Dynamics of Ocean Structures

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Module - 1
Lecture - 18
Two Degrees-of-freedom Systems

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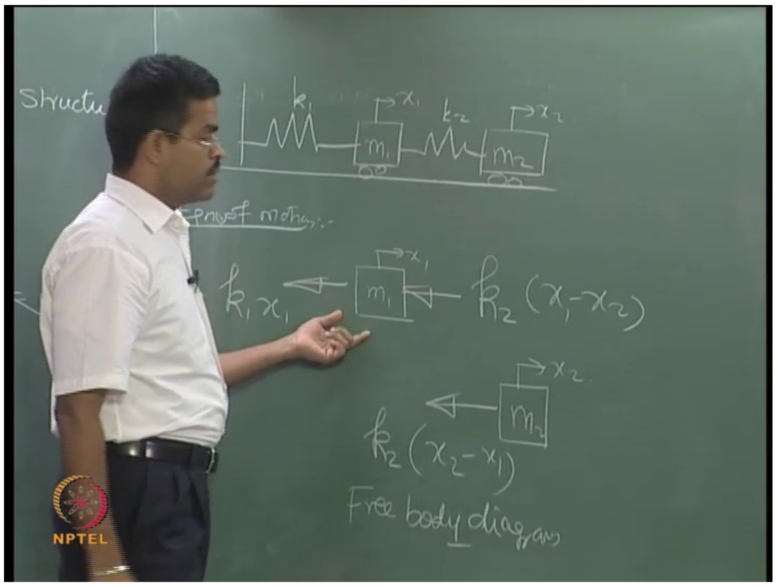

So, today we will discuss about some examples on two degree freedom system models. We will write down the equations of motion and try to solve them. So, first example, we will take up this problem. Let us say, I have two lumped mask points m 1 and m 2. So, from the CGF of m 1 and m 2, I measure the displacements and imply upon the restoring forces, call them as k 1 and k 2. I want to write the equation of motion; it should be equations of motion for the set problem. So, I will use Newton’s law to do this. I will also solve this problem or parallel problem using energy method.

So, I draw a free body diagram for this mass m 1; I give a displacement here. So, when the mass m 1 starts moving towards right, this spring will restore it back and that will be nothing but k 1 of x 1, because constant of a spring is essentially, force or displacements. I multiply the displacements of the stiffness, I get the force that what I am doing here, when I try to move the mass towards right, this spring will oppose and let me mark that force in the opposite direction and this force will be k 2 of simply say x 1 minus x 2.

It is an easy way of remembering. When the mass moves towards right, this spring will push the mass m 1 back, so I put the arrowed direction, take this stiffness and this is connected between two relative displacements. Take this as first and this as second.
If you look at the second mass m 2, the second mass, again moving towards right, which is x 2 and this spring of stiffness will try to pull it back. I simply do the same algorithm. I say k 2 and I am picking up this displacement first and then putting the next relative displacement x. So, these are my free body diagrams based on which now I will write the equations of motion using Newton's law.

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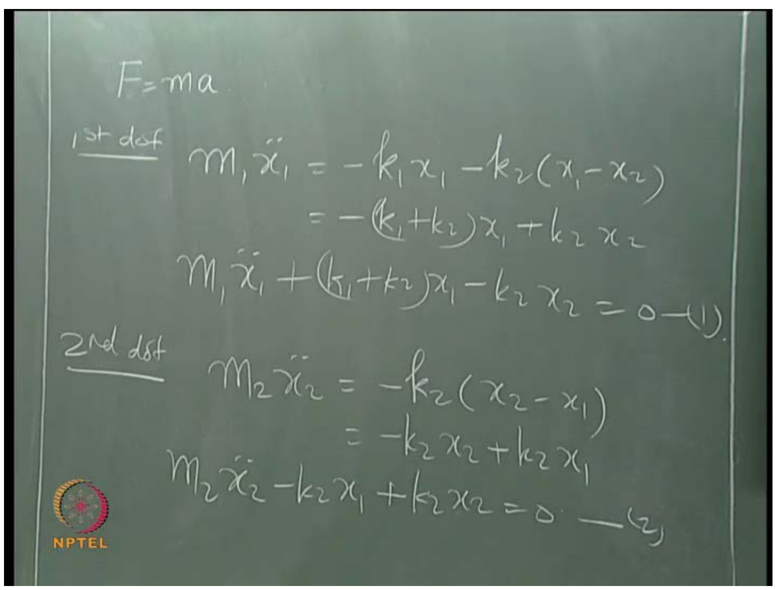

So, Newton's law states, that force is equal to mass into acceleration. So, let us pick up the first case. So, I will say, first degree of freedom or first degree of displacement, which is x 1, so which is going to be m 1 x 1 double dot, that is, the force, which is mass into acceleration of this and it is moving towards right. Therefore, I must say, this is equal to minus of k 1 of x 1 minus of k 2 of x 1 minus x 2. I pick up the x 1 coordinate separately, I say, minus of k 1 plus k 2 of x 1 plus k 2 of x 2. Bring it to the left, so m 1 x 1 double dot plus k 1 plus k 2 of k 1 minus k 2 of x 2 is 0, this is my first equation of motion.

Look at the second degree. The mass is moving towards right, so again, force is mass displace acceleration. So, m 2 x 2 double dot and this is restoring to the opposite direction, so minus k 2 of x 2 minus x 1. So, I can say, this is minus k 2 x 2 plus k 2 x 1. So, m 2 x 2 double dot minus k 2 x 1 plus k 2 x 2 is 0. This is my second equation, this is my first equation.

