Marine Hydrodynamics

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Lecture - 08
Two Dimensional Flows

Welcome you all today the eighth lecture in the series on marine hydrodynamics. And in the last few classes, we have already discussed, basically the conservation laws; one is the conservation of mass, the second law is the conservation of momentum. And then we talked about application of this equation of motion, what is the conservation of mass and the continuity equation, which is nothing but the law of conservation of mass. And then we have talked about several application with this background, today we can talk about various two-dimensional flows, because we have already talked about velocity potential stream functions.

So, easily with this understanding, we can talk about two-dimensional flow. So, the question comes why two-dimensional flows, because everything in nature is a three dimensional, but why we are thinking of two-dimensional flows. One of the obvious question comes, and the when can we make, consider assume that the flow is two-dimensional and still we get accurate relation of the physical model that is under our disposer. One of the fact suppose, I because we are looking at a flow problems, if the component of velocity in the z axis, suppose it is negligible compare to the component of velocity, and other characteristics, motion characteristics in the x and y directions.

Then, we can always neglect the z component, and consider the flow as two-dimensional in such, in such situations, because W the z component, we are neglecting, we can always call, because in a Cartesian coordinator, in the polar cylindrical, polar coordinator always. Let us say Cartesian coordinate here always, we can say that the flow is a function of x and y and this is independent of z and in the process, we consider the flow as two-dimensional. So, only emphasis will be giving on the x and y component of the flow and z component is negligible compared to the x and y component, another aspect is that, suppose I have a situation where to analyze a instead of analyzing a three-dimensional flow, I still can consider a two-dimensional flow. If suppose in the x y plane, the flow is symmetric and I took any plane, and each of the suppose I say z is equal to 0, z is equal to 1, z is equal to 2 any of this plane.

If I say the flow is symmetric in nature, see flow is symmetric in nature, then I can always ignore the z component, and I analyze the problem in the x and y components particularly in two-dimension and generalize it assume that, the similar the pattern remains in similar in nature, in the z direction. In such a situation also, we do say that it is easy it becomes easy to analyze, the problems in two-dimension. The advantages are many of this two-dimensional flow, of a three-dimensional situation, because we are reducing the dimension of the whole problem by 1 means with theoretical and computational efforts we are reducing to a great extent. And in fact, once we consider the flow as two-dimensional, because we can think of functions of complex where, we write at every point z in terms of in the Cartesian coordinators a point in the real plane x and y in terms of x and y or in terms of r and theta in the cylindrical coordinate.

So, because of this even if the, to analyze two-dimensional flow problems, we can take the help of complex function theory. So, today let us give emphasize, show the complex function theory, we can take into account, and then analyze flow problems in two-dimensional. So, now, we will start with a very simple example or before going to that let me say what I mean, in the complex function theory and how I am, I will relate a two-dimensional flow to a complex function theory, to the complex function theory, now, suppose I know that.

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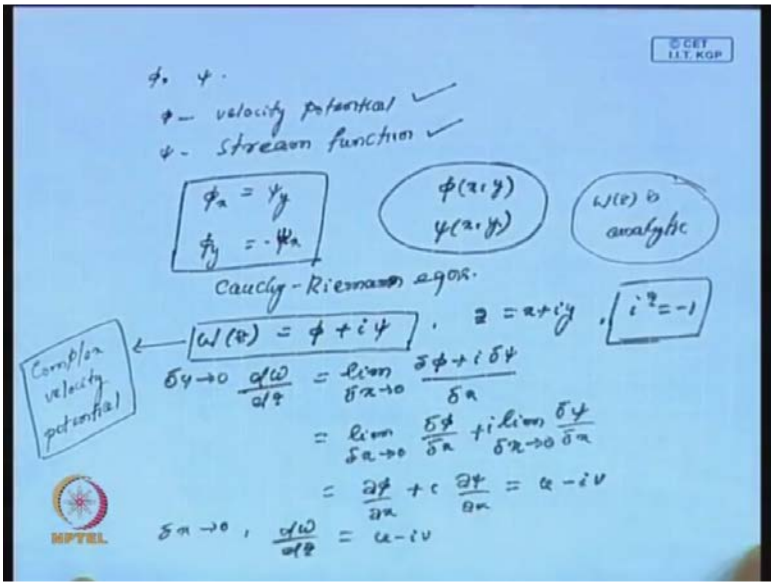

I have the velocity potential phi, and the streamlines rather the stream function psi, phi is the velocity potential, and psi is my stream function and we know if the flow is rotational in nature, particularly when the fluid is incompressible, and flow is irrotational, we have seen that phi x is equal to psi y in two-dimensional phi y is minus psi x, this is what we say, the Cauchy-Riemann equations. We, have already talked about it, I am not going to the results Cauchy-Riemann equations, and then, so from the theory of complex function. If I look at a function W z, which is written as a function of phi plus i psi, where phi is the velocity potential psi is the stream function. Here, we have both phi and psi are function of a x and y they are associated with the flow, where phi is called the velocity potential, and psi is called the stream function.

