**Wave Hydro Dynamics**

**Prof. Dr.V.Sundar**

**Department of Ocean Engineering**

**Indian Institute of Technology, Madras**

**Module No.# 02**

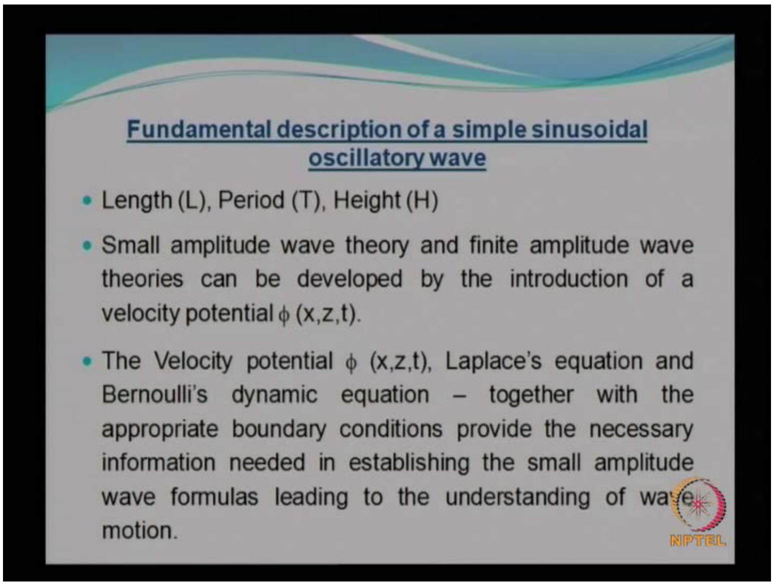
**Wave Motion and Linear Wave Theory**

**Lecture No. # 02**

**Wave Motion I**

So in today's lecture, we will get started to understand the fundamental description of a ocean wave. I mean a progressive ocean wave which is assumed to be a sinusoidal variation. So, as we have seen in the last module two characteristics, two parameters are needed to describe the motion of the free surface which is the wave. The two parameters are wavelength or wave period and wave height. The wavelength and wave period are interrelated, if you can get the wave pe[riod]-, if you are given the wave period you can definitely get the wavelength or vice-versa. So, wave height is a amplitude height of the wave and wavelength is the distance between the any two successive crests.

(Refer Slide Time: 01:41)

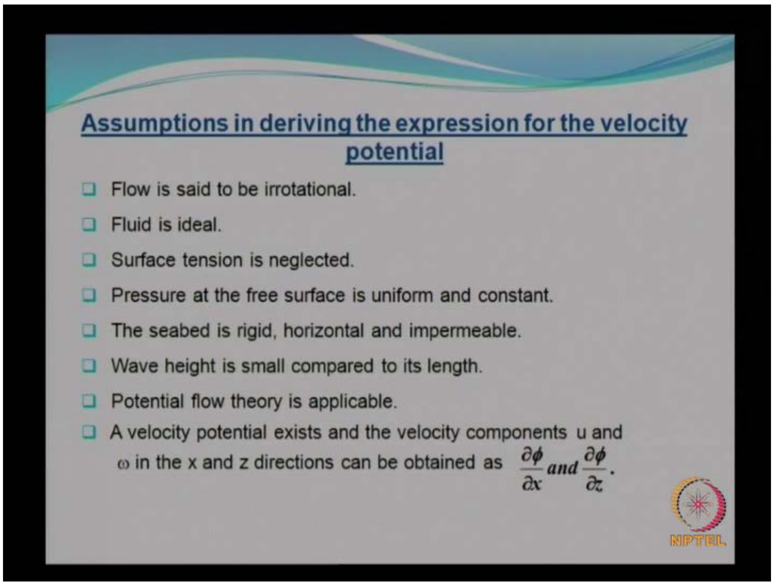



So, here in we have either the small amplitude wave theory or the finite amplitude wave theories to describe the wave motion. To begin with we will try to understand the motion of a sinusoidal wave using the linear theory or the airy's theory please recollect when I say linear theory we assume that the wave height is small compared to the wavelength. We are talking about a wave in motion. So, naturally it has to have some kind of a potential and that velo[city]- and that potential is referred to as velocity potential. You know the definition of velocity potential, if you have the velocity potential you can differentiate to get the velocities in a particular direction of your interest.

You can get displacements, you can get pressures etcetera. So, basic thing is the velocity potential which comes from our fluid mechanics. So, we need to establish a relationship for the velocity potential when a wave is in motion. The velocity potential can be obtained from a governing equation and in this case the governing equation is nothing, but the Laplace’s equation, del square phi is equal to 0. So, when you want to solve for the velocity potential from the governing Laplace’s equation it has to naturally be subjected to certain boundary conditions, and here in we have few boundary conditions which we apply in order to derive the velocity potential.

So, the Bernoulli the laplace’s equation and the bernoulli’s dynamic equation together with proper appropriateah boundary conditions provide the necessary information to arrive at all the wave all the formulas that are needed, for describing the phenomena of ocean waves. Before we get to the derivation part derivation here in we will be looking at deriving the velocity potential and for a progressive wave.

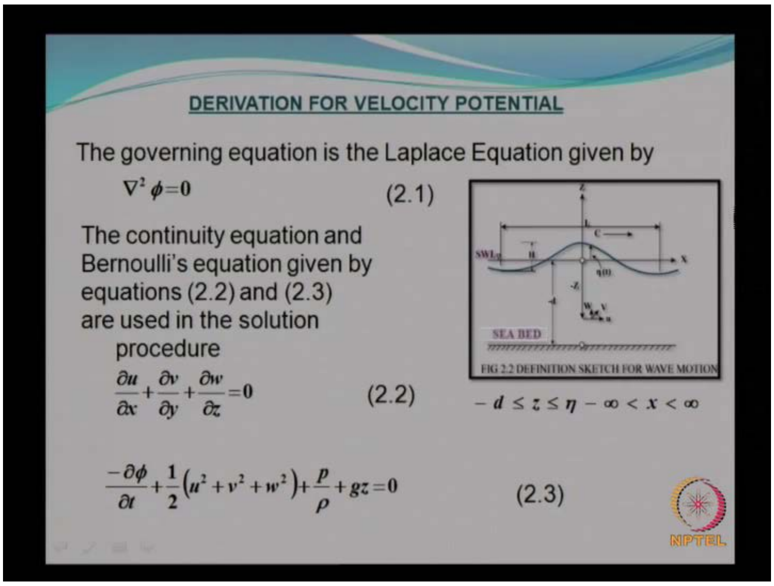
(Refer Slide Time: 04:49)



What are the basic assumptions? Fluid flow is said to be irrotational, fluid is ideal, surface tension is neglected, pressure on the free surface is uniform and is assumed to be a constant, the seabed is rigid, horizontal and impermeable. Mind you that we are dealing with the most simplest case in order to understand the fundamentals. So, you have to digest when I say the seabed is rigid, horizontal and impermeable. The sea bed is not always impermeable , and it is not always horizontal, but we are considering the simplest condition.

