Marine Hydrodynamics

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Lecture - 35

Analysis of Basic Flow Problems (Contd.)

Welcome you to this series of lectures of in hydrodynamics. In the last class we were talking about various examples of viscous fluid, viscous in compression flow, where we have consider that flow between two parallel plates or flow in a channel. In one after that, we have just briefly discussed about, what will happen in there is tube flow in a tube. I will repeat this in again for clarity, so basically will look into the flow in tubes.

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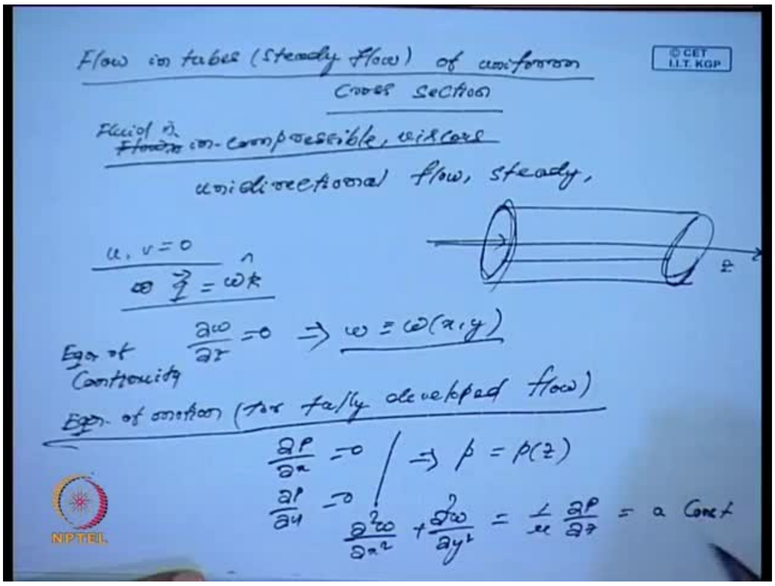

Basically this is a steady flow, flow in tubes of a uniform cross section. So, in this case, I write just, again this are all kind of fully deployed flow, so I will have in the flow is in compression in flow, is in compression, flow is in compression. It is viscous, fluid it is viscous and whether, I will say the fluid is in compressible and viscous and then we have we are considering the unidirectional flow. Assuming the flow is steady, then and I assume that since I am considering a tube.

Since, I am considering a tube, suppose a regular steady wave circular tube I have to start with. So, I will considering the tube the z axis, is allowing a generator of the tube that is the z axis, this crosses an along x and y axis. So, here I will assume that the velocity u and v are 0 and flow is along the z axis that means q is equal to w k at… Further so if I assume this, moment I assume this, then from the equation of motion I will have del w by del z is 0, necessary equation of continuity. One side del w by del z is 0, that gives in w as a function of w x y at independent of z.

Further from the equation of motion because fluid is in compressible in this we said and since we are our set terms are neglected because I have considering a fully developed flow. So, I have fully developed flow, so here in this case. What happen in del p by del z? In del p by del x is 0 that is component gives a because 0 and del p by del y is also 0 and that gives me p as, is a function of p z.

Further I have from the z component equation of the equation of the I will have del square w by del x square plus del square w by del y square is equal to because I am neglecting the gravity term. Basically the body forces or it is a it can be as we have analyzed in case of a fully developed flow and in the motion in steady. So, I will have 1 by mu into del p by del z. I again, I just say that since it is a w z function of x y w z function of x and y and p is function of z. So, this as to be a constant and this constant, let me call this as a constant. So, basically in the process what happened?

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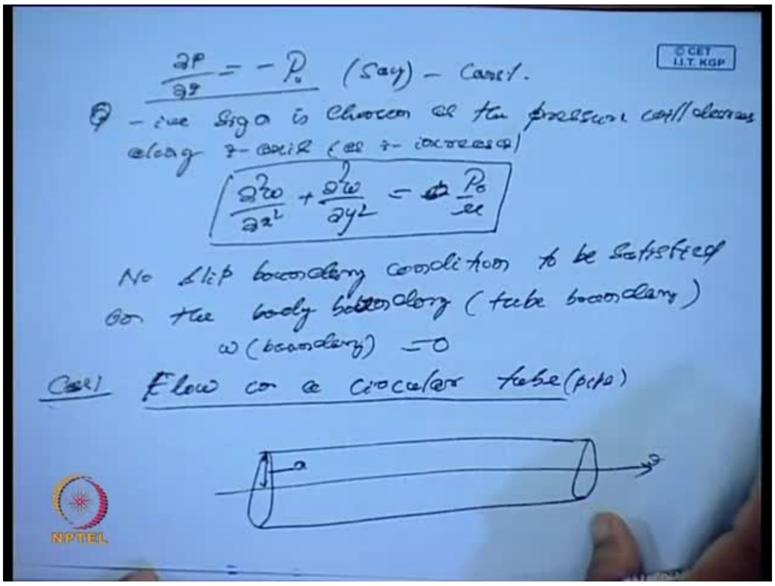

