Marine Hydrodynamics
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Lecture - 39
Solution Methods for Boundary Layer Equations

Well come here to this series of lectures and made in hydrodynamics. In last lecture we have talked about boundary layer theory, basically our concentration was on the general two dimensional boundary layer equation derivations. And we have highlighted that the imitation of the boundary layer theory particularly the reason class of flow problem for which boundary layer theory will not hold good. Today only discuss about the solution processor for the boundary layer equations. Just before that we have seen that the solution procedures are very few problems in discuss associate to this discuss fluid flow has we can get exact solution or the close function solution.

On the other hand and that is basically for the dedicational flows, on the other hand when you look into boundary layer equation, even if in two dimensional boundary layer equation, it is not easy to get a close form solution. The reason is that weather presence of non-linearity in the conductive terms and that makes the problems more complex if there is non-linearity and that. So, because of that we have to the help of numerical methods, so it not that always, all kinds of numerical method we can apply. So, still we try to simplify the problem to reduce the number of variable or to reduce, bring the problem to the form of integral, which will be more easy to axis, handle or may be some times directly go for a numerical method like the final difference one.

Before going to discuss the various solution processor and the limitations, let us see what are various types of flow before we go into analyze in a particular flow problems. When it is better to know what type of flow it is and one has to check, because the movement one is clearable the type of nature of the flow, then it becomes much easy to handle a problem and after use a proper method for it solution.

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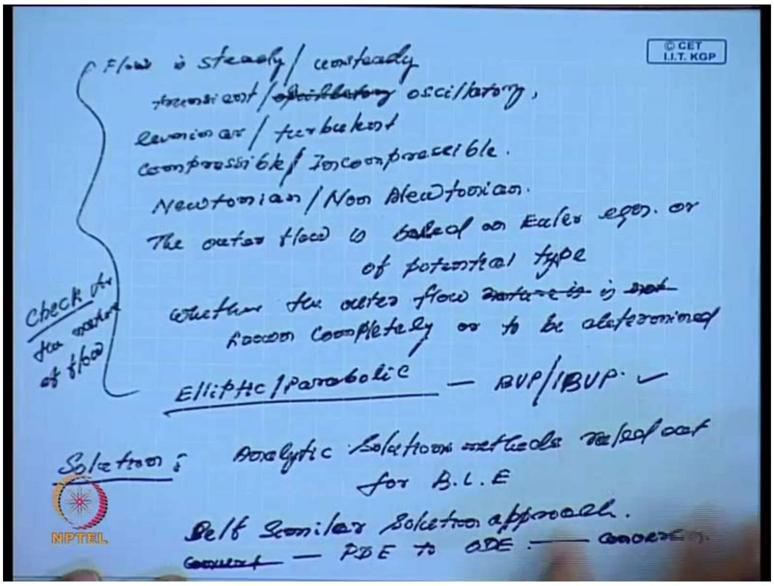

So, let see what are the what are the checks one has to have, before handling going to the solution processor. Basically one has to see that weather the flow is steady or unsteady flow is steady or unsteady, if it is unsteady or non steady flow and weather the flow is a transient or oscillatory transient oscillatory transient oscillatory. Then again weather the flow is laminar or turbulent, flow is compressible flow is compressible or in compressible. Then again the motion is, the fluid motion is Newtonian or Non Newtonian and again, if you have, you are telling which you flow which is boundary layer. The boundary layer theory applicable, particularly the outer flow, that case of outer flow that means, the flow beyond the boundary layer is to the outer flow is based on Ealer equation or of potential type.

In fact in some of the problems the outer flow nature is known and again weather the outer flow nature is known or not. Outer flow is known completely or to be determined, because the movement all this characteristics are unknown. So, because we all deal with him partial deferential equation associated what to say. So, all this characteristic will highlight, sometimes gives us the information, weather the flow is elliptic, the problem with an elliptic problem or a parabolic problem, because depending on elliptic or parabolic nature of the problem, we can say that the weather it will lead to the boundary value problem or initially boundary value problem, even if sometimes.

So, this is the class of problem will come across and one as to go for a check before attempting any problem for solution. Check for this, for the nature of flow, is check is a must like simplify the procedure and also the time one spends for understanding the flow pattern and then if you look into solution processor. Normally, as I have mention that boundary equation then non-linear equations and it is difficult to get a close functional solution, one has to take the help of numerical methods.

So, analytical solution is a ruled out or boundary layer equation, analytic solutions methods ruled out for boundary layer equations. However, what we can do we can always apply the self-similar solution, we can always get a self-similar solution self-similar solution approach. As we have seen that self-similar solution approach we have two things, often we convert the partial differential equations to ordinary differential equations.