Based on these assumptions we will try to derive the velocity potential for the simplest case wave height is small compared to the wavelength and potential theory potential flow theory is applicable. So, when you have the velocity potential differentiation with respect to a particular direction gives the velocity in that particular direction given as dou phi by dou x , and dou phi by dou z.

(Refer Slide Time: 06:40)




Look at the definition sketch, in order to derive the velocity potential. So, this is the rigid horizontal seabed, this is the still water line over which you have the oscillation of the free water surface, this is the crest the difference distance between this is called as the wave height propagating in a water depth d , and we take the coordinate axes at this location, and you see that the coordinate axis this is the direction of wave propagation and this is vertical. So, any point in the fluid medium below the still water is referred to as minus z.

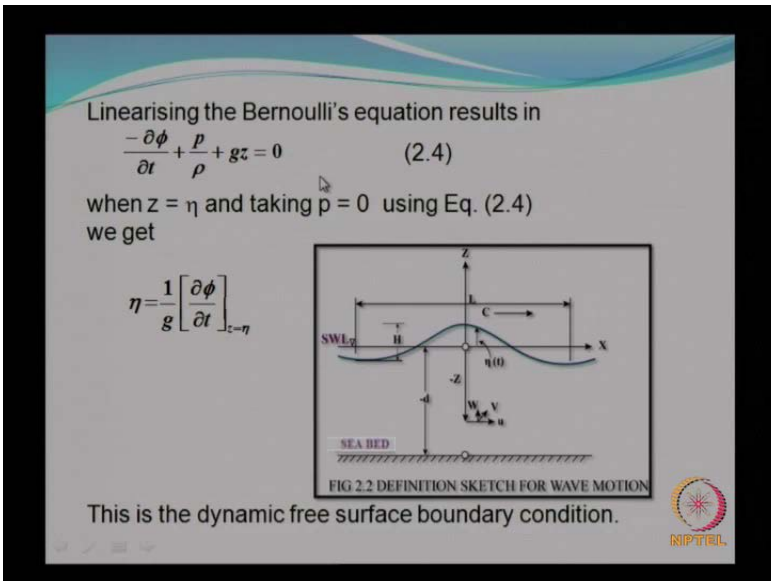
So, if you are interested normally you'll be interested in finding out the velocities at velocities due to the waves or I will simply say the velocity is our pressures etcetera under the wave in the fluid medium which will be within this area, and suppose you are referring to this location this point will be treated as minus z.

And the distance, the wavelength is also indicated here and the governing equation as I said earlier is the Laplace's equation, you have the continuity equation and, this is the what is this? This is nothing but the bernoulli's equation. So, please remember the equation numbers I hope you are making note of the equations. So, that it becomes easier for you when you want to follow because I may be referring back to equation 2.2 or 2.3 or whatever it is, and this equation governing equation will be valid over the fluid medium where extends upto ranges between minus infinity to plus infinity.

And it will be varying from the sea bed which is now referred to as z equal to minus d, d is the water depth. So, here at this location it will be z equal to minus d and it is valid up to near the surface that is z equal to eta. Eta is nothing, but the variation of the height above the main water line. So, eta can be positive 0 or negative, but the wave height cannot be negative. It has to be only positive, because the distance between the crest and the trough.

Now, the boundary conditions as I said earlier please remember the equation numbers the equation 2.1 that is nothing, but the Laplace's equation has to be satisfied in the region which I have already said the pressure at the free surface is 0 when I say pressure at the free surface is zero, that means z equal to eta.

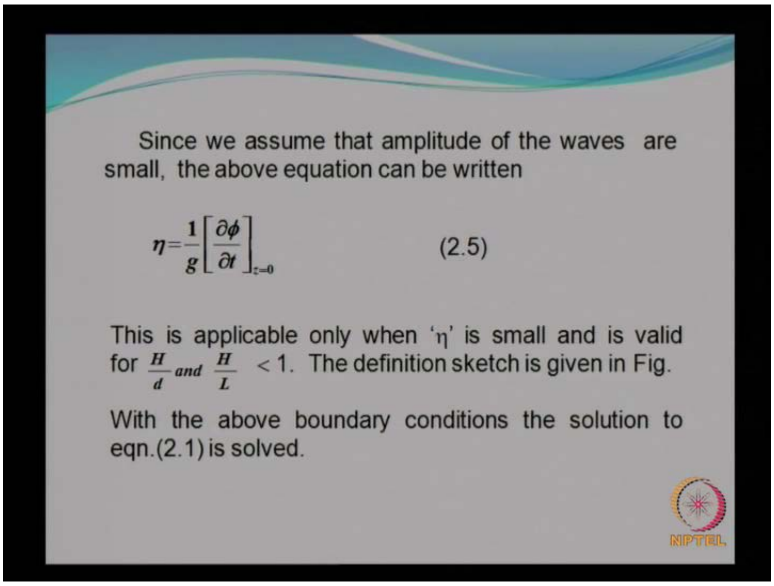
(Refer Slide Time: 11:00)




So, here in we are dealing with the linear waves. So, we neglect the higher order terms and as you have seen here we neglect these higher order terms and hence The laplace's, I mean hence the Bernoulli equation reduces to as shown here in equation 2.4. So, when z equal to eta which is nothing, but at the free surface pressure is 0 this is what is called as dynamic free surface boundary condition.

So, when you put this at z equal to eta and p equal to 0 you get a relationship for eta as shown here that is look at this z equal to eta that is this condition is at this point, any point z can be minus or plus that is what it means here again I am saying that this is obtained by, because of pressure being 0 at the free surface, which is called as the dynamic free surface boundary condition.