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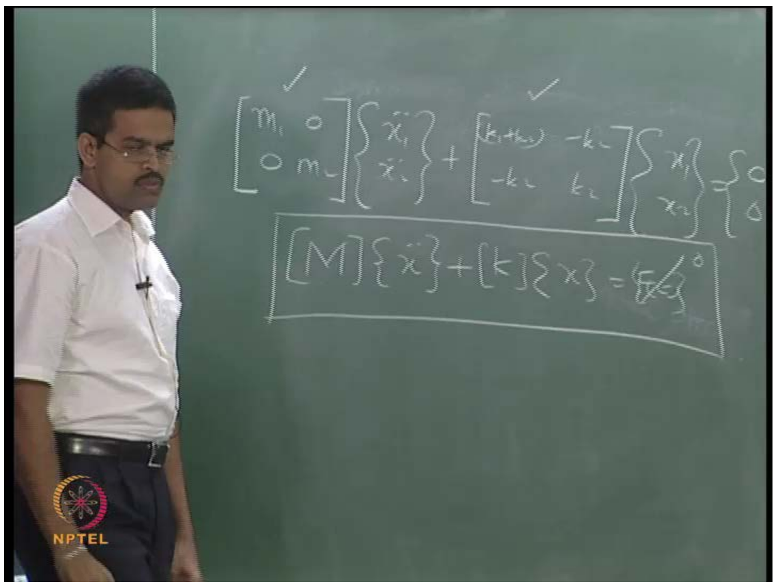

Let me put this equation in a matrix form. There are two points, where the mass is (( )), one is at x 1, one is at x 2. Therefore, the mass matrix will be a diagonal matrix. I will come to this point little later of x 1 double dot and x 2 double dot, I am just converting this in a matrix form, plus stiffness matrix of displacement equals the force value. In my case, it is 0, so with x 1 the coefficient are k 1 plus k 2 minus k 2 minus k 2 k 2.

So, this becomes my equations of motion. In a matrix form I can also write this as M x double dot plus k x as f of t, which in my case is 0. So, this is my equation of motion written in matrix form where mass matrix M is given by this, stiffness matrix k is given by this. So, we can draw very interesting information here. These are very, very important for writing the equations of motion in futuristic problems. Let us see what are those interesting information I can derive from these equations of motion.

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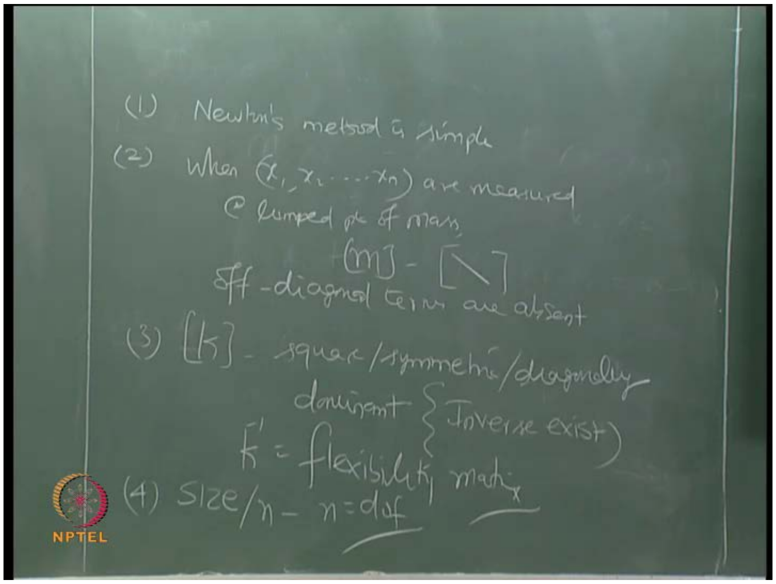

The first information I derive is writing equation of motion. For a two degree or multi-degree, using Newton's method or Newton's law of motion is simple. I can also use energy method, I will show that later. The second inference I derive is when the displacement coordinates are measured at lumped points of mass, mass matrix becomes diagonal. It means off-diagonal terms are absent.

Three, usually stiffness matrix is a square symmetric diagonally dominant matrix. What does it mean is, it is a square matrix symmetric. Diagonal dominant means, any coefficient of this matrix laying along the leading diagonal will be larger than any other element in that row. So, diagonal dominant, it shows me, that inverse of this matrix exist. So, inverse of k is, what we call, flexibility matrix. So, inverse of this matrix exist.

And of course, fourth and very simply observation is, that the size of this matrix will be of order n where n is the degree of (( )). So, two by two, therefore it is… This information is very important. You will see that this is very important information, which I will use later.

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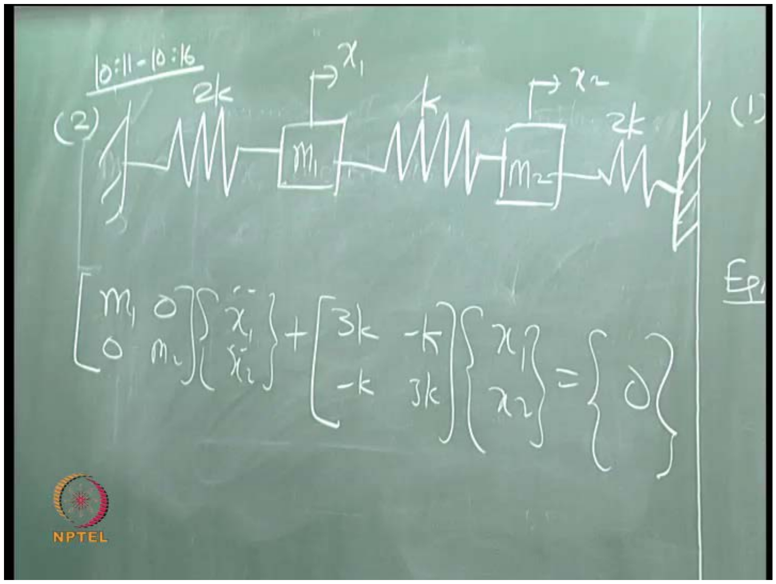

So, I want you to quickly write the equations of motion for the second problem, which I am giving here. I will give you five minutes time. So, m 1, m 2, x 1, x 2, 2k, k 2, so the equation of motion is given. This is how it is obtained. Let us do one more problem. I want you to write the equations of motion for this problem.