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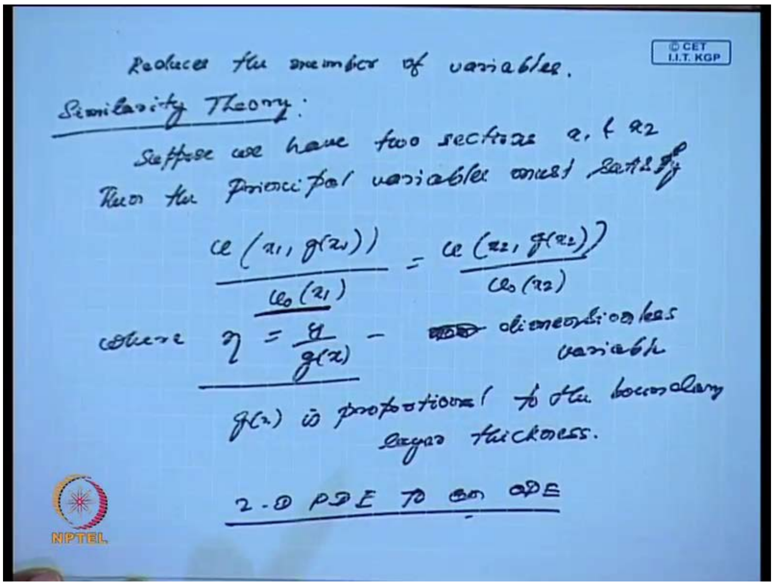

This conversion this conversions helps to great extent, because and also in some situations we reduce a number of variables, also it is reduces the number variables. So, basically what is similarity theory? Let us say similarity, what is similarity theory? Before going to know this similarity theory, the similarity theory tells us for two arbitrary sections, suppose we have two arbitrary sections in the flow flow of two sections let the sections be arbitrary x 1 and x 2. Then the principal variable must satisfy, then they must satisfy (( )) c u x 1, g x 1 by u knot x 1, u knot x 1 is the outer flow, is same as u of x 2, g of x 2 divided by u knot x 2. If this falls and were what is that? Were g is what? eta is equal to y by g x and this is an non-dimension variable, rather call it dimension less variable.

What is g x? in fact g x is of a nature and here what is happening? This eta y by g x and again this g x is proportional to the re nons number to the local boundary layer thickness. Often this g x is proportional to the boundary layer thickness and because of this variable what will happen? This y by g x 1 you put eta and we convert this, because we have to remove it here we have various components. The whole problem by doing this we can bring down the whole problem to the problem of that x and y variable, we can reduce to just a problem of variable call eta.

We can recast the whole problem into from a two dimensional partial difference equation to a one dimensional layer to layer a non-differential equation. So, it is quite possible that in many situation it is possible to converted two dimensional P D E to and an O D E and this principal also will hold good when there is no singularity in the boundary layer flow, but which specific u knot x, which is known. Particularly, u knot x is more side to this, similarity approach is mostly when you not take the solution associate the corresponding potential flow problem beyond the boundary layer. Now, with this, this is the basic assumption on here similarity method will be based on and will come to an example a little later.

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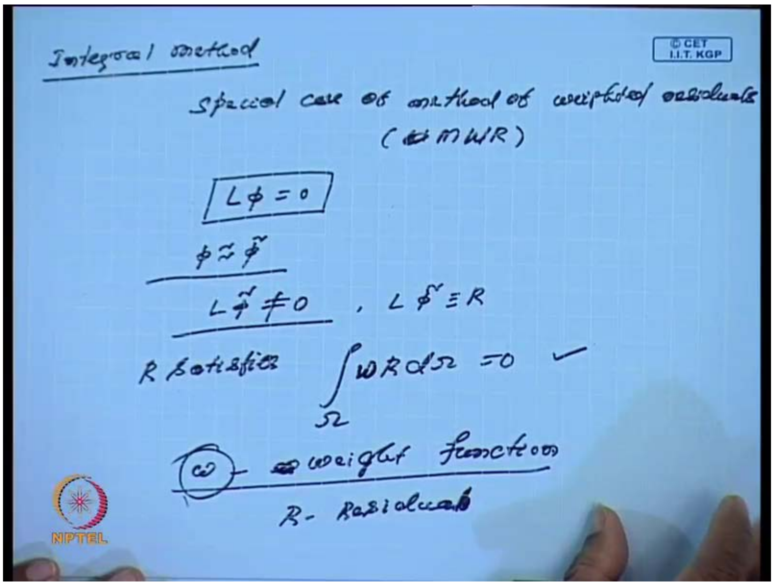

Before that let us go to what is the integral method? (( )) in case of singular, similarity transform, the similarity theory we are we are converting the problem to from a partial difference is equation to ordinary difference equations, basically were reducing the number of variables in the equations. In this case this is another approximate method, which is a special case just special case, this integral method, what we will call it is called special case of the method of the weighted residual.

What is the method weighted residual? Sometimes we call it a M W R method of weighted residual. These methods, suppose I have a linear operator, I have operator L phi that is basically the boundary value problem. If L phi is 0 or unknown function phi L is the differential operators, then we are a looking for a solution always, but we can do is, can approximate 5 phi by 5 Theta approximate by I think that and then we say that if I phi theta approximate solution of this, so it is obvious that because we are looking for phi theta is approximate is approximate solution to a phi.

Then it is obvious that level phi theta will not be 0, because we are this is phi theta is approximate solution. But what will happen? L phi theta will be identical to R and what is R? R is basically satisfy the R satisfies, this is a very simple for, R satisfy over the dominant omega w R d r, d omega is equal to 0 and this is what is called, w is called the weight function w is called the weight function. In fact this is the method of weighted residual and that is called the residual weight function and R is the residual R is equal to residual. Then there are (( )) method best turn on this weighted residual and basically depending on nature of w particularly, the weight function we always the method of a method of a weighted residual changes. So, there are couple of methods are depending on this nature w, that is the weight function.