Now, if I put W z is phi plus i psi, then what will happen where, again z is x plus i y, if I have z plus z is x plus i y W, z is phi plus i psi what will happen if del x, if I say del y tends to 0 what will happen to d w by d z. This I can always write a limit, because my del y tends to 0, I can call it del x tends to 0 del phi plus i del psi by del x, and which always I can write limit del x tends to 0, this I call it del x tends to 0 del phi by del x plus limit del x tends to 0 del psi by i times del psi by del x and del phi by this becomes del phi by del x plus i del psi by del x.

So, this phi x is nothing but u and psi x is minus phi y, and that is minus i into v, here i is i square is minus 1, that is the complex number from the theory of complex. So, what, we have seen that in a similar manner, if I will say del x tends to 0, then also I can easily derive the d w by d z is equal to u minus i v. So, unlike here del phi by del x it will come del phi by del y, and from there I will get I can easily get that, now what will happen, because now so in the process I have brought in a function W, and this is a complex functional of phi plus i psi.

Now, we know from the theory of complex numbers, that if W z is phi plus i psi and its satisfy the question 1 equations then W z is we say this analytic, analytic. Of course, here I assume that the functions are differentiable, so they are continuous once, we say the function, are differentiable because we are looking at both phi and psi. They are irrotational associated with the flow, which is inviscid in a compressible and irrotational and in the process, both phi and psi satisfy Laplace equations, and as a result both phi and psi are continuous, so I can always say that W z is analytic function. This is one of the major thing of introducing, and this W z, I call the complex velocity potential. Now, with this understanding, now if I will come back to, to work out few examples from which it will be very clear, what exactly how this W is representing, the how, they are associated with the flow. So and how will you understand that, this is a flow of a particular type or particular nature?

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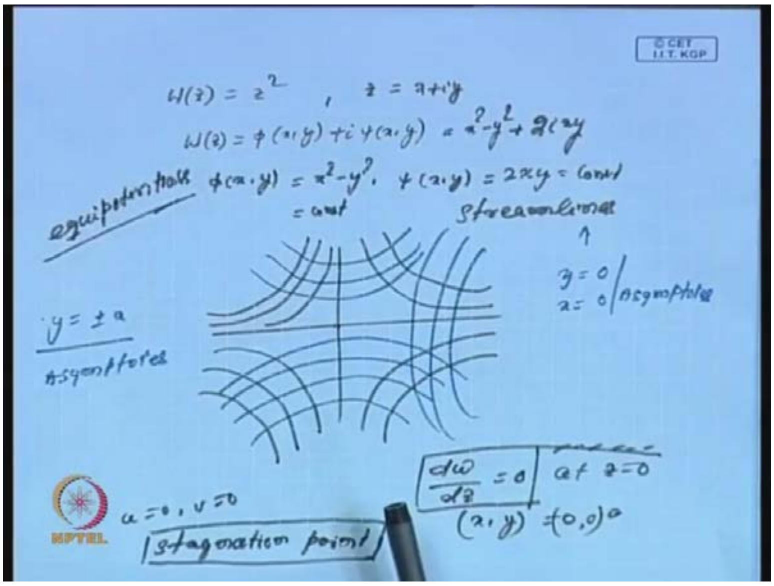

So, now let us say W z is equal to z square, we will emphasis today on various examples. So, if W z is z square then what will happen, then W is equal to W z is equal to phi, I can write at a phi x y plus i psi x y, and this will give us x square minus y square, because I am putting plus 2 i x y, because I am putting z is equal to x plus x plus i y. Once this is there and what will happen my phi x y is x square minus y square, and my psi x y is equal to 2 x y.

So, the equipotentials, if phi is equal to constant, if this is constant I get the equipotential, and if I say this is equal to constant, I get the stream lines. So, this will give me the stream lines, and this will give me the equipotentials. Now, let me draw it, suppose I take the x y plane, and what will happen my psi x y x y is constant, and I can always say that y is equal to from this, I can get y is equal to 1 by x, I always can through get the flow in this way, depending on the constant is positive or negative, then my flow can be in this way.

On the other hand, if I take these are my stream lines, now if I say x square minus y square is constant, I can have flow like this, and here also I can have similarly, in this axis also I can have. So, flow can either in this way or in this way also it can flow and we can easily see that, there confocal hyperbolas, and here in case of the stream lines, the lines y is equal to 0 or x is equal to 0, there the asymptotes, asymptotes, in this case. On the other hand, in this case what will happen to the line y is equal to x, if I look at the line y is equal to plus minus x, then in this case in the case of equip potential there the asymptotes. The another point is here, what happen at 0, at 0 if you look at the 0, d w by d z is 0 at z is equal to 0. So, that means x y is 0, 0 here d w by d z is 0, and once d w by d z is 0, here I will mention, I call this as point this 0 0, because this is 0.

So, I call this as my u component is 0, my v component is 0, so I call this as a stagnation point. So, 0 0 this origin is a stagnation point, so there is no flow after this point, so the point at which there is no flow is a stagnation point, and in the complex in terms of the complex velocity potential this is if d w by d z is 0, then I call this as a stagnation point. Now, with this, so this is clear that in this although, we took just a very simple function W is z square, how it can represent the flow particularly there, how the hyperbolas are rectangular hyperbola, we are able to draw in the flow direction is this way, just is the way.