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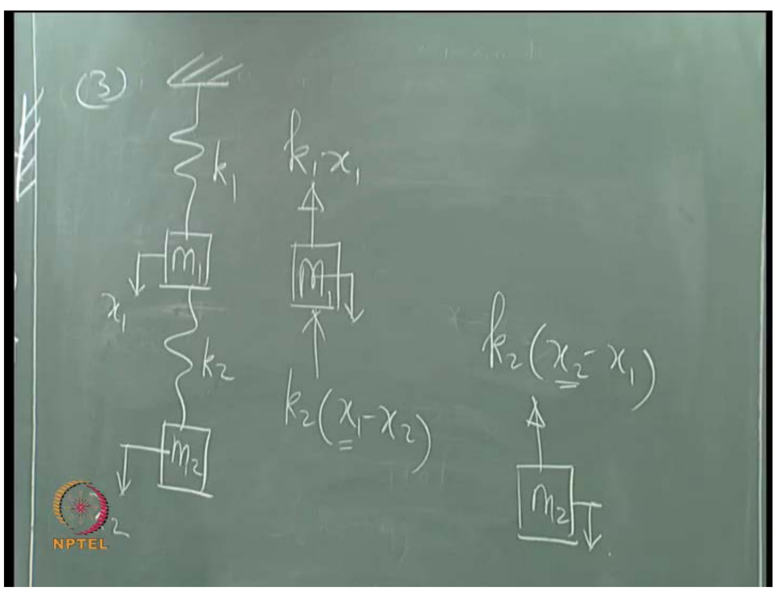

I can draw a free body diagram. I am helping you, if this mass moves down this spring will try to restore it back. So, this will be k 1 of x 1, whereas this spring will try to push it up. So, it is going to be k 2 of x 1 minus x 2, that is, for the mass m 1 and so on.
Same way you can do further, if you do it for the mass m 2, move it down, the mass will move down this spring will try to push it up, those, that is going to be k 2 of x 2 minus x 1.

Always pick up the first value as the place where you are applying the inertia force when you have got relative displacement. I have picked up the first value as the place where the inertia force is applied, and then there is a relative displacement. When there is non-relative displacement, simply use that value. That is all; it is a very simple algorithm.
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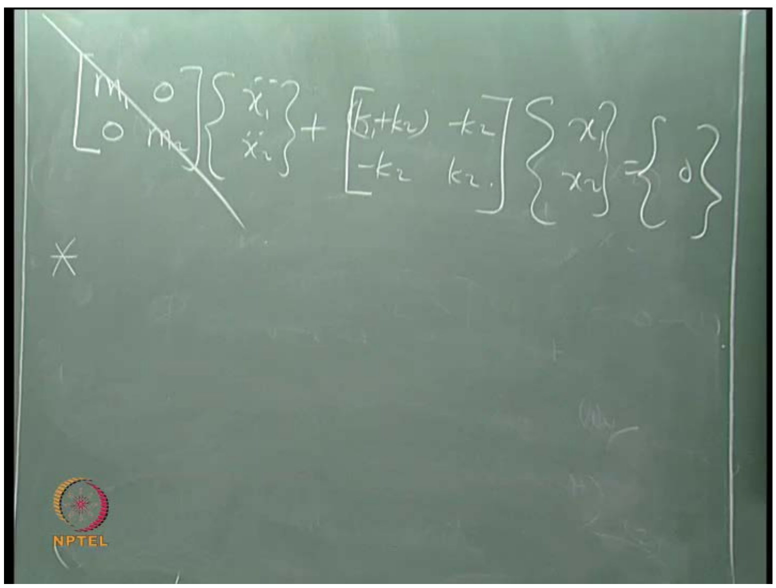

So, I will get the equation of motion as… So, interestingly one can notice that when the mass is lumped at the same point where the displacement degrees of freedom are measured, the mass matrix will be diagonal. We will see an example, now if I violate this, what happens. Before that let us do one more problem of writing equation of motion using energy method.

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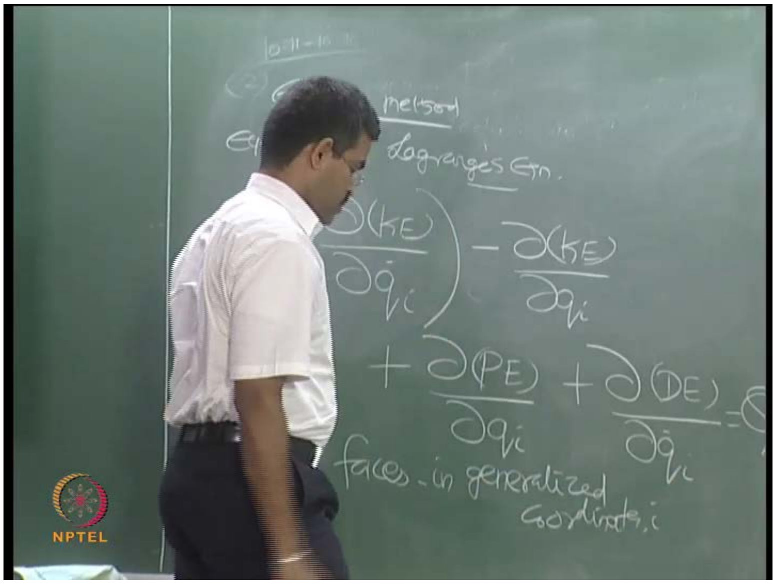

So, I will use energy method now. So, write equation of motion. For multi-degree, equation of motion in energy method is given by Lagrange's equation. Now, what is Lagrange's equation? The Lagrange's equation says, d by dt of partial derivative of kinetic energy with respect to q i dot minus partial derivative of kinetic energy with q i plus partial derivative of potential energy with q i plus partial derivate of dissipation energy with q i dot, should be given by equal to q i, where capital Q i is the forces in generalized coordinates of I.