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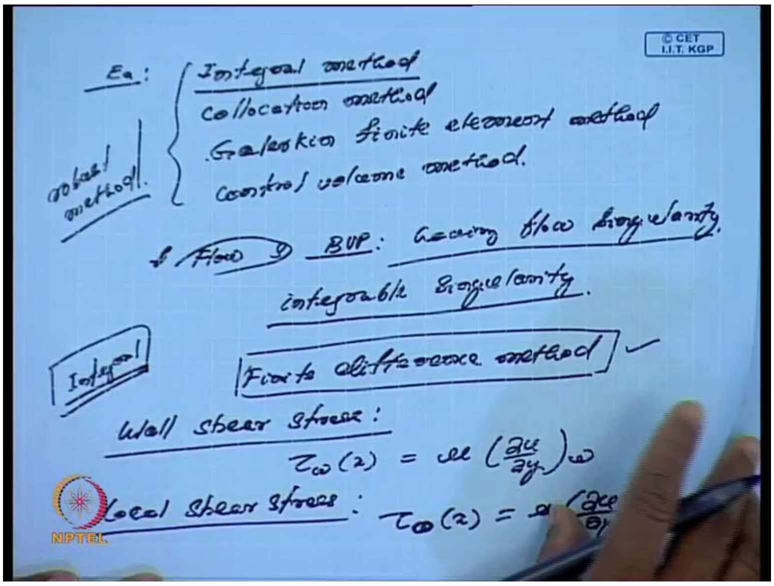

Basically, the methods come up and then from this we call four methods basically, integral method, the other method is called the collocation method, this are part of the, and then sometimes we call it as galerkin finite element method, the other one is called control volume method. In fact all this method, this integral, this method of weighted residual is a very robust method. One of the robust method and one of the advantages is that, because particularly in we have corner points, we have singularity and the boundary, while basically while dealing with boundary value problems and having flow singularity, having flow singularity.

This is a one of the robust method and here this flow singularity when I say in mathematical terminology, we may say that they have integrable singularity. This will be obvious when you come to problems and see and which is otherwise not possible by the standard finite difference method. It is very difficult to handle a singularity in a flow. Being even if the singularity integrable in nature by the finite difference, because conversions very slow by the finite difference (( )). On the other hand we shall be and other advantage that when you think of anything of integration will go for a summation kind of things.

So, there are associate totally much less and easy to handle, so almost all the boundary foundations are initial foundation can be always club together with the integral in this method, that is one of the u t of integral method of the, what we call the method of weighted residuals. So, with this understanding will go only discuss our impression will be an integral method and again before going to there is another method which is used for the boundary layer equation that is basically the finite difference method and that is the direct method, I am not going to the details of this when a finite difference method. All of most of you have knowledge in this, on this finite difference method, I am not going to the detail of this.

So, before going to, so this are the three methods, which are one is the method of a similarity a similarity theory, the other is integral method and in the third one is finite difference method. So, with this I will impassive basically in the give example, to illustrate the role of role of similarity theory and integral method in solving boundary layer equations. Here before going to solve a problem, let us define few terms what is wall sheer stress. I think while I have talked about, discussed I have already told the wall shear stress, Tar w is dependent mue Del u by Del y at on the wall boundary by w and sometimes we call it a local shear stress, if the wall is at 0.

Then if the wall it as 0, then we call it some time I call it has w x, which mue Del u by Del y at y is equal to 0. If the wall is at y is equal to 0, then we call this as Tau 0, that is call the local shear stress and then what is skin? I have already highlighted this motile again.

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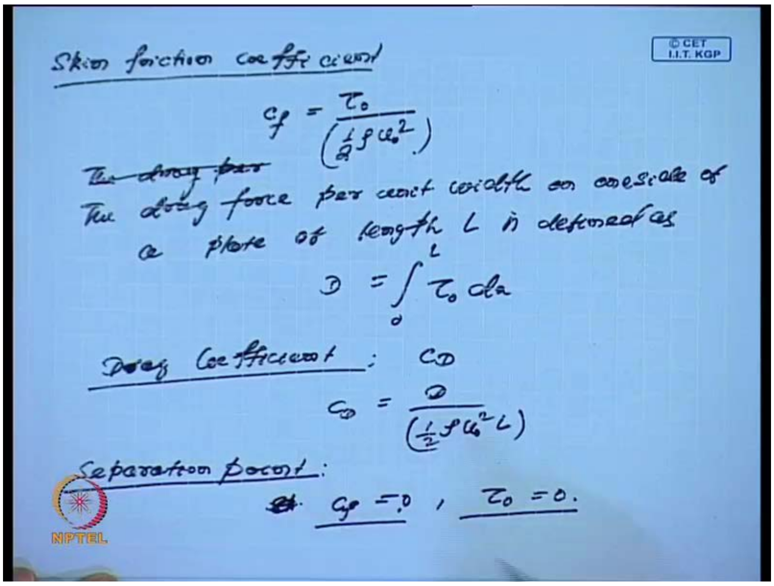

So, the skin friction coefficient the skin friction coefficient it is depend as the c f and this is given by Tau knot, that means global shear stress by half of 2 u and knot square. Then we define what is the drag per unit, basically call that drag force per unit width on one side of the of a plate of length and it is defined as D is equal to 0 to L, here already Tau knot tau 0 x d x, this is drag force per unit with one side of a plate. Then we has called drag coefficient and this is denoted by C D and C D is nothing but and is D by half of row u knot square by into L, this is called the drag coefficient.