Then I will just go to another example, because today, we will be spending most of the time on working out several examples, to understand the importance of the complex function theory, because this in the early days, when the there was a viscous viscous flows are not well studied those days, by using particular when it started with the theory of continuous mechanism, when potential flow theory was developed, those days the concept of because those days today’s high speed computers are not available. As a result many problems were analyzed by using the function theoretic approach, and those but it has far reaching effects, because that has given a good understanding, about various flow characteristics at, at in various situations. Whether it is flow through a tube or whether it is a flow in a ocean or in a stream or in a channel. So, even if in aero dynamics when it has played a very significant role, so this two-dimensional flows.

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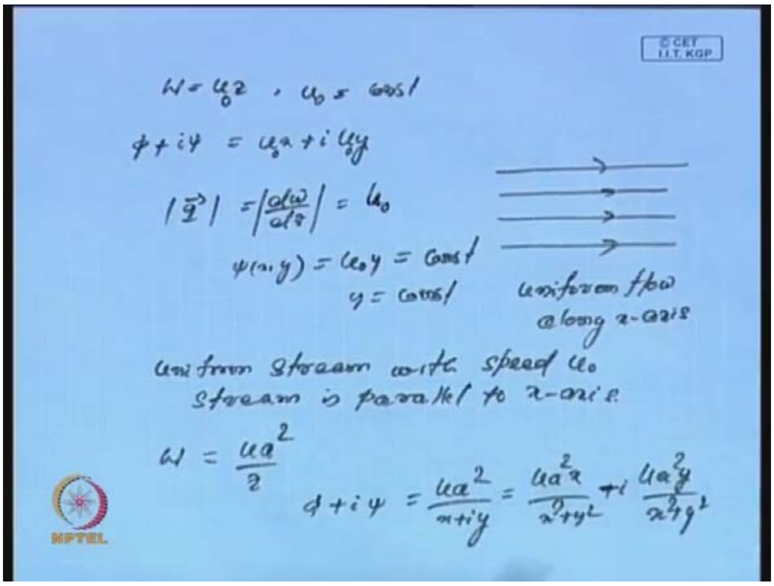

Now, suppose I take W is equal to u z, where u is a constant, u is known constant and z is equal to again x plus i y. So, what will happen my W is phi plus i psi, then that will be u x plus i times u y and here, another thing I will say that, what is q the speed it is nothing but d w by d z modulus. So, and if I say this modulus is the speed, then what will happen in this case, d w by d z becomes u either take it u naught to differ, differentiate with the x component velocity small u. So, q bar is so speed is uniform, the speed is constant, and if you look at psi y psi x y here it becomes u naught y, and that is constant, if psi is constant that means y is equal to constant.

So, the lines it is parallel to the x axis, will represent the direction of the flow, and here the speed of the flow is always constant, that is 0. And we call this as a uniform flow along x axis or sometimes, we call if the flow is parallel to the x axis, because the stream lines are always lines, which are higher the constant y is becomes a constant, so the lines which are parallel to the x axis and. So, and in the process, we say W represents the motion of an incompressible uniform flow, uniform stream rather, we call it sometimes, uniform stream with speed u, speed u naught.

And the stream is parallel to x axis basically, I mean the flow is parallel to x axis, I will take another example, suppose my W is equal to u a square by z, then as usual my phi plus i psi, I can write it has u a square by x plus i y and that will give you u a square x by x square plus y square plus i times rather I will say, this is minus, minus i times u a square y by x square plus y square, and if I say that phi is equal to constant.

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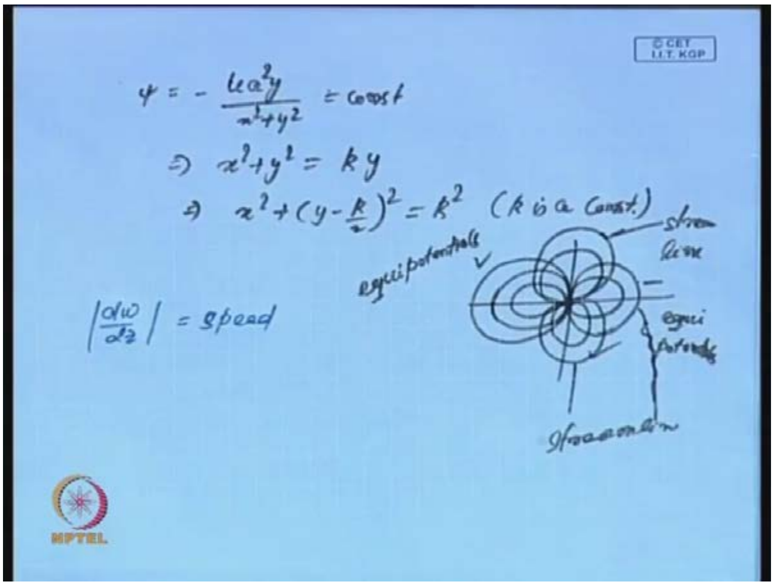

Phi is equal to in this case minus u a square y by x square plus y square, and if I say this is constant, that means it will give me x square plus y square is equal to a constant times K times y. And that always gives me x square plus y minus K by 2 square equal to K square, so depending on the value of constant K is a constant. So, depending on the value of the constant, and there what will happen, I will get different stream lines. So, and this here, I can always see, that this will be like this, this is the way the flow look like, even if it can circles, all are circles touching the, so these lines will be the stream lines, and this will give me the potential lines, equipotential equipotentials.