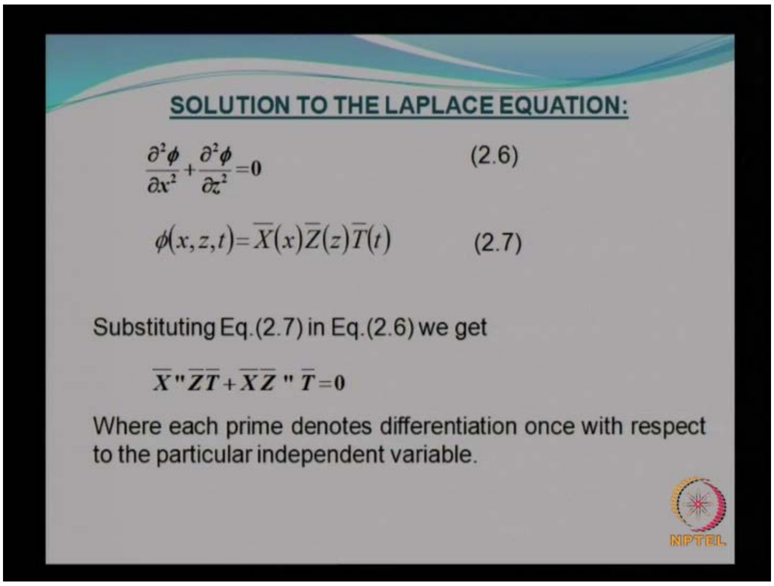
(Refer Slide Time: 12:27)




In the beginning itself, we said that the amplitude of the wave is much less compared to the wavelength hence, do not you think that you can apply this boundary condition at z equal to 0 itself, yes we can apply because the height of the wave is small. So, we are using z equal to 0 instead of z equal to minus eta, because of which the problem simplifies, as I said earlier this will be valid for wave parameters as shown here,d is the water depth H is the water height L is the wavelength.

The definition sketch we have already seen earlier with the above boundary conditions the solutions to equation is now try to we will try to obtain the solutions.

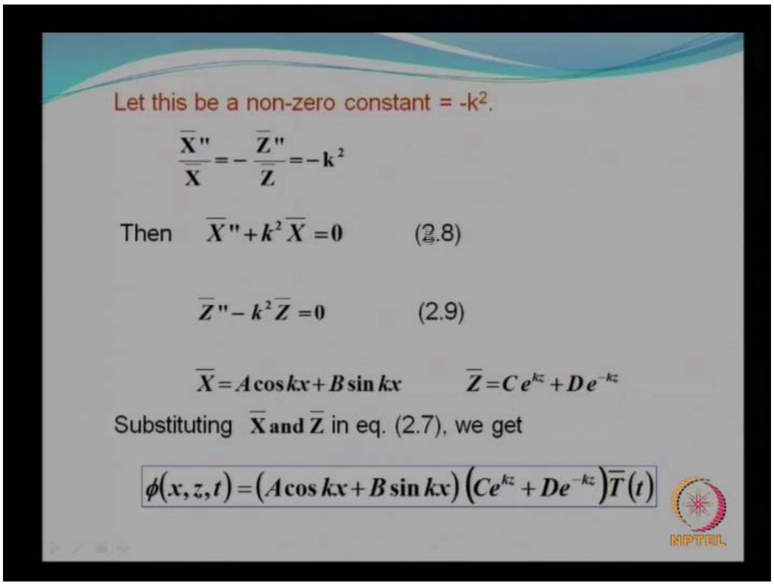
(Refer Slide Time: 13:30)




We have one more condition which is called as the kinematic bottom boundary condition what does that say, that says that at the seabed the vertical velocity will be 0 which we will look into it while we are deriving the Laplace's equation. Then you try to obtain the solution for Laplace equation.We are dealing with two dimensional flow. So, it is a boundary value problem two dimensional, and hence I can represent the velocity potential as a function of X Z and T which will be assumed to be a product of x which will be a function of x z a parameter which will be function of z and T which will be a function of T, when I assume the velocity potential as given here and then substitute in this Laplace's equation, I obtain an equation as shown here.

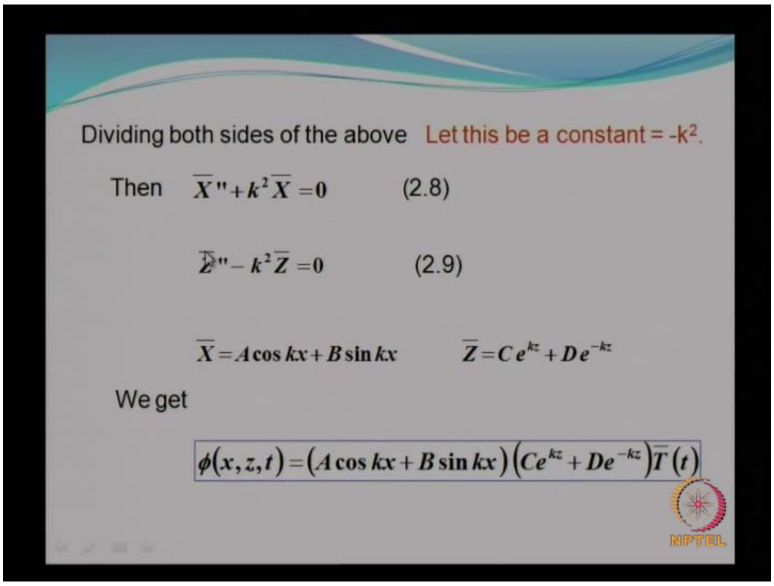
Here in each prime denotes differentiation being done once. Since we have double derivation we have two primes now, this is the equation which we have got from the Laplace's equation.

(Refer Slide Time: 15:40)




And let us say that it is equal to a constant approximately equal to is a constant minus of k square when I do that equal to minus k square then I can write down as shownhere as an equation in terms of X and an equation in terms of Z.