So, this are the some, then what we say that method of a separation point. The point where the skin separation co-operation is the point where the skin co sufficient is 0, that means C f is 0 and in other words that wall shear stress c f 0 in other words we can say the wall shear stress, a local shear stress Tau knot is 0. This is the point where c f is 0 or we can say Tau knot is 0 and often this point is called a separation point, it can be seen that flow separation occurs in the region were pressure increases. So, flow separations occur at a region at there is an increase in separation boundary layer.

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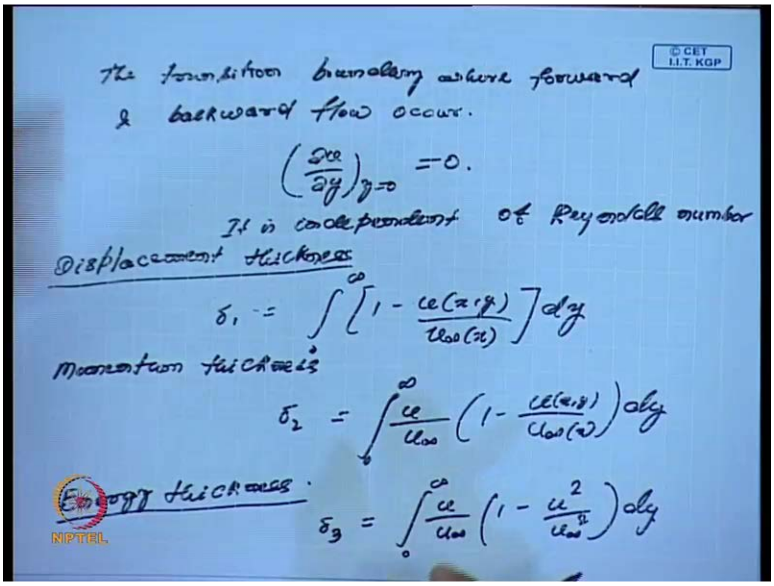

Another point is that, often the separation point it is defined as a point where there is a transition between at the backward and forward for take place. The transition boundary the transition boundary where forward and backward flow occur and as usual here you will have because (( )) will be 0 Del u by Del y, y is equal to 0 is 0. In fact it can be a really seen that it is a independent of the separation point in, it is independent of

Reynolds number. Now, with this I will think of few more things terms, that is what we call the displacement of thickness displacement thickness.

Displacement thickness if I call it as a Del 1 it is defined as 0 to infinity 1 minus u x y by u infinity x. This is the flow in the outer region of the boundary layer and this is d y and again, but called the movement of thickness or this is called Del 2 c is equal to again 0 to infinity, u by u infinity 1 minus u y infinity. Basically, u is a function of x y and this is function of x, because this is the behavior of flow infinity at a large distance and this is a d y. Similarly, we have an energy thickness; in fact there is class of problem, instead of going direct of u.

Many times we are interested in the calculation of this displacement of thickness, movement of thickness, energy thickness and that helps us in a training various physical content of interest, which will be obvious when you will work out on problem on this. This will call 0 to infinity this is called the energetic thickness, this again u by u infinity, u is function of x y and 1 minus u square by u infinity square d y, this is called displacement thickness. So, this quantity as a of boundary layer theory particularly when we solve by movement term integral method, then we this quantities (( )) significant role.

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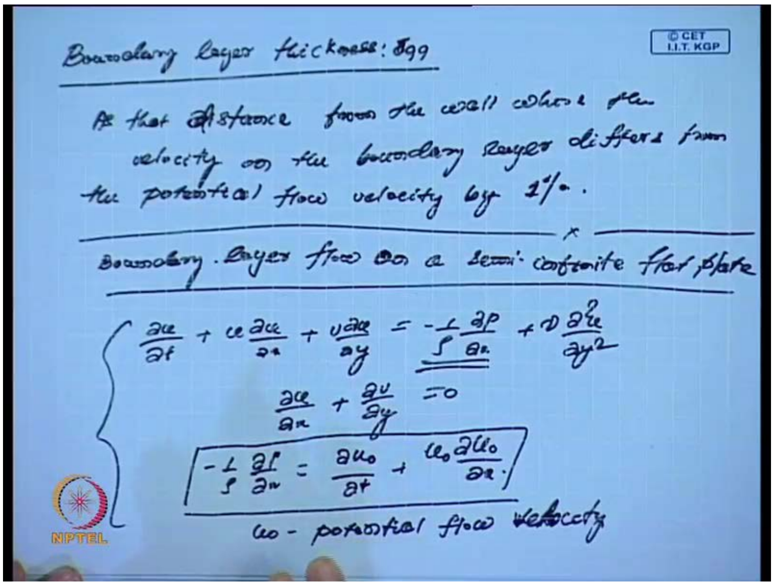

Then we have something called boundary layer thickness and that is called Del 99 that is boundary layer thickness. It is basically difference, it is regarded and the distance between, so it is regarded as that distance as that distance from the wall where the velocity on the boundary and the velocity on the boundary layer differ from the potential flow velocity. It is that point that its basically the distance from the wall were velocity boundary layers differs from the potential flow solution, potential flow velocity by 1 percent. In fact this definition is quite, it is not a very advertary, but it is one of the definition which is widely used Del 99 what we call the boundary layer thickness.

