this problem? So, I can extend the same principle of solving this for multiple degrees, three, four, etcetera, which we will demonstrate later. But I will also equivalently touch upon some of the approximation methods, which are very popular in the literature, to find out the mode shapes and frequencies because we are interested to find out the mode shapes. Can I have a question here asking to people, why we are interested in finding out the frequencies of the given system? Why we are interested to find out the frequencies of the vibrating model? What is our necessity? It is not to avoid, basically one is interested to know at what frequency band the model is vibrating. So, one will know, depending upon the excitation bandwidth of the frequency of the forcing function, will the model resonate or not.

So, one would like to know, remember advantageously, the fundamental frequency of a given system is the property of the form, it is not depending on a forcing function at all. You may not know at which location on the c state you have got to install this platform depending upon what is the rings modulus of the material, what is the sectional dimension and what is your mass of the member and what is your cross-sectional property, rectangular, circular, etcetera. Your fundamental frequencies are fixed in the beginning itself.

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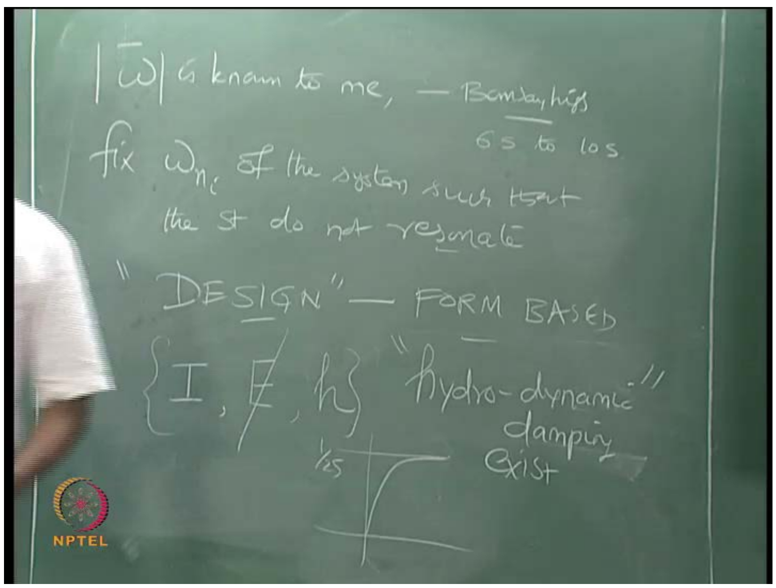

So, if you have got a jacket structure, if you have got a TLP, if you have got a span, if you know omegas of these in advance, on the other hand if you know the (( )) state bandwidth of frequencies where they are going to be placed, that is, the excitation frequency is known to me, if the excitation frequency omega bar is known to me for a given (( )) state, I can always fix omega NIs of the system such that the structure do not resonate. This is, what we call as, design because this is known to me. I am not saying omega bar, I am saying omega bar is a band, I know this band.

For example, if you look at Indian Ocean conditions, for example, let us say Bombay High, I want to design a platform in Bombay High. Now, the period of waves, approximately, vary from, let us say, 6 seconds to 10 seconds, so I know the period. Therefore, (( )) omega band of the forcing function of the wave alone, I pick up my omega n i, natural frequencies of my system, in such a way, that wave do not fall in the band of my operation. So, how do I do that? Why I call this as design? Because omega n depends on: the cross-sectional dimension of the member, which you are selecting; the Young's modulus of the material and of course, the span of the member.

So, you select in such a way, that the omega n i's of a given system does not fall in the bandwidth at all of your excitation frequency. This is how I initially said, that offshore structures are form based design, so there is no functional characteristic here. Whether I am going to do drilling, production, offloading, storage, I am not bothered. I am only bothered about, that this form should not fall in the band of my excitation frequency, that is why I said, it is formed based design. Form means, cross-sectional dimension, E and I and h. Of course, material is almost fixed; almost fixed.

People use steel, one can use composites, one can use concrete, etcetera, different kinds of material, but E may not have much freedom, but I and h have a tremendous freedom. That is why we said, it is a form based function, I mean, form based design, not function based design. So, design is essentially form based. Interestingly, we have also studied, that even though my natural frequency will fall in the bandwidth of excitation frequency, as long as hydrodynamic damping exists for offshore structures, the upper bound on my response will be limited within 1 by zeta, I will not go to infinity.