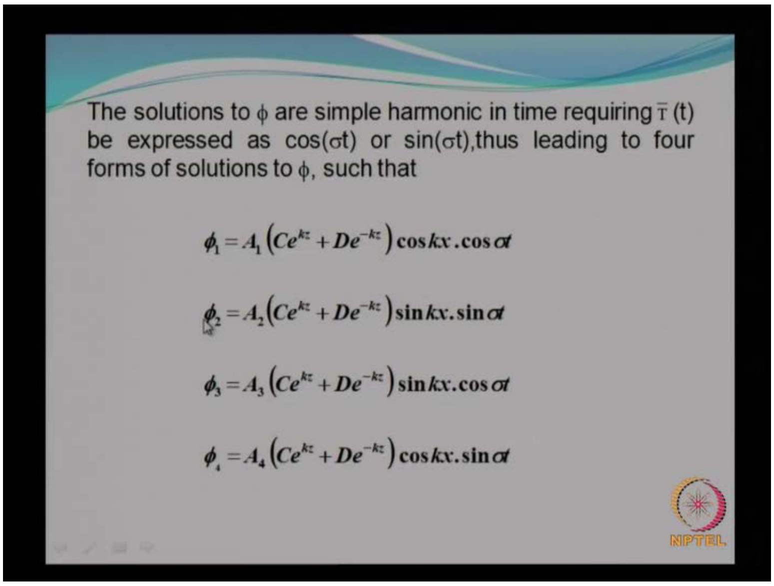
(Refer Slide Time: 16:14)




From the basics of partial differential equations using the method of separable,you have to have a look at this method when you have an equation of this nature the solution to this equation can be written in this from, and the solution to this kind of an equation can be written in this form, because I am trying to skip some of this information, because this are this this information is available in standard text book for partial differential equations.

So, when I substitute for X and Z in this equation then I get a solution to the velocity potential, which will be a function of x z and a few constants four constants A B C and D these two have been obtained from these two variables, and now this will remain this is the time variable, which we consider the periodicity of the wave.

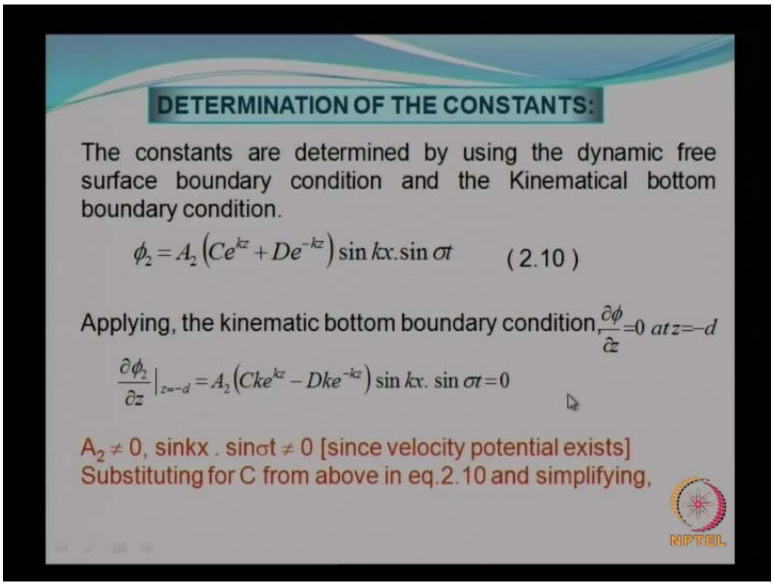
(Refer Slide Time: 18:29)




So, when you have this form of a equation, this equation from the velocity potential can have four forms, because you see there are products this one ,this one, plus this one product of three terms I would say which which will give rise to four forms of equations ph[i]-phi 1, phi 2, phi 3, phi 4 again we are not done with the constants what does this equation give the solutions of velocity potentials are harmonic in time.

We know that we are dealing with a simple harmonic motion,that means, the T has to be express in terms of t can be expressed the T can be expressed as a form of cos of sigma t or sin of sigma t. Hence you will have for this equation four forms of velocity potential phi 1, phi 2, phi 3 and phi 4 have a clear look at the combinations cos k of x into cos of sigma t sin k x into sigma sigma t then sin k x into cos sigma t. So, all these combinations are possible. So, we need to consider all these combinations and arrive at some kind of an expression in order to eliminate the constants, for which we resort to applying the boundary conditions this is the basics of any partial differential equation you want to apply a partial differential equation for any kind of a problem.

(Refer Slide Time: 20:25)



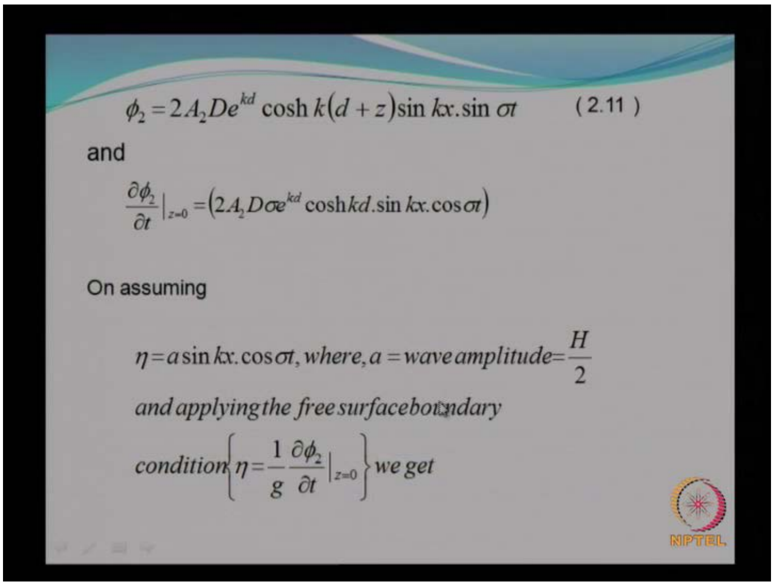

So, now we are going to be the process of determining the constants that is going to dictate the variation of the velocity potential. So, now the constants are determined by the dynamic free surface boundary conditions which I have earlier said and also the kinematic bottom boundary condition, what does the kinematic bottom boundary condition say, the bottom boundary condition says that the vertical velocity at the sea bed is zero.

Here in the horizontal velocity or the velocity in the direction of wave propagation is termed as you that will be the in the x direction and in the vertical direction you will have vertical velocity referred to as w and that will be in the z direction, in the open ocean you are going to have one more component which will be perpendicular to the screen, if you consider that velocity component then we call it as three dimensional wave that we are avoiding rightnow in order we will understand step by step.