This is a stream line, then this again streamline, this is again, we will say, these are again, this will give me equipotential, this is equipotential lines, equipotentials. And here, we can see that, they the stream lines touches the, whereas, the stream lines, they they touch the x axis rather equipotential lines, they touch the y axis, and they all this, now another aspect here is that, what will happen to q here. If you look at W, d w by d z modulus that will give us the speed, now I will go to another example here, third example, the third example I will take of a line vertices, we have already talked about vertices, let us look at a line vertices.

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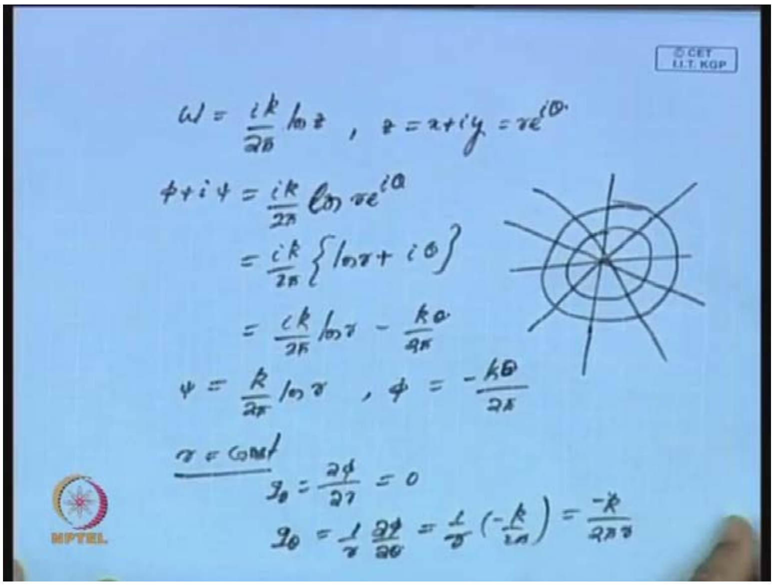

In the complex potential how it will look, suppose is say i K by 2 pi log z, then here my z is equal to x plus i y, and if I say phi plus i psi i K by 2 pi log, so this also I can write it as r e to the power i theta. So, then I can write this as r e to the power i theta, and this is i K by 2 pi into log r plus i times i theta, so it implies i by i K by 2 pi log r minus K by 2 pi into theta. So, here if this is phi plus i psi my psi will be constants psi is equal to K by 2 pi log r, and my phi is minus K theta by 2 pi. Now, here again I will see if I say that, this is a K by 2 pi log r psi, so that means if psi is constant, that will give in the stream line, so r is equal to constant and the stream lines. So, if r is constant then I will get various circles in fact, we have already talked about this in my earlier classes, and then what will happen to the phi is equal to constant, that will be theta is equal to constant once theta is constant, the flow is always in the radial direction.

This, these are the equi potentials, so these are the equip potential, and then the circles center at origin, that is the, that gives me the stream lines, that is the flow direction, and here again if I look at, I can easily, my since in the, from here. I can always find, what will happen to my if my phi is given by del phi by del r, if my del phi by del r that is the radial component of the velocity, and that will be 0. Here again, what is my tangential component of velocity, that will give me 1 by r del phi by del theta, and that gives me 1 by r del phi by del theta is minus K by 2 pi, and that is gives me minus K by 2 pi r. So, if I say my what are the 2 things I have looked into, these are the 2 components of the velocity, then what will happen to the my, I will say rather I will consider 2 separate cases.

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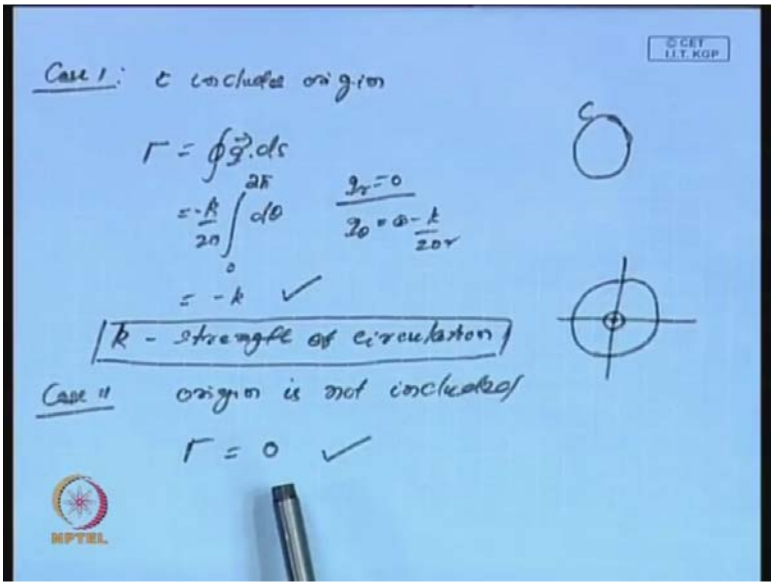

Because if I look at the circulation case 1, what will happen, if I draw a close core, that is C, and that is the flow direction, which is because this is the direction, in which the fluid is flowing, and this is the point, which includes the origin C includes origin. Then if C is origin is included, then what will happen gamma, because I can always say q bar dot d s, and that will give me, because d s has component. So, if I look at this, then because the flow is in the radial direction, and in the radial direction, we do not have q r is 0. And I have already seen my q theta is a 2 pi by r minus K by 2 pi r. So, we can always see that this will give us, q r bar dot d s that will give me a by 2 pi r d r d theta r d theta, that will give me, because x radial this radial component is 0.