We already discussed this by derivation, that for any given value of zeta other than 0, the upper bound will stay at 1 by 2 zeta even at a resonance band where omega n matches omega bar. So, we have no problem because in our case, for ocean structures, hydrodynamic damping will always exist. That is why, in engineering design of ocean structures when you do dynamic analysis, we always go friendly with the viscous damping models because viscous damping is, what we talk about, hydrodynamic. Of course, there are friction damping also, coulomb damping also exists in the intersection of members, we do not consider that serious. We talk about only the viscous damping because we are talking about the hydrodynamic. Damping exercised on the members by water body surrounding the members, which will be always present for an offshore structure.

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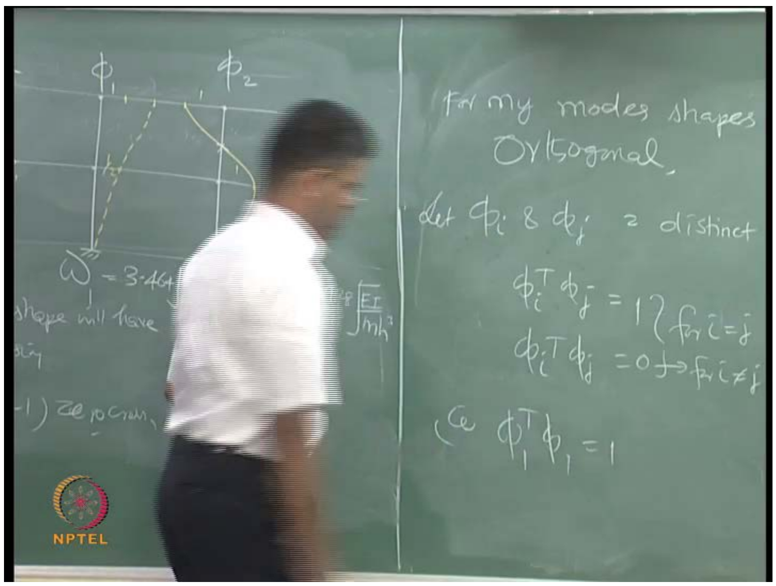

Now, interestingly, if my mode shapes remain orthogonal, they have got one more advantage. So, let us see what is orthogonality? This is a very interesting part of this lecture. Suppose if my, for my mode shapes remain orthogonal, I derive many advantages from this, I will come to that, first let us understand what is orthogonality? So, if you have got two mode shapes, phi i and phi j be two distinct mode shapes, by the way what are mode shapes? Mode shapes are relative portion of the mass displaced when the system is vibrating at a specific frequency.

For example, phi 1 will give me the relative portion of all the mass points in the given structural system when the system is vibrating at omega 1. Similarly, phi 2 will give me the displaced relative portion of all the mass points when the structure is vibrating at a frequency omega 2, and so on. So, if you have got two distinct mode shapes, phi i and phi j, if phi i transpose phi j is true, then the modes are orthogonal, that is, phi 1 transpose phi

1. And let us say, phi 1 transpose phi 2 should become 0, if this condition is satisfied, I can say, that phi 1 and phi 2 are orthogonal.

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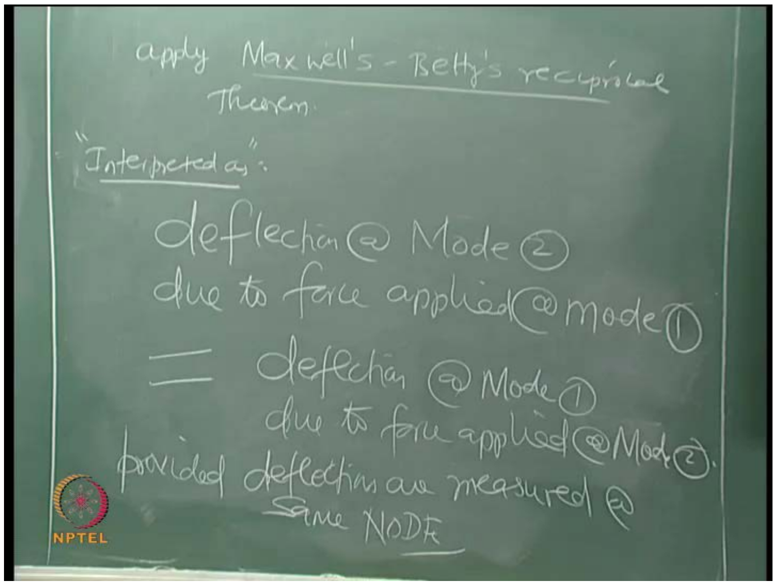

Now, one may ask a question, that why orthogonality is important? The moment this mode shapes becomes orthogonal, then the advantage starts from applying, you can apply the Maxwell’s-Betty’s reciprocal theorem directly to this problem if the mode shapes are orthogonal. Now, what is Maxwell’s-Betty’s reciprocal theorem on dynamics? Because this theorem is on statics, statistics, how this theorem can be applied to dynamics?