Now, with this understanding of some of the resica terms, now will go to the, to analyze the boundary layer flow, boundary layer equation on a semi infinite flat plate. Basically it is a two dimensional problem and if I look at the organic equation, we have Del a by Del t in a two dimensional boundary layer equations plus u Del u by Del x plus b Del u by Del y, that is minus 1by row Del p by Del x plus u Del square u by Del y square. This is plus we have the (( )) equation is Del u by Del x plus Del v by Del y is 0. Here by this minus 1 by row, this terms minus 1 by row Del p by Del x is nothing but Del u knot Del t plus u knot by Del x, so this is and unit is the potential flow velocity, this is the potential flow velocity.

Then if this is the total, the two dimensional boundary layer equations, now what will happen, what will happen to the steady boundary layer flow equation? If the motion becomes steady, then we call it as the Blasius solution, what did Blasius did 1908?

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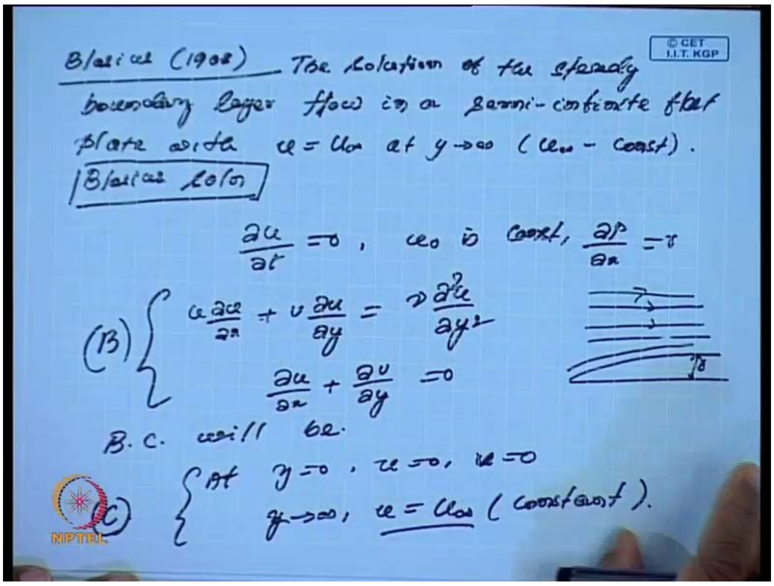

So, basically it is a solution of the steady on, provides the solution of the steady boundary layer equation in a semi infinite flat plate, in semi infinite plate with u is equal to u infinity at y times infinity. This u infinity is constant; these solutions were as called the Blasius solution. In fact call the first problem which has (( )) boundary layer equation when it was derived, this was the first problem for which a solution was obtained and in this case, obviously Del u the motion is steady we have Del u by Del t is 0 and again, since u knot is constant, then we have Del p by Del x is 0.

Hence, the boundary layer equation will be reduced to u Del u by del x plus v Del by Del y is equal to mue Del square u by Del y square and Del u by Del x plus Del v by Del y is 0 and here the boundary (( )). Boundary (( )), because that is y is equal to 0, at y is equal to 0 basically it is a flow. So, beyond this is the flow is like this, so know at y is equal to 0, u is equal to 0, v is equal to 0 and at y equal to infinity as y tends to infinity at u is equal to u infinity, this is a constant a stream line flow v is equal to infinity. Sometimes we call this as the, beyond this is pre stream, it is called the press stream velocity. Now, the solution of B, basically if I look at this one I called it as the B subject to C, to obtain the solution of this subject of C, what will do? I say that here we will follow the similarity method.

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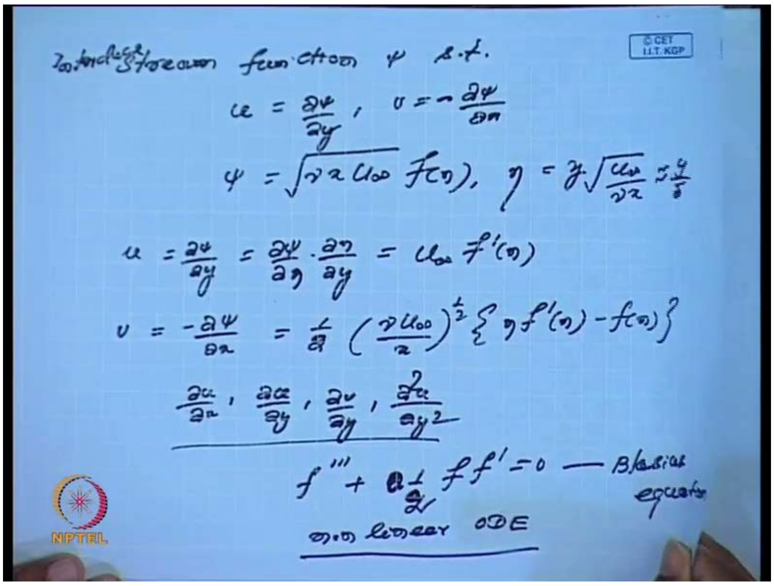

In similarity method what will do? We will introduced the stream function side, only introduced the stream function psi and in psi when you are introduced, such that u equal to Del psi by Del y and v is equal to minus Del psi by del x. Then if you set size is equal to nue x, U infinity root over into F of Eta, were Eta is equal to y times U infinity by nue x and this is almost y by Delta, Delta is the boundary layer thickness. In fact when I have talked about similarity theory, I have said eta is some y by g x and g x is a proportional to boundary layer thickness, in fact it is follow this and what I will do, I have brought in that a function F Eta.