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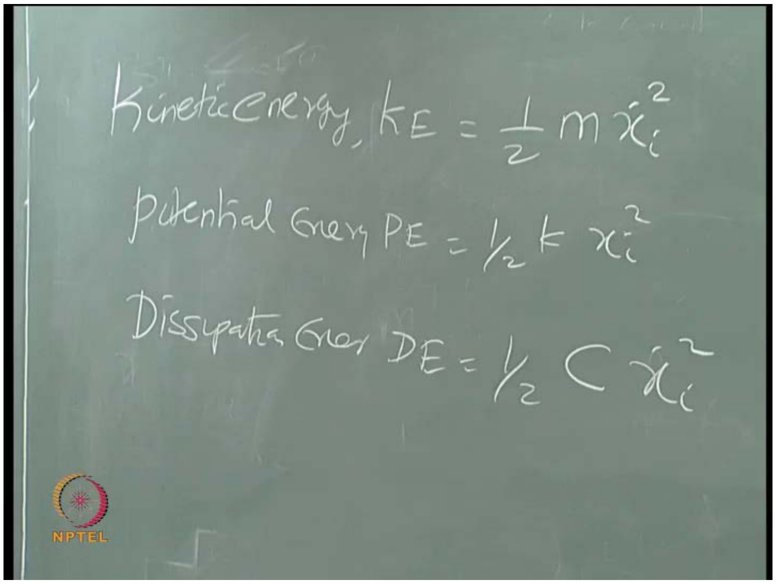

Whereas, kinetic energy, potential energy and dissipation energy are given by the following equations: kinetic energy, half m x i square; potential energy, half k x i square; dissipation energy, half c x i dot square. So, use these expressions now and let us try to write the equations of motion for a two degree freedom system, as I am going to show you now.

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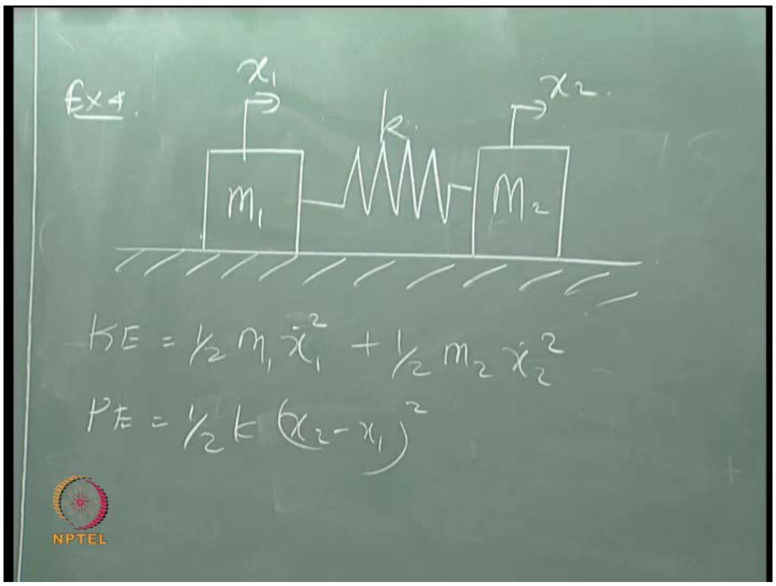

So, we will neglect the frictional forces between the mass and the contact points. So, kinetic energy, half m 1 x 1 dot square, that is, half m 2 x 2 dot square; potential energy, half k relative displacement. So, we will use Lagrange's equation to write down the equations of motion based on energy method. The Lagrange equation says, first find the partial derivate of the kinetic energy with respect to each one of the degree of freedom.

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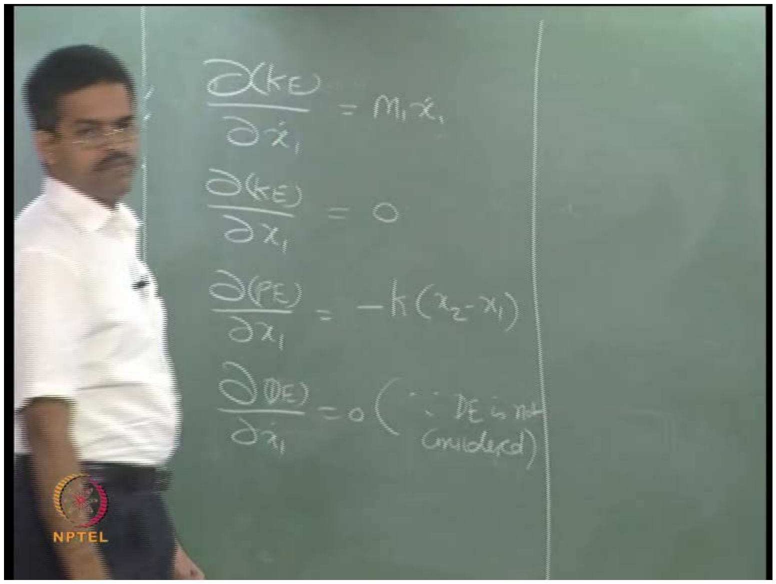

So, let us try to do that. First, potential kinetic energy with respect to… Instead of q i, I am using my coordinate generalized as x i. So, dou x 1 dot, that is what I am going to do here, partial derivative, because kinetic energy is a function of x 1 and x 2, both. I take a partial derivate of this. This will become simply x 1 dot. Similarly, dou kinetic energy of dou k 1, which is the second term here, obviously it is going to be 0. The third term, potential energy by dou x 1, that is the third term here. So, this is going to be minus of k of x 2 minus x 1, is that ok. Partial derivate of dissipation energy with respect to k 1 dot is 0 because dissipation energy is not considered. I am not considering any frictional force between the mass and the plane of motion.

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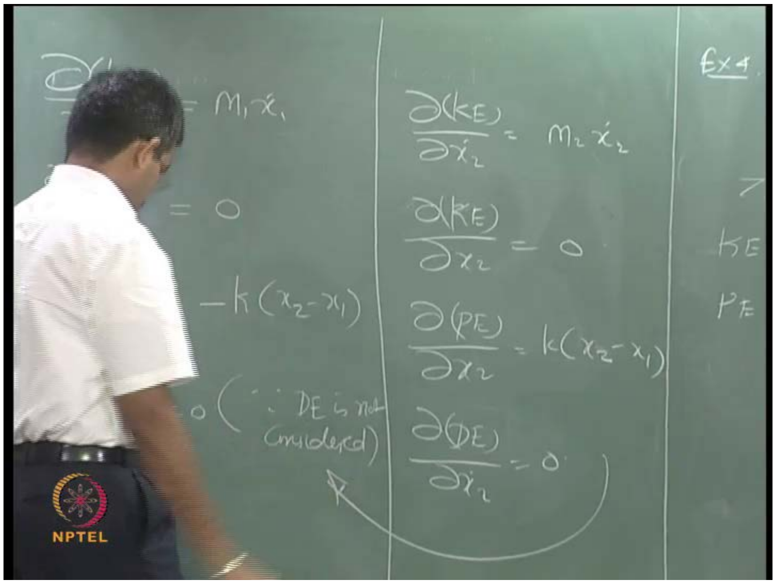

Similarly, I can do for the second case. So, dou kinetic energy by dou x 2 dot is m 2 x 2 dot dou kinetic energy by dou x 2 0 dou potential energy by dou x 2 is k of x 2 minus x 1 dou dissipation energy by dou x 2 dot is set to 0, because of the same reason.