This theorem can be interpreted as, remember it is our interpretation, Maxwell’s- Betty has never given this theorem for dynamic application. This theorem can be interpreted as deflection at mode two due to force applied at mode one, will be same as deflection due to at mode one due to force applied at mode two provided deflections are measured at same node. So, these are all modes. This is a node, there is a difference, and thereare two points to be understood here, let us quickly see what are they?

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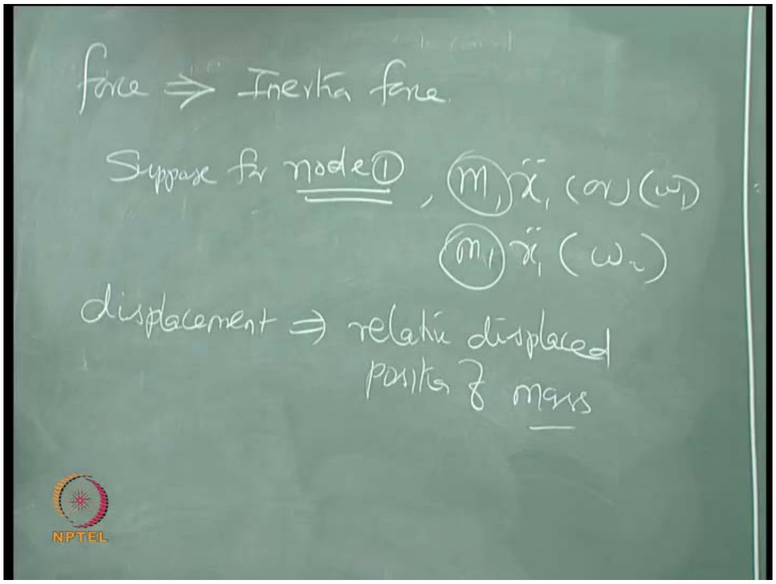

We are talking about some force, what is that force because we say, that deflection at mode two, we are talking about forces, the force what we refer here corresponds to inertia. Suppose, suppose if you take node 1, node 1, that is, node 1 if you take, node 1 you must apply, m 1 x 1 double dot or m 2 m 1 x 1 double dot, corresponding to either omega two or… You must apply m 1.

The displacement what we talk about, the displacement of the deflection what we talk about, corresponds to the relative displaced position of mass. So, what do you understand by this statement now? Look at this figure, I have got a two degree freedom system, which is vibrating at omega 1 and omega 2 displacing the mass positions this way, provided these two modes are orthogonal. We have not even discussed it yet provided these two modes are orthogonal.

Then, the force applied or the displacement measured at node 2, because of the force applied at node 1, will be as same as displacement measured at node 1 because of the force applied at the mode 2 provided you measure the displacement at the same node, may be at 1, may be at 2, may be at 3 and so on. It is because it is very easy for me.

For example, somebody asks me, what to be the displacement in the second mode shape, the displacement of mass m 1 in the second mode shape when it is vibrating at omega 1? What I should do is I must find out the displacement of the second mass when it is vibrating at this and that will be equal. I can keep on interpreting it provided these two mode shapes are orthogonal.

So, first we must understand whether to check they are orthogonal. If they are not orthogonal, how to orthogonalize them, what we call as normalization of the mode. I want to normalize these modes. The advantage is, once I have got normalized modes or weighted model matrix, I can straightaway apply the concept of this theorem on dynamics. This theorem does not configure to dynamics. We are interpreting this theorem like this, directly it can be applied, so the interpretation of the displacement, position of every mass point at any specific frequency can be easy. It means, if you have got multi-degree freedom system, which has got about ten frequencies and ten mode shapes, you need not have to find out all the ten mode shapes. So, find a few and interpret the remaining displaced position of the mass from the initial values itself. Is it not very easy, provided the mode shapes are orthogonal?