So, what I will do, I will do by using this one function F Eta the problem which is u v and the variable say x and y, so recast the whole problem, that is in v into problem in F Eta, that is one difference equation little come up. So, that will help us significantly to solve this problem. So, what will do? If will take u and that is nothing but Del psi by Del y, u is Del psi by Del y and this Del psi we can call it is as a Del psi by Del Eta into Del Eta by Del y and this can be easily checked, that this is nothing but u infinity F dash Eta.

In a similar manor will kill v is minus Del psi by Del x and it can be checked that this gives me 1 by 2 nue u infinity by x to the power half into Eta F dash eta minus F of Eta. Just follow this in procedure, one will get in a similar manner will first of (( )) Del u by Del x, already Del v by Del y, Del v by Del y and Del square u by Del y square, just follow the simple procedure. If you do that and then substitute for u x, u y, b y Del square u in v, then in terms of F will get an only a single equation and F triple dash plus half F into F dash that is equal 0.

This becomes the two, the boundary layer equation reduces to this and the boundary conditions that will be reduced to and this is a non-linear differential equation, ordinary differential equations and which is known as double Blasius equation. So, as I mentioned in the beginning that by using the self (( )) theory, similarity theory, we can always introduced a function, so that will transform the boundary layer equations into just single ordinary differential equations.

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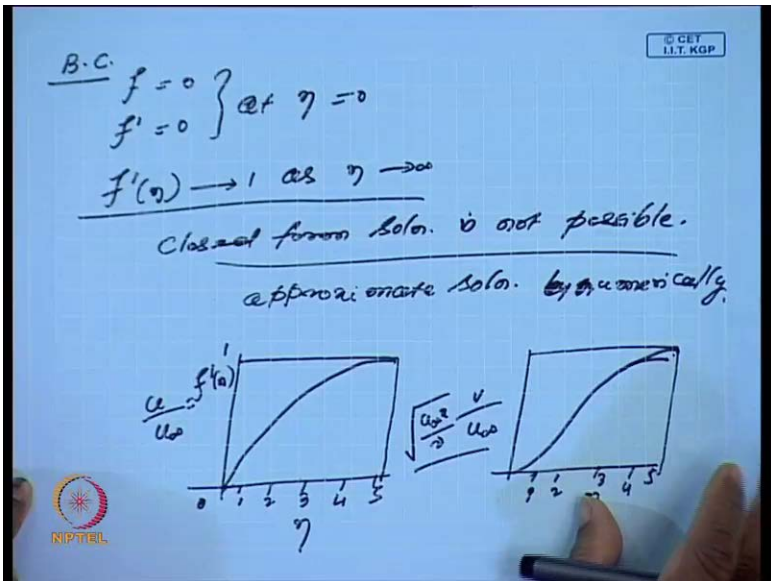

However, this ordinary differential equations is highly non-linear and it is not easy, difficult, it is not possible to get a close consolation and what will happen to the boundary equation? The boundary equation will be F will be 0, F dash is will be 0 at Eta will be 0 and you can see that F dash is Eta will return to 1 as Eta terms infinite, so this will be my the boundary condition they are the boundary conditions. So, that we are solving this non-linear difference subject to this and then it is not always possible, as I mention that it is this solution, closed form function is impossible, is not possible. So, what will do? We will go for approximate solution that means approximate solution by numerical methods.

One has to and in fact this solution method and obtained by applying the shooting method, hence the results I will just plot the result how it is. So, my u y, u infinite and u by u infinite which is nothing but F dash Eta and a plot will be like this, if it is 0, 1, 2, 3, 4, 5, then m dash Eta will be, if I call this, this point is 0 and this 1. Then may flow will be, this is the nature of the flow, u by u infinite will be applied this. In the other hand if I have another plot that is, this the longitude flow directions, along which longitude flow and if I look at the transfers flow v by u infinite, then and my v will be v by u infinite into u infinity x by nue, this is the transfers flow reduction.

Then I will have Eta on this side, this is Eta and then the flow reduction only this is also like this, so this will be, this results will be obtained at the end of the day. So, you can say 0, 1, 2, 3, 4, 5 so and in this case it is not possible, once we get to u and v will always get in a numerically, we can also get, so all shear stress (( )) coefficient and other physical quantity of interest again numerically. So, this is one example, one of the famous examples which is known as the Blasius solution, which is obtained by the use of the similarity theory. Another example will work out on boundary layer equation, here the solution of the will apply the integral approach.

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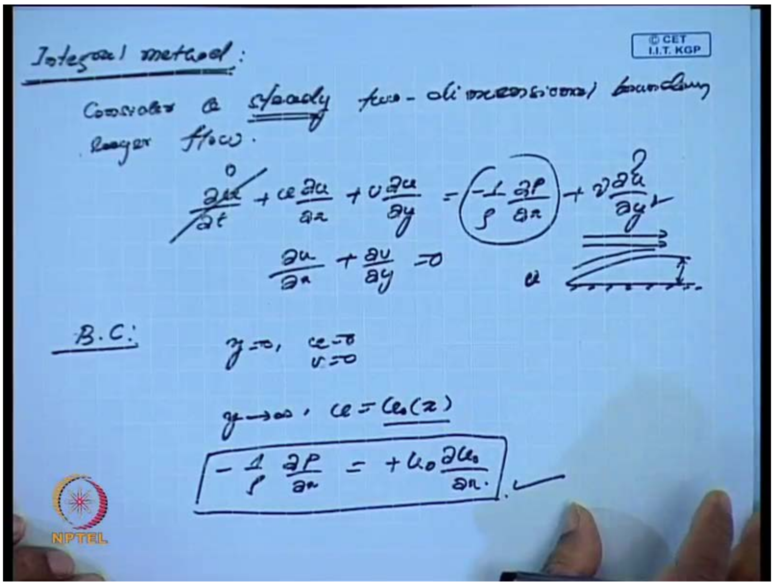