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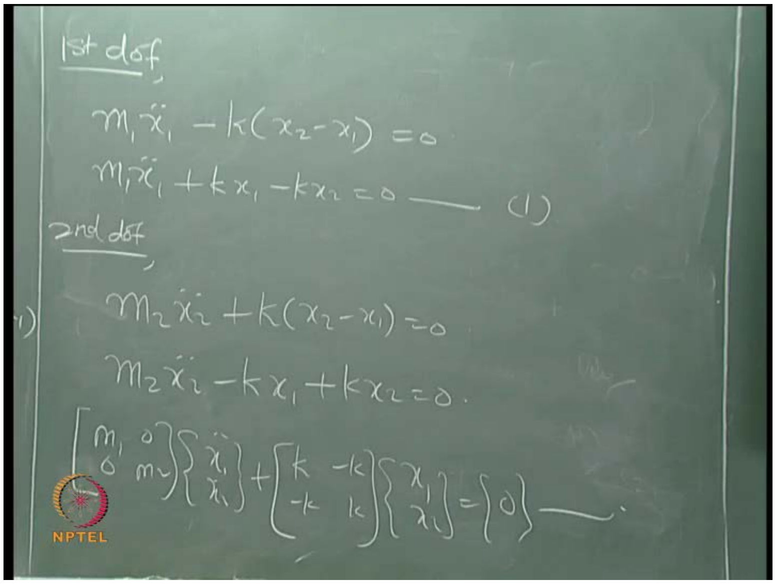

So, for first degree of freedom, I may say, d by d t of this equation where all i will become 1. So, I must say, d by d t of this, which will give me m 1 x 1 double dot. I am differentiating this with respect to time minus k of x 2 minus x 1 will be 0 because there is no q i acting in the model. So, I should say, m 1 x 1 double dot plus k of x 1 minus k of x 2 is 0 will be my first equation of motion for this problem.

For the second degree, again d by d t of this expression with respect to the second degree of freedom, which I have here, which will be m 2 x 2 double dot plus k of x 2 minus x 1 is 0. I am putting 0 here because there is no q 2 term in the given model. So, I should say, m 2 x 2 double dot minus k of x 1 plus k of x 2 is 0. I can convert this in a matrix form as m 1 0 0 m 2 of x 1 double dot x 2 double dot plus k minus k minus k k of x 1 x 2 as 0, which is my equations of motion in a matrix form.

You will again notice, that mass matrix becomes diagonal, stiffness matrix becomes diagonally dominant and symmetric and square because I am applying the mass lump at the same point where I am measuring the displacement. Let us quickly check whether I do the same problem with Newton's law and get the same equation of motion.

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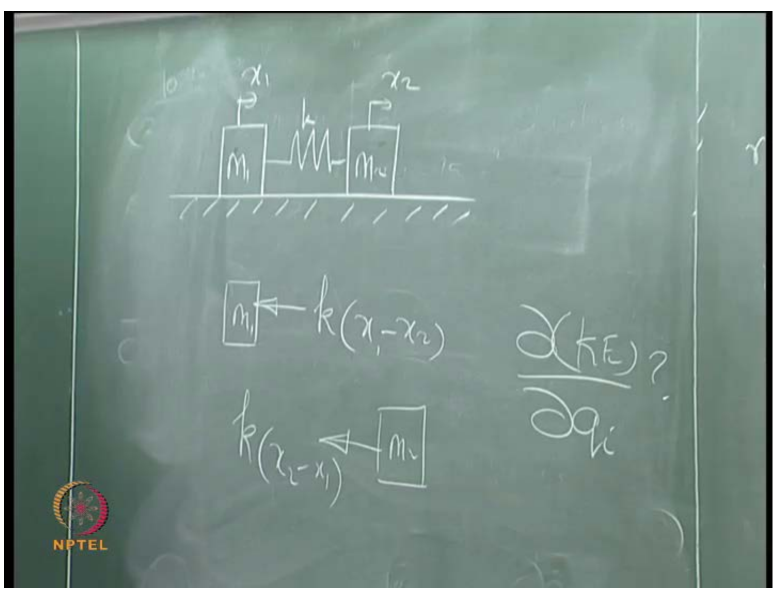

So, if I take mass m 1 separately and draw a free body diagram, the mass moves to the right and this spring will push the mass back, which will be x 1 minus x 2, as I said already. Take the component of the stiffness here, use this coordinate first, when it is relatively connected pick up the second mass m 2, the mass moves to the right, the spring will bring it back. So, it is again, k of x 2 minus x 1, it is very simple algorithm to remember. So, draw the free body diagrams and then write down the equations of motion based on the Newton's law, F is equal to ma and check, whether you are getting the same equations as you already have.

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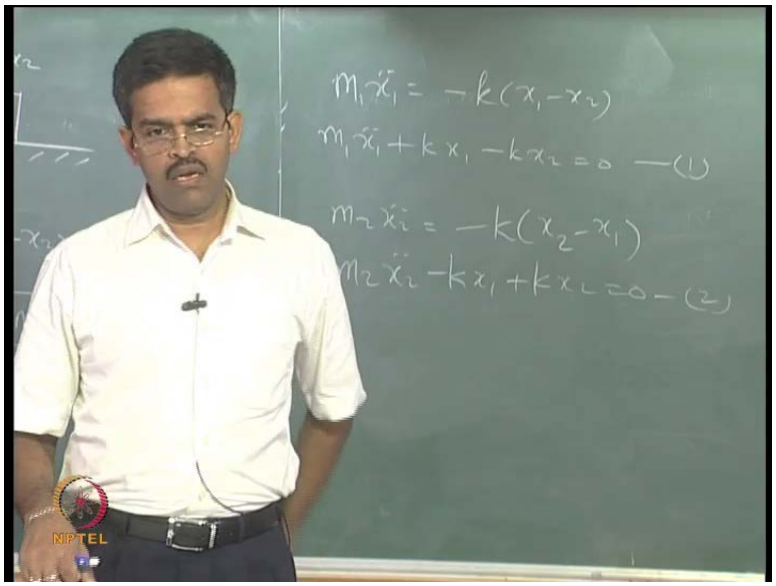

So, m 1 x 1 double dot is going to be minus of k of x 1 minus x 2, which will give me m 1 x 1 double dot plus k x 1 minus k x 2 as 0. This is my first equation as same as what I am having here. Similarly, m 2 x 2 double dot is going to be minus of k of x 2 minus x 1, which will give me m 2 x 2 double dot minus k x 1 plus k x 2 is 0, which is as same as the same equation, which I got here.