Why it has got to be orthogonal? This can be mathematically proved; I will show that mathematical derivation is there. If you may, if you make the modes orthogonal, then this proof can be applied directly. The advantage physically is, if you have got ten mode shapes and ten frequencies, I need not have to find all the ten frequency mode shapes. I can easily find the displaced position of the mass at nth mode caused by, is as going to be same as the mass at that first mode itself at the specific node.

I can easily interpret this, that is why, you see, generally in dynamic analysis people do not consider the necessity of higher modes with lower modes itself, you can do the analysis. So, we will talk about this later, when how this can show, that the mode shapes will remain orthogonal. We will talk about this, but remember very carefully, this is only an interpretation of the Betty’s theorem. Betty’s theorem does not give anything on dynamics, these are all modes, and this is node. Node means point, a mass point.

So, we have got only two minutes, we will not be able to do the orthogonality check, so I leave this as homework to you. You have got phi 1 and phi 2, please check whether they are orthogonal or not. I have given you the condition for orthogonality, you can check phi i transpose, phi I should be 1 if i is equal to j, otherwise it should be 0. Please check this and tell me whether these modes are orthogonal or not.

So, we have learnt two things from this class one for a two degree freedom system like this, how did we get the mass matrix, the stiffness matrix, why the mass matrix was diagonal, why the stiffness matrix was not diagonal, how to obtain a stiffness matrix, how to write an equation of motion for this and using a classical Eigen solver theory, how to find natural characteristics, that is, natural frequency and mode shape of the given system as we have seen. When we solved this, we never knew whether this is omega 1 or this was omega 1. We had then tried to find out the corresponding mode shapes, then we identified that for the first mode shape, there will be no zero-crossing if this is the first mode shape, the corresponding frequency is the first frequency.

Similarly, this is the second mode shape; the corresponding frequency is the second frequency. How can we say the second mode shape? Any number will have minus 1 of this value as a zero-crossing. If you have got a second mode shape, there is one zero-crossing, therefore this is considered to be the second mode shape. So, corresponding frequency is omega 2 and the corresponding frequency is omega 1. Then if you presume, that these modes are orthogonal provided this conditions are satisfied, which we discussed in the last few minutes, then I can interpret the Betty's reciprocal theorem in terms of my dynamics because this is going to give me an advantage of not working out all the mode shapes and frequencies. I can work few of them and interpret. I am saying interpret the remaining part based on the application of reciprocal theorem.

So, it solves or it condenses or it simplifies the dynamics of these multi-degree freedom system models, so check this. We will get back to you in the next class to show this, as well as to orthogonalize it, again we can normalize it. How to do this few minutes, we will discuss this, then we will talk about the multi-degree solutions for omega and phi. Now, the necessity, the foremost necessity in dynamic analysis is for a given structural system, derive the mass matrix, stiffness matrix and find out omega and phi, that is the first (( )). We must know the natural frequency and mode shapes of a given structural system to know, that you should know omega m, I mean, k and m. You must know how to derive k and m as we have demonstrated in the previous lectures. Any questions, any doubt here?

So, we leave it here and we are not going to have anyway the test on next Monday as I said 27th is the schedule, 25th is the schedule for slot A for quiz 1, we will not have it, I do not want to waste that class, so you can tell me whether you want to have it on 23rd, this Saturday or may be next Friday. Is it possible to have it on Friday at 4 o clock, Friday, that is, either on 22nd or on 1st of March? It can be either on 22nd, that is, this coming Friday or on next Friday, 1st of March, you can decide and tell me, you can consult and tell me. You can either have it on 22nd or 1st, but we are not going to have it on 25th, I do not want to miss the class. So, we will run the class, but we will have the exam and quiz on some other day. Whatever I discussed till the day before the quiz will be appearing in the paper, so it is up to you when you want to have test.

So, I am giving you good time. If you allow me to do two weeks, I will move on to even dynamics applications of ocean structures also. I will pick up some models, physical models of ocean structures and I will do dynamic analysis for that, which can also appear in the paper. I need another two more classes for multi-degree, I will do some examples, I will give you some (( )) also. You can solve them, if you still have difficulty, we can wait and again discuss it again, no problem. I do not want to do many number of problems here, I will do of course couple of problems for you to familiarize, you solve it yourself and then let me know if you have any difficulty, we will again resolve.