So, the boundary condition when I said at the sea bed at the sea bed is nothing but z equal to minus d, and dou phi by dou z is the vertical velocity equal to zero. So, let me take this form of equation phi 2 first and then I apply this condition in order to obtain an expression when you differentiate this and equate to 0, I will get this equation.

If you look at this equation can A 2 be 0 A 2 cannot be 0, because the velocity potential has to have some amount of potential, I mean some amount of magnitude. So, A 2 cannot be zero , can k x sine k x and sine sigma t be equal to 0 no because we are dealing with the simple harmonic motion. So, naturally out of this product only this has to be equal to zero.

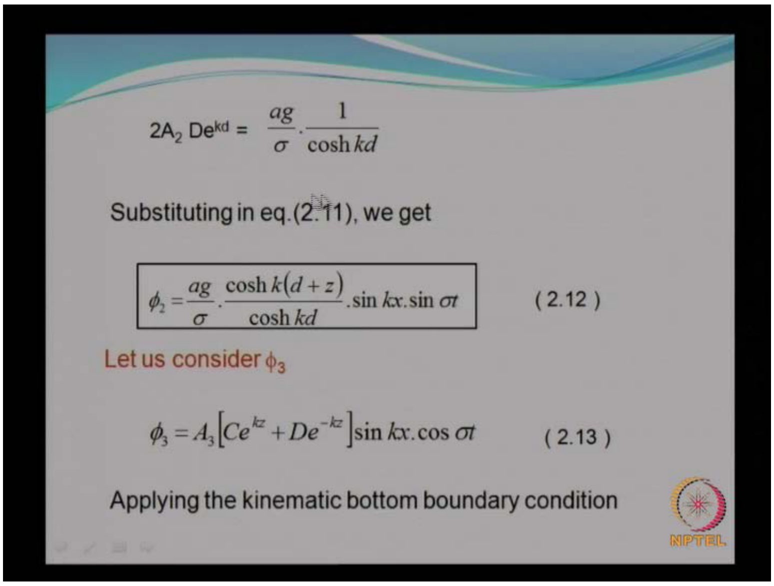
(Refer Slide Time: 23:22)




So, we said this equal to 0, when we said this equal to 0, this is the kind of equation we will have. So, now and dou phi divided by dou t, at z equal to 0 what is it dou phi divided by dou t, at z equal to 0 that is coming from the dynamic free surface boundary condition. Is it clear? what was the dynamic free surface boundary condition please recollect it is eta equal to one by g into dou phi divided by dou t at z equal to zero, that was a dynamic free surface boundary condition which will lead to this kind of equation.

Now, you see that phi 2 as the form as given here. Now let us assume that eta is equal to a into sine k x into cos of sigma t here, a is nothing but amplitude of the wave, wave height divided by two, and the free surface boundary condition is given here, we apply the free surface boundary condition to get this expression. So, now you see that we are trying to tackle 2 of 1 time.

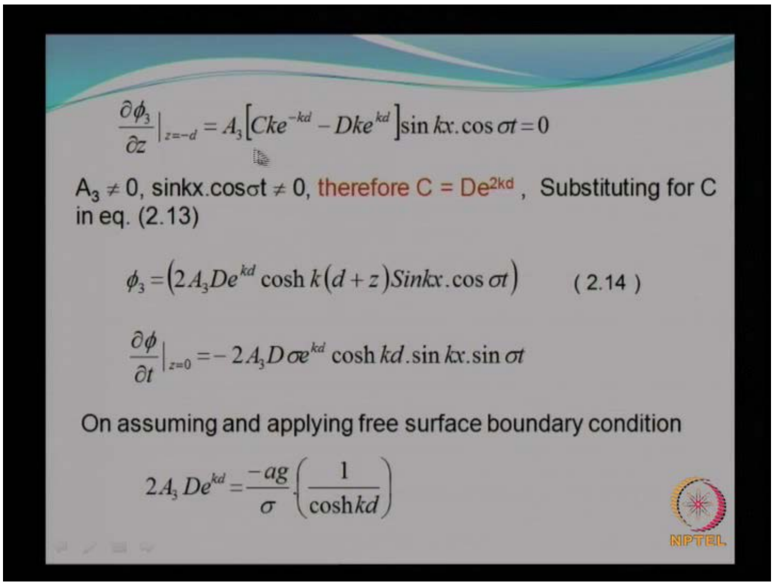
(Refer Slide Time: 25:39)




So, because we get from this free surface boundary condition this is equal to now, substitute so, this is the equation. So, we substitute for this and then we get an expression for velocity potential what did we do we have eliminated the two constants a and d now we have an equation, as a function of amplitude that is a k d is the water depth z is at any particular location which shows that the velocity potential will be varying within the fluid medium from top to bottom, but still we have not got a kind of solution what we would like to have.

So, let us consider phi 3 go back so, phi 3 is given here, we took phi 2 now, we are taking phi 3 when we take phi 3 apply the kinematic bottom boundary condition I am going to get we will get something like this.

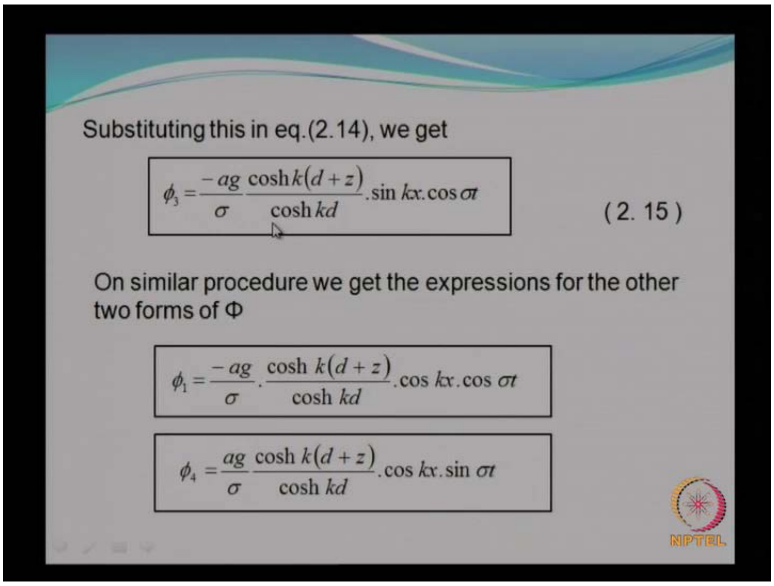
(Refer Slide Time: 27:14)




So, again I repeat the amplitude cannot be zero, this cannot be zero because there has to be a phase therefore, I can get c equal to D into e to the power k d, now substitute for C in equation 13.