So, q theta r d theta, and this will be 0 2 pi d theta, and that gives me minus K, because I have taken q theta is minus K by 2 pi r. So, once it is minus K that means, if I look at the circulation, so K is a constant, and this constant is the strength of circulation, on that is K. On the other hand, if I do not include, if origin is not included, then what will be a gamma, that means I have to avoid the origin. So, if I have to avoid the origin, then I have to take a small circle around the origin, and then I will calculate. If I do this, and then I will easily, I will easily get this gamma as a 0, because I take a small circle, how small the circle is, if it makes a because we have seen here this is independent of r. So, whether how big the circle or how small the circle, it is independent of r the radius of the circle, so it will be 2 pi K K minus K will be and that will be 0.

So, here we see that, the stream lines are concentrated circles, and here which origin this gives a value K whereas, without origin this gives me, the gamma is 0. In fact, in one of my earlier example, I had left this as an exercise, that circulation when you include the origin, we have a similar example here, we are trying to calculate the flow. And then, we had gone to calculate this gamma particularly the circulation and we have seen that in both the cases, this will be different and the earlier case, earlier discussion, I have left it, now it is very clear that gamma will be different, if you include origin or we do not include origin. In this case, now with this again let us look at, a different types of flow, now already, I know that, suppose I take another example.

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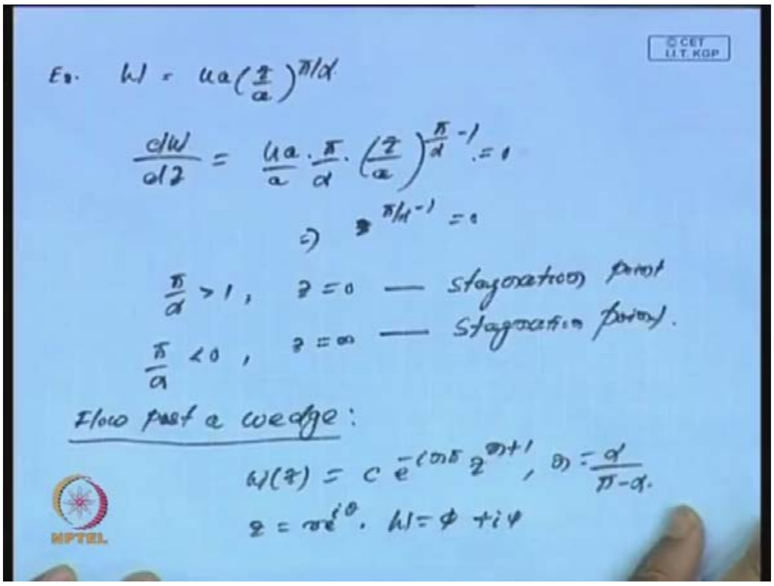

Suppose W is equal to u a z by a to the power pi by alpha, which is the idea of introducing to various complex velocity potential here, is to have a clear cut idea, about the various types of flow, and how this simple complex functions, represents very complicated flow patterns. Now, what will happen d w by d z, if you look at d w by d z this is u a and n to pi by alpha into z by a to the power pi by alpha minus 1. And then, we will have 1, a will come 1 by 1 1 by a will come by, and if I say d w by d z is 0, which implies my z to the power pi by alpha minus 1 is equal to 0. There are 2 cases, if pi by alpha is greater than 1, then z is equal to 0 only represent is a stagnation point.

Now, if pi by alpha is less than 0, then you can easily say z is equal to infinity only will represent the stagnation point. So, see even if this is a very little complex, the flow is, but although it depends on a constant, how the flow pattern looks like. Now, I will go to another example, in the next example basically I often call this flow past a wedge, it will be. Suppose my W z is given by C, it is a minus i n pi z to the power n plus 1, here my n is alpha by pi minus alpha, and then, we put z is equal to r e to the power i theta, and W is equal to phi plus i psi, if you do that and again, what will happen to this.