Integral method and this problem will analyze to obtain the solution integral method, but will do solution by a integral method, so will go for a plane incompressiable boundary layer, so solution of a plane name incompressiable boundary layer, problem will obtained by this method. Let us considered steady, consider again will go for a steady motion, consider a steady two dimensional boundary equation, boundary equation or boundary and steady two dimensional boundary (( )), but what are doing here I assume that, so I will have a basically equation will be Del u by Del t plus u Del u by Del x plus v del u by del v, which is equal to minus 1 by row Del p by Del x plus nue Del square by Del square, see then we have Del u by Del x Del v by Del y is 0.

Since, and this term is 0 this term is 0, since I am telling I am looking at a staedy problem, then what are the and let me see that look at the classer problem, this what are the boundary conditions. I look at a problem were this is subject to the boundary condition, that as usual I look at the infinity plate or plate is infinity and here u is equal to. So, here y is equal to 0, u is equal to 0, v is equal to 0 and y tends to infinity, I say u is equal to u knot x, so it is not a constant of previous example in the case of Blasius solution. So, what will do, again we have another think there is 1 by row Del p by Del x, because this term this will satisfy minus u knot Del u knot by Del x, this is plus.

This is another compressive guidance satisfy this u knot and since here it is, again the motion is steady, so u is equal to u knot, so in the process is p satisfy this. Now, to see farther what will I do? Will past integral this equation, that mean the boundary layer equation, if we integrate the boundary layer equation 0 to h, were h is point which is beyond the boundary layer, outside the boundary layer.

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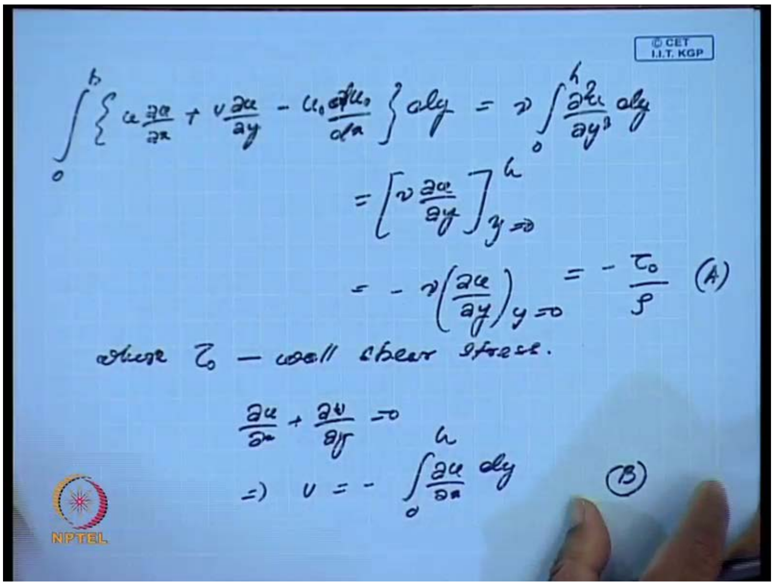

So, I integrate over 0 to h the boundary layer equation, that is u Del u by Del x, because I have the steady I have already taken Del t by Del t 0 plus v Del u by Del y minus u knot d u knot by d x, d y. I have told that h is a point, h is some were beyond the boundary layer and this will give me the right side nue 0 to h Del square by Del y square d y and the right side what will happen? This will be integrate by parts this will give a new Del u by Del y, this is y is equals to 0 to h and since h is beyond the boundary layer, h is beyond the boundary layer, so Del u by Del y will be 0.

So, this will give me minus nue, because beyond boundary layer u is equal to u knot and h is some were, so h is u knot x function of x it will give minus nue Del u by Del y, that is at y is equal to 0 and that is nothing but minus (( )), this is called the shear stress or wall shear stress, were Tau knot is local shear stress or we call the wall shear stress. Further as I am already mention that Del u by Del h, Del u by Del y at y is equal to h 0, because it is beyond the boundary layer and beyond the boundary layer u is equal to u knot x and (( )) that is 0. Now, if I look at the second equations, the second equations is Del u by Del x plus Del v by Del y is 0 and from this I will get my v minus Del u by Del x, b y this is 0 to h, so this is may be. Now, what I will do, I will substitute for this is v this is A, I will substitute for B particularly v here, if I substitute in A in value of B, then what will happen?