So, you can adopt any method for writing equations of motion. They have demonstrated it using a Lagrange's equation, which is energy principal as well as Newton's method, which you already know, but generally, the equations of motion are comfortably written using Newton's method because they deal with forces, whereas Lagrange's deals with the energy principles when you want to minimize or optimize or talk about response control in terms of energy mechanism. Then one can write equation of motion using energy methods, otherwise Newton's law, which is adopting the force technique, will be simpler. Remember, very interestingly, here it is not depending upon in which way you want to solve the problem and these are all time variant, x 1 double dot, they are all time variants and so on, any questions here?

So, one can solve or one can write equations of motion in any preferable method except the few, which we discussed in the first chapters on single degree of freedom system models. Now, throughout these examples we understood, that when the mass matrix or the mass is lumped at the point where displacement is measured, mass matrix becomes diagonal and diagonally dominant and off diagonal terms are generally 0. But is there case or are there any cases where we do not have the lump mass at the point where the displacement is measured?

We will take an example exclusively on that form and see how to derive mass matrix for that situations because it is very important in our offshore structural systems. It is not necessary, that the force can be applied at the same point where the mass is concentrated or displacements need not be measured in the same point where the mass is (( )). I gave you a very simple example. I have got a floating, let us say, FPSO. The CG of the floating FPSO are somewhere located above the mean sea level, let us say, but I do not want to measure the displacements at the CG of the FPSO. I want to measure the displacement at the deck, it means, I am not interested in measuring the degrees of freedom or marking the degrees of freedom at the point where the mass is lumped.

So, what will happen? How do I compute my mass? Not necessarily you should always want to know the displacements in the place where the mass is lumped and there may be instances where the force may be not applied at the same point where the mass is said to be lumped. For example, aerodynamic forces, they have got the derricks, they have got, let us say, cranes where wind force are acting on the top side of the structural system in offshore, whereas CG of the system is not at the point where the force is acting. Therefore, you will have to measure the displacement of the point where the force is acting because you want to know the maximum displacement but your mass is lumped at different point.

What will happen in such situations? How do I write my equation of motion because I do not know how to derive my mass matrix? In this case it is simple because mass matrix becomes diagonal. There are cases where stiffness matrix also will become unsymmetrical, let us see. There are special cases, as we come, as we move on, in general stiffness matrices are generally square and symmetric. They may still remain square, but they may not remain symmetric. There can be access symmetric terms, which will influence the response behavior of the platform in certain cases, we will see.

I have a very interesting question for you let me see who can answer this. In Lagrange's expression I have one term, which is dou kinetic energy by dou q i, the kinetic energy is always a function of velocity. Why this term exist in Lagrange's equation? Kinetic energy is always function of velocity, but why I have a term here? Think it over; any doubt? We will move to another problem where we will violate this standard and see what happens.

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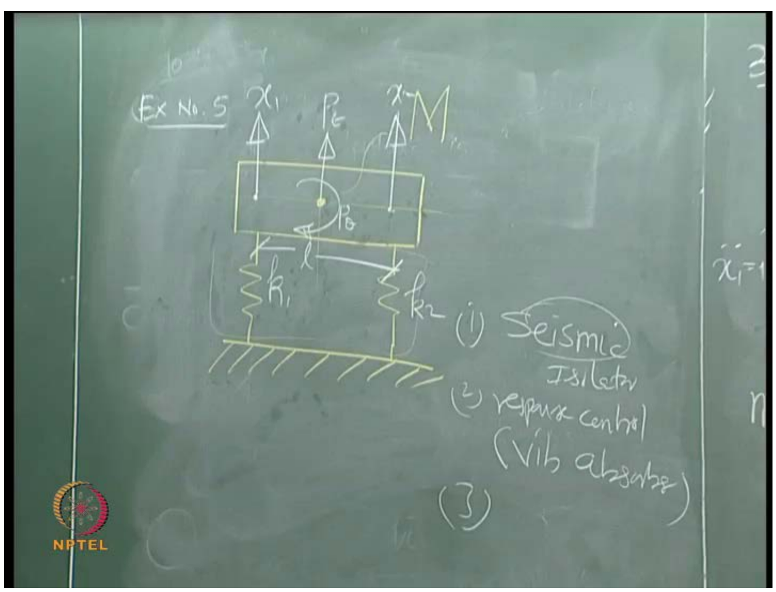

Take out a problem like this, example number, is it 5 or 6? See that will never settle, it is a democratic country, I think we should take OT. I have a deck, which rests on a system, which is supported by springs. It is a very typical model, there can be practical applications of this model, I will come to that slightly later. Let us say, this is my mass center, the total mass of the system is capital M, total mass. So, I have forces applied here as p t and p theta. These are the forces, which are applied on the system. Obviously, these forces are expected to be applied at the mass center, whereas I am crazy to measure the displacements at these locations where the spring is connected. So, I am violating the system saying, I do not want to measure my displacements at the point where the mass is lumped. What will happen to my equations of motion? This is my k 1 and k 2, I am deliberately varying them so that we see the derivations clearly. So, to find the stiffness matrix, first it is an axial stiffness, to find the stiffness matrix first.