Because that C is going to be in the form of D. So, I get an expression for the velocity potential that is phi 3 as shown here, if I differentiate that with respect to t in order to apply the dynamic free surface boundary condition I get an expression as shown here this is obtained by applying the dynamic free surface boundary condition, and then when I use this I get an expression I can easily solve for phi 3. So, now, you see that the procedure is quite straight forward in order to get the velocity potential phi 2 and phi 3, but both phi, phi 2 and phi 3 we have got the amplitude part fine.

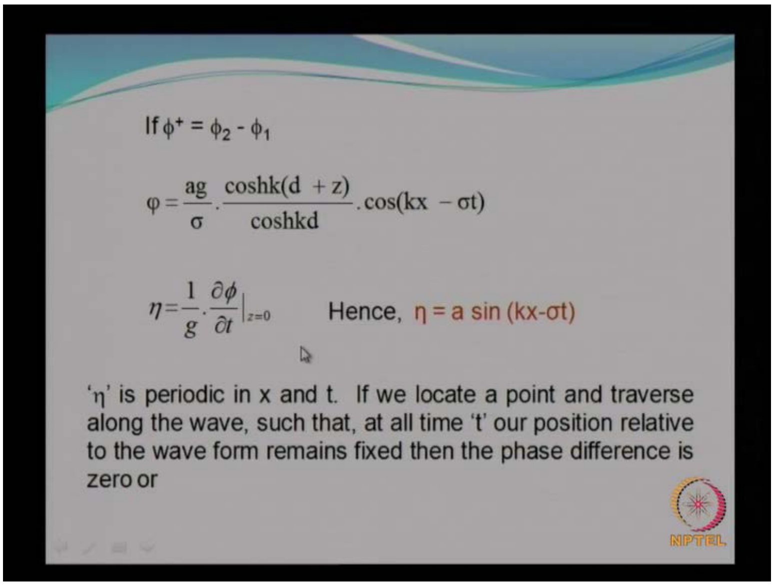
(Refer Slide Time: 28:39)




See when you look at simple harmonic motion you have, what how do you define the simple harmonic motion amplitude into sin of some phase. So, this is going to be a amplitude, but this is going to be the whole thing has to be reduced to the phase. So, we have started from the fundamentals and trying to eliminate the constant which we have just now finished. Finally, we should try to get the form withwhich we are comfortable and with which we can represent the simple harmonic motion. So, I suggest after the lecture you go through all these equations and I am sure that it will be finding it quite comfortable. So, on a similar process, similar procedure you can also get expressions for phi 1 and phi 4 the other two forms of the velocity potential and all the constants are now eliminated.

So, now if I assume that the positive velocity potential see velocity potential can be negative or positive.

(Refer Slide Time: 30:28)




So, if I assume that the velocity potential is a positive, I just take this form because phi 2 is already there phi 1, phi 4 and phi 3, now phi 2 minus phi 1. I take combinations of any two forms of the velocity potential. In this case I take these two, which can easily be reduced as shown here, this is simple trigonometry.

So, you can easily sit and derive this, know while doing this. So, this is the total velo[city]- one form of total velocity potential positive, what was the dynamic free surface boundary condition? this is what is the dynamic free surface boundary condition. So, initially we assumed an expression for theta eta that is the water surface elevation as shown here. So, we assume eta and try to derive the velocity potential, after deriving the velocity potential we try to reestablish what should be the variation of your eta.

So, according to this we have got an expression for velocity potential which does not involve any constant, but involves only the variables associated with our wave propagation problem. Now I use this eta equal to one by g dou phi by dou t at z equal to zero, and my velocity potential now is this one with which I get eta equal to a into sin into k x minus sigma t.

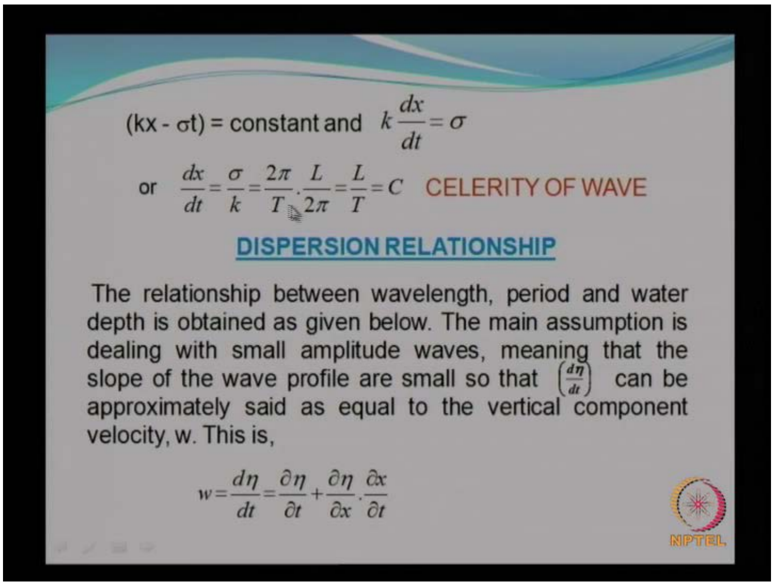
Now, we have got the basic equation of a sinusoidal wave, but what is this, this is the amplitude of the wave, and the here we have x, and we have t when we want to understand the characteristic of a wave we have to consider one full cycle, how do you consider one full cycle, you have to consider both in variation with respect to space as well as variation with respect to time, and that is what is brought up in this within this bracket. So, x is the variable that takes care of the space.

What is the maximum space within one cycle? one wavelength and two pi is the total, in terms of radiance. So, k is two pi by l now what will be the value of k value of x, x will vary from zero to l. So, you are covering one full cycle which is equal to two pi. So, k is called as wave number and it is given as two pi divided by wave length. Now what is sigma, sigma is two pi by capital T , capital T is nothing but wave period.

What is the value of small t, small t will be varying from 0 to capital T. So, one cycle for time is also taken into account. So, this phase takes considers both variation in space as well as in time. So, that is what I am writing here eta or the variation of spa[ce]-variation of the wave elevation is in both space and time, how do we understand this, now you assume simple thing is you assume a channel and a wave is moving from one end to another end, you try to at locate a point on the wave.