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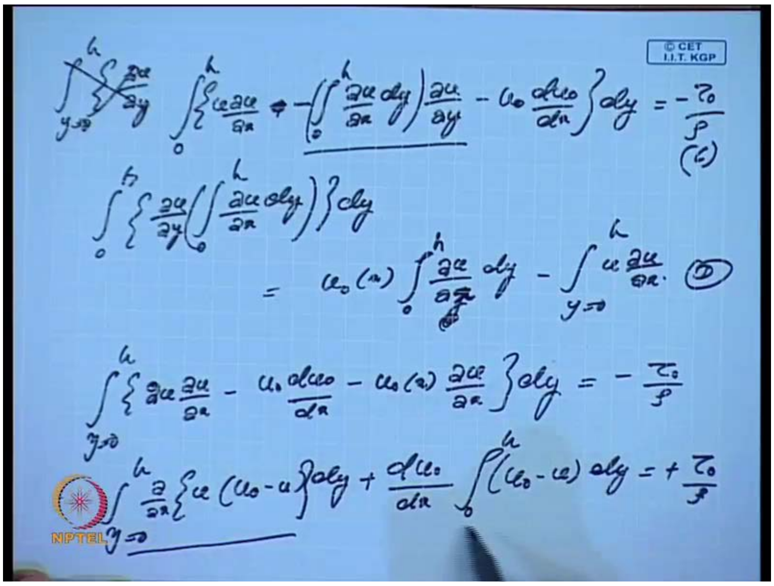

If I substitute for B then it will be y is equal to 0 to h, this will be Del a by Del y, u Del u by Del x sorry, 0 to h u Del u by Del x plus b is nothing but minus 0 to h Del u by Del x d y into Del u by Del y minus u knot d u knot by d x, d y minus Tau knot by row. Now, let me analyze this term, this term is nothing but 0 to h, this is Del u by Del y into 0 to h Del u by Del x d y and d y. This easily can we seen that this will give me u knot x, 0 to h Del u by Del y, just a integration by parts and d y minus y is equal to 0 to u Del v by Del x.

Now, I will substitute for this, this is my we call it C, substitute for D for the second term from D in C and if I do that I will get it y is equal to 0 to h and will give me 2 u Del u by Del x minus u knot d u knot by d x minus u knot x Del u by Del x. This will what I will get and d y, but this will come here u knot x Del u by Del x this is called Del u by Del x.

So, this Del u by Del x this term is there, this is there and unit is there, so this is what I will get and this is nothing but minus Tau knot by row. So, now this equations if I introduced the, I can re-write this is h, y is equal to 0 to h Del by Del x, this is u into u knot minus u, d y plus d u knot by d x into 0 to h minus u knot minus u, d y and is equal to minus Tau knot by row, rather this is plus Tau knot row.

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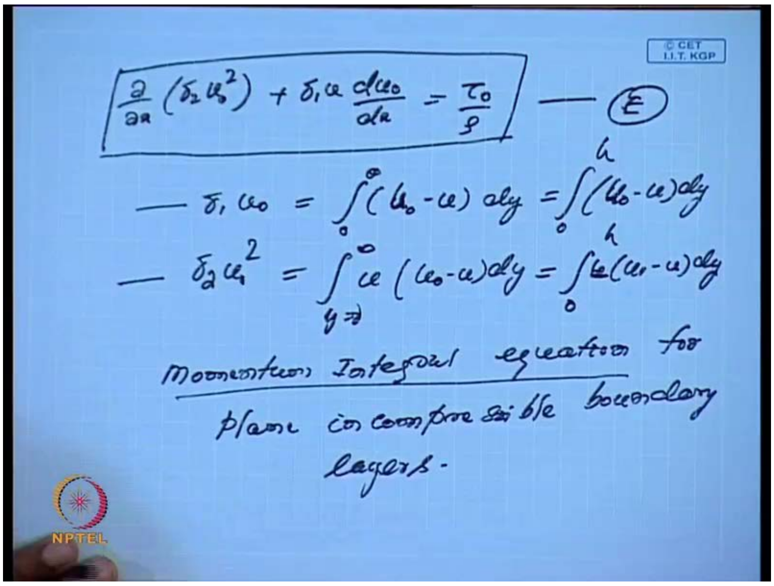

This equation, if I now apply the definition of displacement thickness, this equation will give me Del by Del x, Del 2 u knot square plus Delta 1 u, d u knot by d x is equal to Tau knot by row, in fact the whole equation and where I will say my Del 1, u knot is 0 to infinite, u knot minus u, d y and which is same as 0 to h, u knot minus u, d y. The reason is that because h is beyond the boundary layer, so beyond h contribution will be 0 because u will be u knot, so that is y 0 to infinity same as 0 to h is infinity. Similarly, Del to u knot square is nothing but y is equal to 0 to infinite, u knot minus 0, d y and that is nothing but 0 to h u into u knot minus u, d y.

Same thing this two definitions I have given, this is called the displacement thickness and this is called the momentum thickness and this equations is called this is equations E, this is called the momentum integral equations this is called movement term equations or plane in compressible boundary layer. So, this is what I was telling that final we have brought it in a form of differential equations in terms Del 2 and Del 1 and if we solve it everything use sign of integration here. So, once u is in sign of integration, so we will call it as integral equations, since momentum term is this displacement is enrolled we call it as momentum integral equations. In the next class will see how the solution of this equations can be obtained, because initially I had to deal with here step up partial difference equations subject to certain boundary equation.

Now, everything is converted to just one integral equation, what I was telling the beauty of integral method. In next class will see how will obtained the approximate solution of this integral equations and in fact this is one of the robust method to deal with a flow problems were singularity, close singularity may occur at just nicely a solution only. Here if there is some closed similarities here, this u or, because all these are integral, as I told that there is a singularity there in terms. So, these terms will take care of those things and in the process is easily get the solution. In the next class I will stop here and in the next class will talk to will talk about how will get the solution of this in an approximate manner. This I will stop here.

Thank you.