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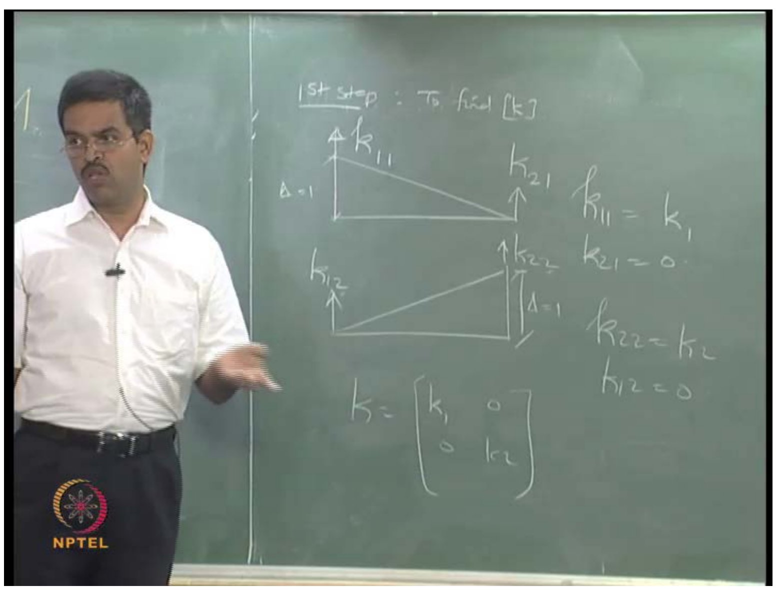

So, first step is to find stiffness matrix. What will be the size of the matrix? Whether there are two degrees of freedom? Not because they are two sets of forces acting, two degrees of freedom. It also improvise a very important conclusion, degrees of freedom are not the points where the mass is lumped. Here I have got only one point where the mass is lumped; there are two degrees of freedom. I think we have cleared it very carefully in the previous definitions. So, you can always have a shortcut saying mass points are said to be the degrees of freedom, but it is not true always. For example, in this case, so I want to derive the stiffness matrix for this, so I give as usual, unit displacement at this end I put delta as 1. I do not give any displacement here; it is going to be a liner line because it is actually a hinge.

So, the force, what I will get, which is responsible for causing this unit displacement is my stiffness, which I call as k 11. Force in the first degree because of displacement given in the first degree itself and obviously, this will be equal to k. Force in the second degree, because of displacement given in the first degree, remember stiffness matrix is always derived column-wise. The second number will be always same, so always column-wise, flexibility matrix is always row-wise.

Obviously, from this figure one can easily understand, that k 1 is directly equal to k because that is the stiffness, because stiffness by definition is force by unit displacement. I am giving in displacement, therefore this value will be directly equal to my k 1 and what will be my k 21? No displacement, no force, is it not? Similarly, what will be this coefficient called? k… Always the second coefficient of the second number will remain same because column-wise, so k 22 will be equal to k 2 and k 12, 0. k matrix now is going to be k 1, 0, 0, k 2.

Now, interestingly, you see what happened to my k matrix? It has become diagonal technically and dynamics literature people call this as coupling and uncoupling of stiffness and mass matrices, we will come to that later. Application parts terminologies, we will come slighter later, but what you can interesting see here, stiffness matrices become diagonal, whereas mass matrix will not become diagonal. If they become diagonal what will happen? The system will become… I will remove this, think it over. If stiffness and mass both becomes diagonal what happens? See, if these questions can be answered extempore, I think you can directly take the third module of these classes.

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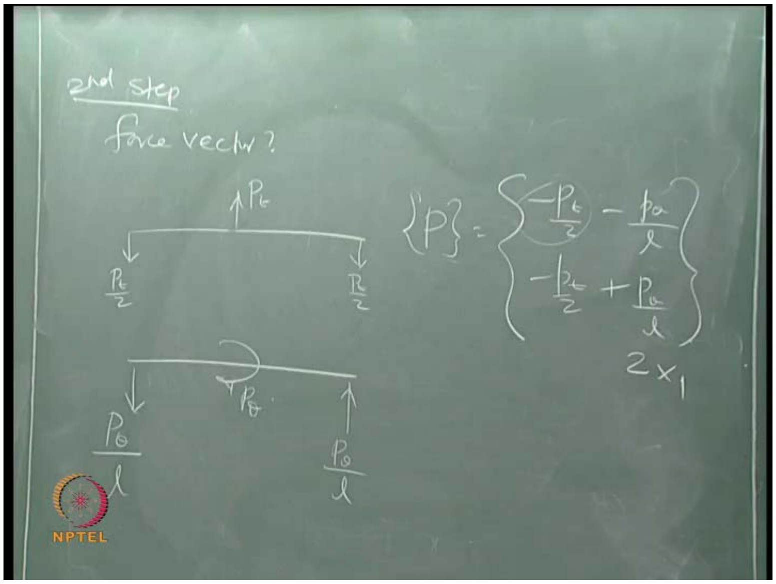

So, I want to now derive the force vector. Now, one may wonder, so why I am deriving the force vector. Remember very carefully, here I am applying forces at the point where the displacements are not measured, so I want to find what is the equivalent force of these components at x 1 and x 2. In the previous examples we always assumed, that either the force were 0 or the force was expected to apply at the point where the mass are concentrated and fortunately, the points where the mass were lumped, masses were lumped, we had also the degrees of freedom measure, so we had no confusions at all in the previous examples.

But here we have got second conclusion, that the force are not applied at the point where displacements are measured. I will work out another problem where displacements are measured at this point, but the force are applied at this point, we will talk about that, that is also possible. You must visualize a physical importance of these problems, they may look like a simple application, I will come to that. Once we finish this I want to find the force vector, what would be the basis to find out this? I pick up this deck and apply let us say p t and the component of this will be because of symmetry p t by 2 and p t by 2 and I apply p theta, which is clockwise, so there is a clockwise moment. I would like to make an anticlockwise couple to counteract this that is nothing but p theta by l where l is the distance between these two points.

So, now I have a vector p, the force vector, which is nothing but minus p t by 2 minus p theta by l. Similarly, the second one, if you look at, is going to be minus p t by 2 plus p theta by l. It is negative because the force, what we have here, is opposite to that of my direction of displacement and so on. So, I have got my force vector, which is a vector of two rows and one column. The third step will be to derive my mass matrix. Any doubt here? I will remove this.