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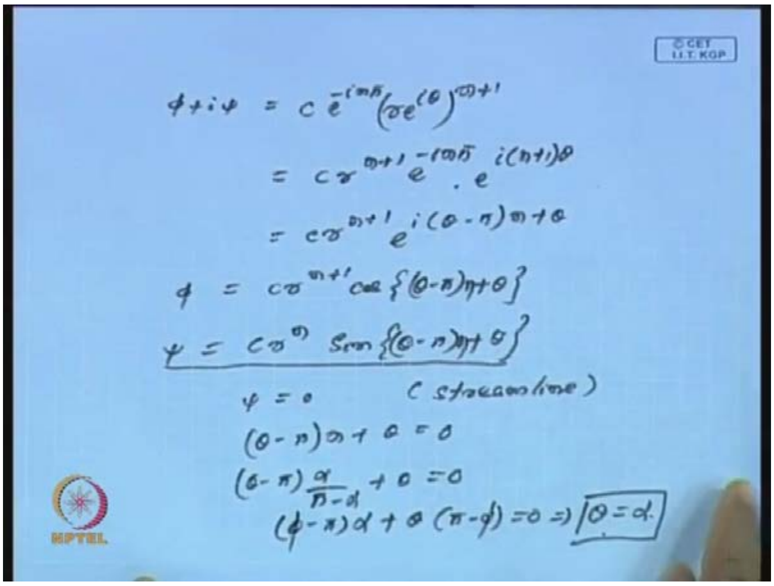

So, you can get phi plus i psi equal to C e to the power minus i n pi, and z is r e to the power i theta to the power n plus 1, and that gives me C, this is r to the power n plus 1 C r to the power n plus 1 into, into where minus i n pi e to the power i times n plus 1 into theta that gives me C r to the power n plus 1 e to the power i times theta minus pi into n plus theta. Then our phi will be C r to the power n n plus 1 cos theta minus phi plus theta, and my psi will be C r to the power n sin theta minus pi n to n plus theta.

Now, psi is the stream, this is the stream function, if I say psi is equal to 0, that will give me a streamline, will give me a streamline, and what happen when psi is equal to 0, psi is equal to 0 means, when r is not equal to 0. That will give me theta minus pi into n plus theta is 0, and if I substitute for because I have taken my n is equal to alpha by pi minus alpha, so this is theta minus pi into alpha by pi minus alpha plus theta is equal to 0. So, which gives me, from which I can always get, theta minus pi into alpha plus theta into pi minus alpha will be 0. So, here theta alpha minus, theta alpha this will get canceled, and then, we have pi alpha minus theta.

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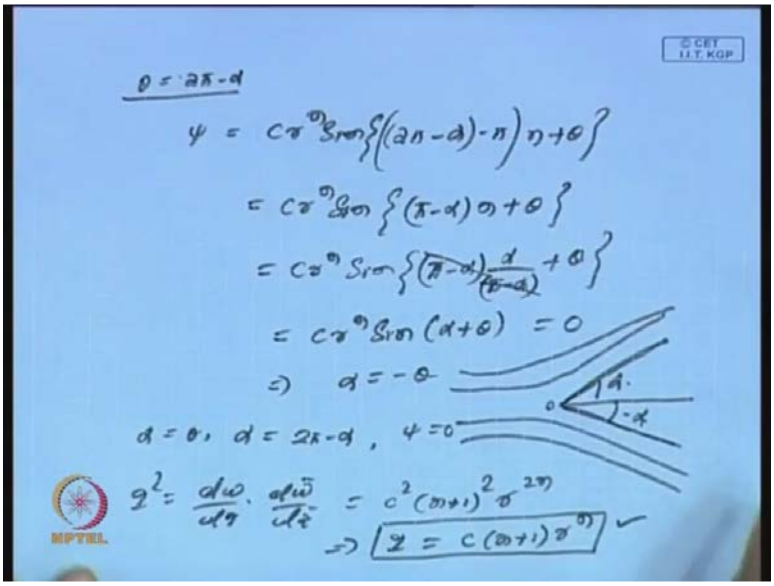

So, that means theta is equal to alpha, so this is one criteria, now the second one again. If theta is equal to 2 pi minus alpha, if theta is 2 pi minus alpha again, we can see that what will happen psi, psi will also theta is equal to 2 pi minus rather, we can also say that rather, we will see that C r to the power n sin, let us say theta is pi minus alpha, so this is 2 pi minus alpha into minus pi theta minus pi into n plus theta. And this becomes 2 pi minus pi is become C r to the power n sin 2 pi minus pi is pi minus alpha into n plus theta, and this is equal C r to the power n, sin I will put for n, n is alpha by pi minus alpha, alpha by pi minus alpha plus theta, and thus pi minus alpha pi minus alpha, so C r to the power n sin alpha plus theta.

And if psi is equal to 0 this gives me obviously, that one of the criteria is alpha is minus theta. So, if I plot this because 1 there are 2 2 things I am getting psi is equal to 0, so for alpha is equal to theta or alpha is equal to 2 pi minus alpha, I am getting psi is equal to 0. That means, that will act as a stream line, and alpha is 0 means now theta is alpha, so if I at any point this angle is alpha, and then this angle is minus alpha. So, you have a fluid is flowing, so my fluid will be because this will act as a stream line, so because of this, because this line is behaves like a stream line, and again I can see that because of my characteristics of W. I can always say that at z is equal to 0, d w by d z will be 0, so z will act as a stagnation point z is equal to 0 will act as a this origin, will act as a stagnation point.

And this flow is like a it is a way swept, so the it is showing that as if the, it is the flow past a wedge that is why this is represented, further if you look at q square, q square is a d w by d z to d w bar by d z bar. And if I simplify this, we can get this as C square in plus 1 square into r to the power n 2 n and that will give us q is equal to C into n plus 1 r to the power n, that gives us the speed at which, this will give us the speed at distance r, from the origin. So, this is a flow past a wedge this will represent, now with this I will only, now let us look at a uniform flow, we all know that in case of uniform flow.