So, when you try to move along the wave, because you have identified a point and when you try to move along the wave, in such a way that there you always in line with that particular point which you have identified, in if you are able to achieve that then what do we say that there is no phase difference between me and the wave, or the phase is a constant. In order to accomplish that, I have to move with certain speed only then I can accomplish so, that we and the wave are in the same phase or we do not have any phase difference I hope this clear. That is what I am trying to say here.

(Refer Slide Time: 37:01)



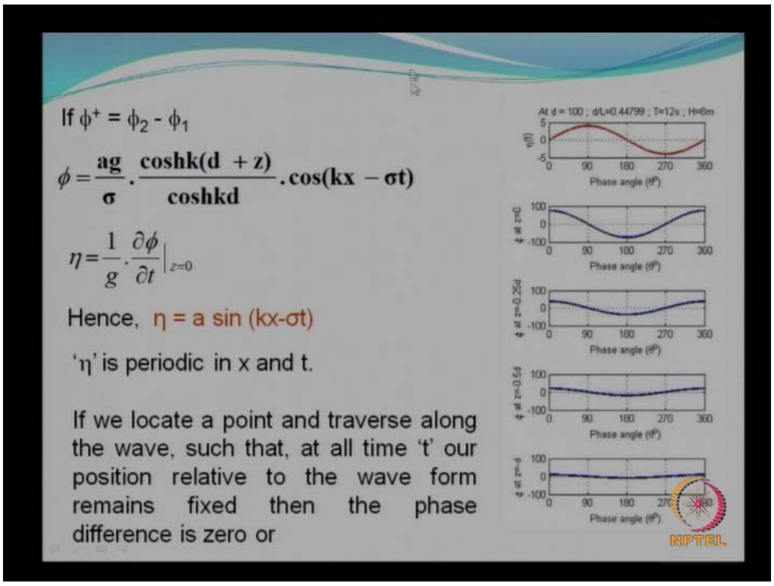

So, if I can do that then I can consider k x minus sigma t is equal to a constant that is there is no phase difference between me and the wave. When I say that this is a constant k as I said earlier, it is 2 pi by L and sigma is nothing but 2 pi by t. So, when I carry out the simplification I will get a value d x by d T equal to it boils down to L by T.what is L by T, L by T is nothing but d x by T which is nothing but the speed. So, now, you understand that the speed of the wave is the celerity it is also referred to as celerity.

Now, let us move into now before I move into other aspect let me let us try to understand a bit more about the velocity potential. The velocity potential now here what we have considered is a cosine curve, if you assume the whole thing as theta. So, this is a cosine curve for a sine wave for the time being you assume that k x minus sigma t is equal to some theta.When the variation of the wave elevation each in the form of sin curve the potential will vary as in the form of a cosine curve.

How does the velocity potential vary, potential the velocity potential will be a function of wave height, will be a function of wave period, because sigma is nothing but two pi by T it will be a function of wave length, and this will be varying as a function of z, from the still water as it go up to sea bed what will happen the velocity potential will be varying, and what kind of a variation it is,it is a hyperbolic variation.

So, when you take a fluid medium, and when the wave is moving. So, I am considering only a sinusoidal wave then, at each elevation the velocity potential will be varying as a cosine curve at each z. What will be the magnitude, since we are dealing with hyperbolic function this magnitude will be higher near the free surface as you go down below, because of this hyperbolic function z is remember that z is negative. So, as we go down towards the sea bed, the value of this magnitude will decrease. One of the reason why we say that it is a surface phenomena more dominant near the surface the velocities velocity potential is higher close to the surface.

(Refer Slide Time: 41:30)




So, it varies along the surface upto the sea bed. So, now, I have just taken an example water depth as a hundred meters, wave period as ten seconds, and a wave height of eight meters. So, this gives me the relative water depth of about 0.44. Now in this slide I am just showing you the variation of eta here, and this is a sinusoidal variation.

Now, the variation of the velocity potential at z equal to 0 that is at the still water line is shown here. This will be a cosine variation as we have already seen the variation of velocity potential and the variation of eta now here the phase variation is the phases just phase of the velocity potential is just of by about 90 degrees with respect to, of by 90 degrees from the eta now youlook at the magnitude of the velocity potential. So, this is near the still water, at the still water and slightly below it that is 25 percent of the water depth, and this gives the variation of magnitude which is less than, at the still water line.

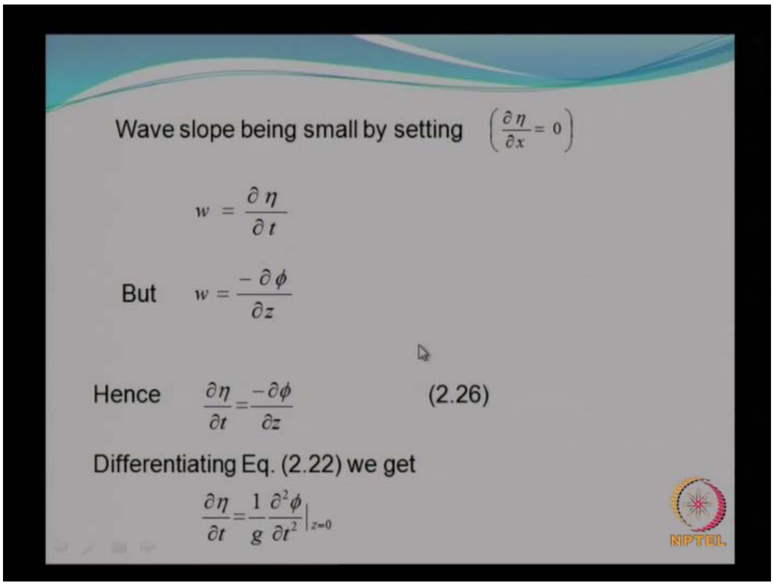
Now, at a distance of about 50 percent of the water depth below the still water line. You will see that the velocity potential further reduces. So, you see the phase variation and how the velocity potential varies or reduces as you go towards the sea bed. So, I hope you have understood what happens when a wave is propagating the velocity varies the velocity potential varies when velocity potential varies naturally, the velocities have to vary which we will see later.