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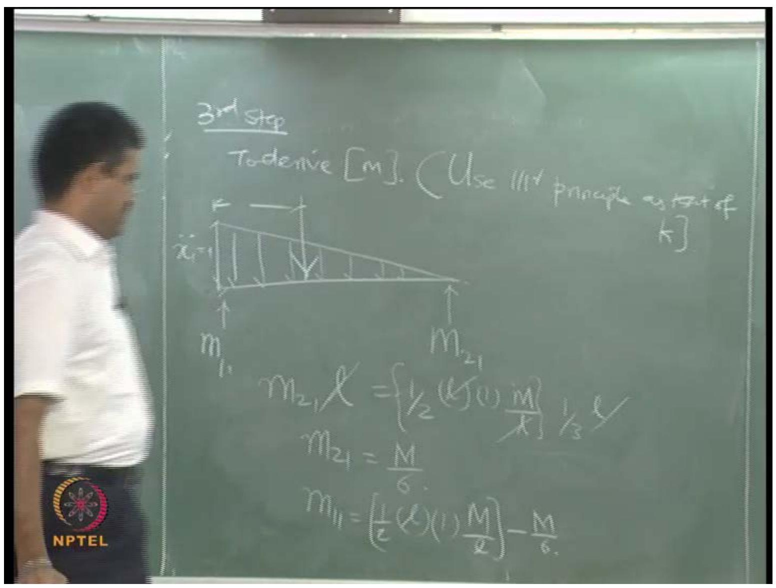

Mass matrix, I have used the similar principle as that of k, use similar principle as that of k. I am taking very great care to write properly, but my handwriting is extraordinary. So, you have got to follow that carefully. It is one of the non-identifiable characteristic of my this thing, so you have got to follow it carefully, so keep all your organs on. So, I will use this, I will give unit acceleration, I call this as, let us say, let us say, x 1 double dot as unity and set this to 0. So, when you do that it will create mass and acceleration and this will be opposed by two reactions, I will call this as mass value 2 because applied at 1, I call this as mass value 1 applied at 1. Same concept I am using for stiffness, similar one, so let me take moment about this point, so this is going to act down, take moment about this point. So, M 21 of l will be half l height of M by l. I am finding out M per unit length.

Now, M is the total value, is it not? And this is the force, one third of l, that is, the CG of this from here, so this will give me M 21. So, l, l goes away, l, l goes away, is it M by 6? So, M 11 will be the total area of this triangle or the total force or the total inertia of force minus the reaction force of M 21, which will be half l height M by l minus M by 6, which will be M by 3, is that ok.

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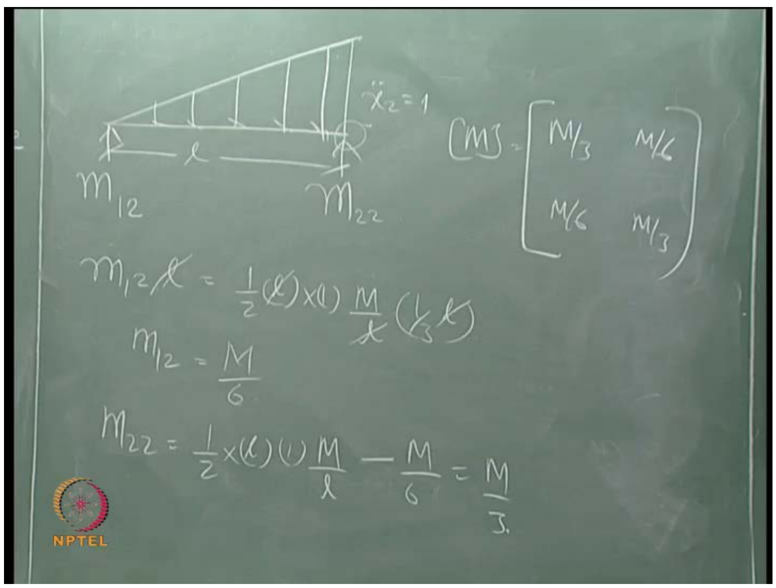

Now, I want to find the second row, sorry, second column of the mass matrix. Same algorithm apply here for x 2 double dot as unity and apply these forces and find these reactions at the distance l and so on. I call this value as M 22, I call this value as M 12, so can you quickly find out these two values? So, taking moment about this point, so M 12 of l will be equal to anticlockwise of half base height M by l of one-third of l, this gives me M 12. l, l goes away, l goes away, M by 6 and M 22 will be half base height of M by l minus M by 6, which will be M by 3. So, my mass matrix now becomes M by 3, M by 6, M by 6, M by 3, which is not a diagonal matrix. So, by this way we can keep on working out all probabilities of putting the forces elsewhere, measure the displacements elsewhere and keep on doing it.

Now, what is the practical significance of this problem? There are many applications of this particular example. Number one, this can be a problem of a seismic isolation where I raise my whole mass on a system supported by springs, so it is a seismic isolator. You may say why seismic, it can be even response control mechanism? I can give a very simple example of this practically. You must have seen in olden days people were using typewriters. Typewriters are normally not kept on the desk, directly on a table because what will happen, when you keep on using the typewriters, because of the vibration when you are using the typewriter, the legs of the table will get shake, so they put a pad and keep the typewriter over that. That is a vibration absorber. Similarly, here the response control, I can say vibration absorber.

The third is an interesting geotechnical model where the super structure resting on the foundation on (( )) springs, where (( )) springs are represented as equivalent series of springs like this. So, there are many practical applications of a land diagram, what you see here, whereas forces are all applied to the mass center. The displacements need not be there at the mass center. This also proves very interesting in the one thing to us, that the degree of freedom is nothing related to the mass point. Here it is 2, whereas the mass is said to be constant only at 1 point.

So, all confusions related to degree of freedom are point where the mass is concentrated are all gone. We already defined it very carefully and very clearly, that it is nothing related to mass concentration lumped mass points, but crudely one can remember like this, that wherever the mass are lumped, these are the number of degrees of freedom, but not always you see here. So, be very careful in marking them and always visualize a problem where are you measuring your displacements where the mass is concentrated, where are the forces applied. So, always equation of motion cannot be simply in the same form what we saw in the previous examples. It can slightly vary, so be very careful in marking these elements of or the dynamic characteristics of a problem. So, we have any questions here? I think we will stop here.