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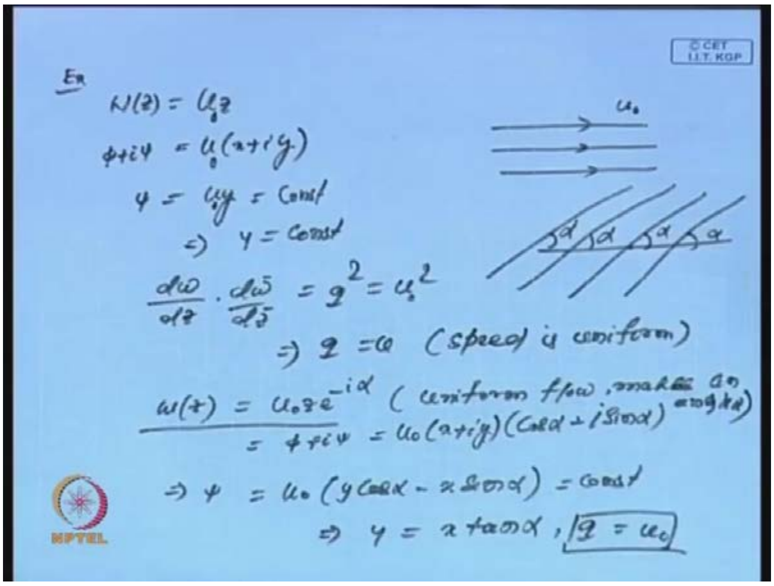

The flow is always parallel to the x axis, if I let us see how, we can represent in terms of a complex velocity, suppose I say I have a uniform flow with velocity u naught. So, if I write as W z is equal to u z, it is very simple, this is can be written as phi plus i psi u x plus i y, and that will give us psi is equal to u y, and if you see this is equal to constant, that will give you the streams line, that means y is equal to constant, these are the stream lines. So, this is when fluid flow and it can be easily say checked, what is d w by d z into d w bar by d z bar, and that is q square. And in this way, it is also u square, so which implies q is equal to u, so the speed is constant speed is uniform, U is a uniform, along the x axis, it is and the fluid is uniform, it is parallel to the x axis, and the speed is also u is equal to u naught t by I have taking at the u naught.

So, this is speed is constant, now the same thing if I just twist a little, suppose I say W z is equal to u naught z e to the power minus i alpha, then it can be easily seen that, this if I say this is equal to phi plus i psi, and which is nothing but u naught z is x plus i y into e to the power minus, this we can also write cos alpha minus i sin alpha. And if I put it then I will easily get my psi as y cos alpha, u naught y cos alpha minus, this will be pi i y cos alpha, then minus this will be x minus x i naught. And that, if this is constant that will give us the stream line, and that constant gives me implies y is equal to x tan alpha and once y is equal x tan alpha, if I draw line, and here this makes an angle alpha, this is the way, the fluid will flow, all these angles are alpha. So, this is a uniform flow, which certain substantial angle alpha with the x axis, and this is the way and if you look at again here what will happen to q, it can be easily find that speed is also that same as u naught.

So, here also, we call this as a, this example, we call this, uniform flow, this is also situation of uniform flow, which is making an angle r, which makes an angle, angle alpha with the x axis. So, this is horizontal whereas, in this case, this is making an angle alpha with x axis. So, this example suggest us, that we can always find several situations, where our flow is by using the complex function theory, we can represent the flow characteristics represent various flows describe various flows. Now, there are some flows which are very important, so I will, I will come to few more cases, let us look at a very important thing.

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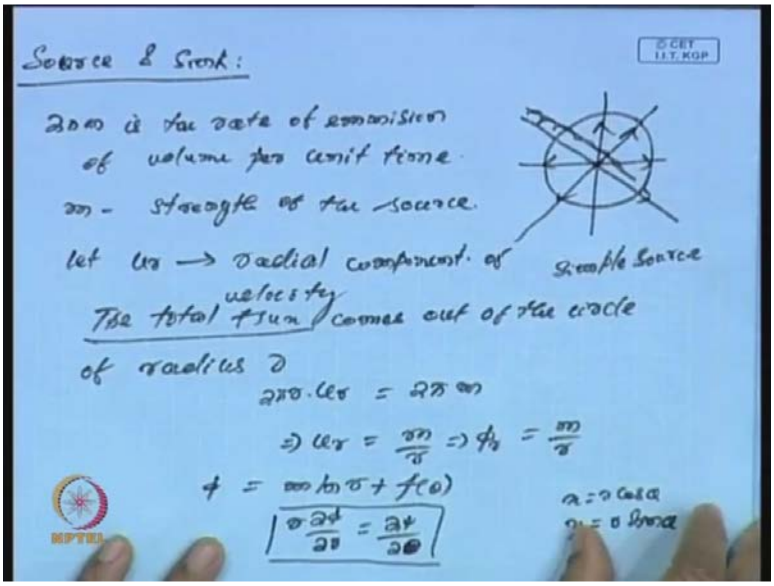

In marine hydrodynamics is source and sink, what is a source and what is a sink, in physical terminology always, we understand a source is something from where something, we say source, source means there is something from where, the things will be out. And if you say sink as if something is entering there, so in this same manner, if when a fluid flow, in a particularly, in a two-dimensional fluid flow a slope consist of a point from where the liquid will emerge out, and in the, it will emerge in the symmetrical in the radial direction out. It will go through from the point source, and it will symmetrically will come out, in the radial direction, and then this point will call this as a this point, we call this simple source.