Now, before going into further, we have looked at the variables, wavelength, wave period, water depth, etcetera. How do we get all these information because mostly the wave height and wave period are given to us. They form the basic characteristics to define a wave, but when you look at the velocity potentials see look here k is nothing, but wave number which is two pi by l, that wavelength we need to find out.

How the wavelength and the wave period are dependent. This is what is called as the dispersion relationship the main assumption is dealing with a we are talking about the small amplitude wave, meaning that the slope of the wave, the slope of the wave being very small. We can assume the vertical displacement with respect to vertical displacement of the water surface, when I say water surface water surface elevation, when I say water surface it refers to eta that is variation of the displacement of the water surface with respect to time which is given as d eta by t ,can be approximately said to be equal to vertical velocity, I have basic definition.

So, now basic definition is right, but then what how you can split this total derivative this total derivative can be split as dou eta by dou t partial derivative with respect to time into the special. What is this dou eta by dou x eta is elevation and x is the distance. So, dou eta by dou x is nothing but the slope. We have said that the wave slope is small, because we are assuming that the wave height is small. So, what will happen w is going to become now just dou eta by dou t.

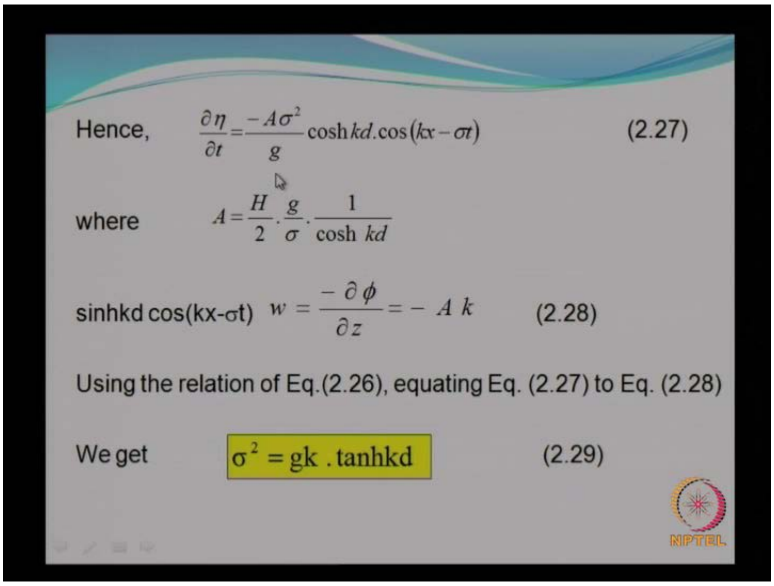
(Refer Slide Time: 46:39)




What is w? w is nothing, but the velocity in the vertical direction, that is the reason why we have got our velocity potential known to make use of the velocity potential to evaluate the particle velocities. What we have seen earlier which is celerity is the speed of the wave itself. Now we are trying to understand when the wave is moving what is happening under the wave. So, you know that the velocity potential is going to vary within the fluid, within the fluid medium. So, is the velocity. So, horizontal the vertical velocity will be minus dou phi by dou z.

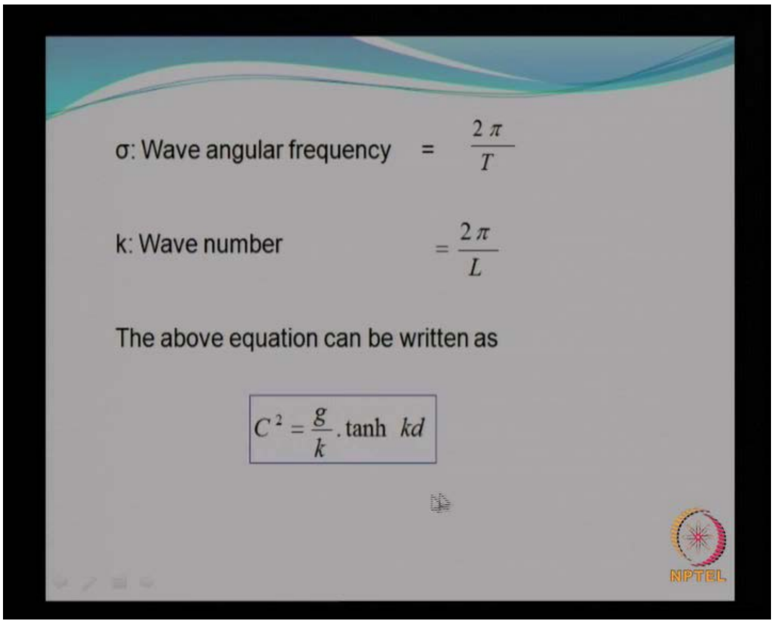
So, I can equate these two, when I equate these two, I get this equation. Now I differentiate this as shown here let us see that equation, I have skipped something. So, this dou eta by dou t what about your velocity per free surface boundary condition ,this is the free surface boundary condition. I can differentiate this with respect to T then I have a double derivative here, and that is what I have done here dou eta by dou T using the free surface boundary condition, I get this equation.

(Refer Slide Time: 49:06)




Now, I can as well equate this and this, and the velocity potential is already known to you this is the velocity potential, and when you do this I get I can evaluate the constant and then finally, when you simplify you get an expression for relating t that is the wave period and wavelength. So, I have explained the methodology because you know this w, you know dou eta by dou t and then equating using the relationship into 2. 26 you can and equating these two equation 2.28 I can get an equation or an expression which is called as the dispersion relationship.

(Refer Slide Time: 50:04)



And what is sigma here 2 pi by T and k is 2 pi by L, and this above equation can also be written in this form, because once you substitute for sigma, and the wave number I can get an expression as shown here, now if you do not substitute for if you want to retain your k and only sigma you want to eliminate and I can just use 2 pi by t sigma. So, I can write the dispersion relationship in this form or in this form, but this is widely used, or the speed at which the waves moves, the wave moves. So, this is nothing but, what this is nothing but the speed of the wave.

So, earlier we tried to find out that speed is nothing but L divided by T. Now we have derived the expression for the celerity. This is a generalized equation for the variation in celerity. Now with this I will we will just go into the next module in the next class.I hope things are quite clear today. So, you need to do some additional homework also, although we have given all the information so, you should look at some of the references books and try derive it yourself in order to become more and more familiar with this topic.