Again if I will say, if 2 pi m is the rate of emission, of volume per unit time then, we say a minus the strength of the source, a minus the strength of the source. And if I say, if let u r with the, radial component of, radial component of velocity, because here the fluid flows in the radial direction, there is no flow in the tangential direction. So, what will happen the total the flux out of the total flux, which total flux, if I consider which comes out of the circle, of the circle of radius r when, we look at a circle, because fluid is emerging out, if I look at this how much total fluid, that will be emerging out, that the total flux will give me 2 pi r into u r, because I have considering a two-dimensional flow.

And that will be same as the total rate of the total emission, that is 2 pi into m, if I equate both the things, and my u r will be m by r, and what is ur? Which is nothing but phi r that this is the radial component of velocity, this is m by r which gives me phi is equal to m log r plus f of theta, where theta f of theta is a arbitrary function. Of a now again we know that, r del phi by del r is equal to l psi by el theta, because the velocity potential, and the stream lines they are related in a cylindrical polar coordinate, and x is a r cos theta, y is r sin theta. Then if phi is velocity, the potential and psi is the stream function, then r del phi by del r is del psi by del theta, and if I substitute for phi that gives me, that gives me del psi by del thetas.

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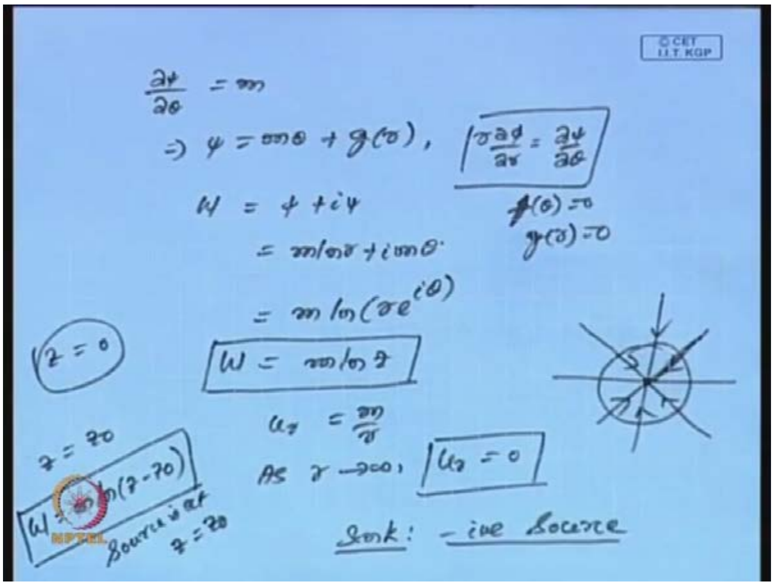

Is equal to m, and that gives psi is equal to m theta, plus g of r. Now, so what will happen, my phi plus psi, so W, I can easily get, that is phi plus i psi, and it can be easily set again be using r del phi by del r is del psi by del theta, that will, if I utilize this characteristics my g theta is 0, and my f theta is 0, and g g r is 0. And that will give you phi plus W is phi plus i psi, that will give me m log r plus i m theta, and that will give me if I put it, m into log r e to the power i theta, and which I can write as m log z.

So, if I have a source of strength n, then my complex velocity potential is W is m log z, now some of the characteristics of the source, is that ur, we have seen that u r is m by r, as r tends to infinity ur is 0. So, that means source at large distance, the velocity radial component of velocity is 0, whereas, no tangential component of velocity, in this case. That means, there is a low flow only in the local distance, local region there will be a fluid which is merging out but when it goes too far filed, there is no such flow, and again I will just say, if this is when the source z is at 0.

Now, if I said the point the source is at z is equal to z naught, then my W will be m log z minus z naught, I am just shifting the origin. So, this becomes the complex velocity potential, if the source is at z is equal to z naught, another point here I want to mention, a sink in the same line a sink is nothing but is a negative source, means that in a sink of the fluid, in case of a source the fluid goes out where, in case of a sink. The fluid will emerge to a particular point, from all direction in the radial direction, it will, if I just say the fluid in all direction, it will come to this point. This is, and this point will becomes a sink, and is so it is, often called a sink as a negative source.

In fact in a any potential flow problems, the source and sink concept is used very much and particularly, when you look at the fundamental singularity, in a flow problem particularly, if you look at the potential flow problems and sometimes, this sources or sinks are called the fundamental singularity, in a potential flow problem. And I will stop here, in the next class we will again discuss various problems in the presence of sources and sinks, and other singularity or in the presence of boundaries again using the same complex function theory approach, basically the complex velocity potential. With this I will stop today.

Thank you.