I assume that del p by del z is equal to minus p naught, say and these p naught is constant, I have taken a negative sign is as usual. As the pressure gradient will decrease as pressure will decrease, a pressure will decrease along z axis. Basically as z increases, if you move along the generator, basically in the pressure decreases, so taken this as and this is again a constant. So, then what will happen? My my equation in w del square by w by del x square plus del square w by del y square is equal to minus p naught by mu.

In fact like in the last time whatever I have considered the, in case of one dimensional or even if in these, this is a kind of proportional equation overall in a electric equation as a partial difference in equation. It is an type partial differential equation because of this form of lesson on these p naught by mu is a constant. So, these kind of flow are a elliptic in nature, often we ask this question that what kind of flow it will be, if in the flow is a fully developed and steady, then it is elliptical in nature. So, since it is the flow in elliptic in nature, it will be Poisson's equation will have a boundary conditions method equally sides.

In the other words will have the boundary condition will mostly the boundary condition, that will be satisfied on the boundary of the tube. Boundary condition has to satisfied, to be satisfied on the body boundary. In this case in the body boundary is a tube, boundary of the tube is the surface of the tube and this and this will be a… So, that means on the w on the boundary has to be 0, now I will compute this specific example. Suppose, I consider the cases, two cases will consider. Case one couple of cases will consider, the case one.

Flow in a circular tube suppose, let me consider as a flow in a circular tube. If I look at the flow in a circular tube, basically or called a pipe this is the direction, this is a large pipe as I say the generator is along the z axis. And then the pipe if j of radius r are is equal to a. This is called the radius of the pipe and this is the boundary. So, what will happen on and again in this case I have two things. In this pipe, what will happen?

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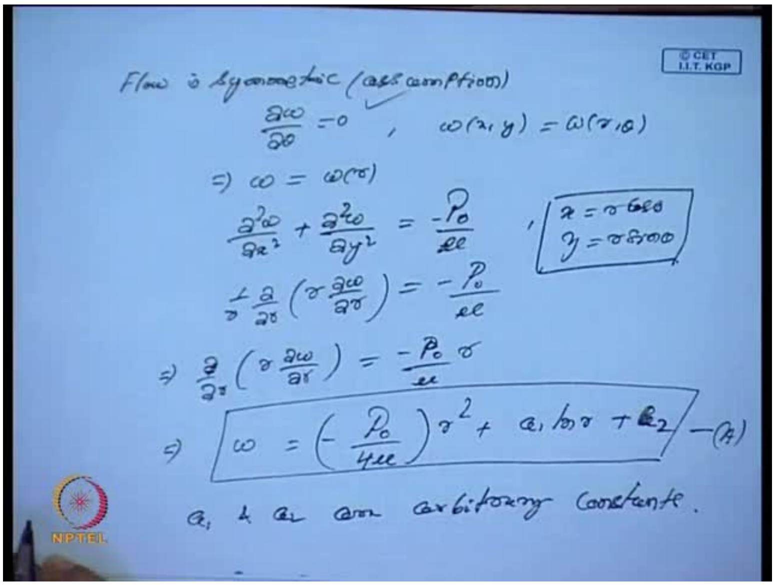

Then we have, since the I will assume the that the flow is symmetric. Once I assume the flow is symmetric that means del w by del theta is 0. This is the assumption, once w by del theta is 0 because our w as a function of x y and which is a function of again r theta and del w by del theta is 0, means w is equal to w r. And if you look at the because by Nic equation, by Nic equation by an equation is del square w by del x square plus del square w by del y square minus p naught by mu.

This becomes if I substitute for w is equal to w r basically, so in the co-ordinate system then this will give me 1 by r because del w by del theta is 0. So, it will give me 1 by r del by del r, r del w by del r minus p naught by mu. This is of the converting x y because here x is r cos theta, y is r sine theta. If you substitute for these, then utilize this characteristic that del w by del theta is 0 because of the symmetric flow is symmetric, characteristic of the flow then will get this.

Again p naught by mu is a constant, so that will simplify is to, if I do that 1 by r naught will give me del by del r, r del w by del r minus p naught by mu into r and if I integrate it then, I will get. You will get w will get minus p naught by 4 mu r square plus a 1 log r plus b 1 a 2. So, this will give me the general form and it has flow in a tube. Let me call this as a.

Now, I have two conditions a 1 and a 2 are arbitrary constants and these constants are written by unknown slip boundary condition. This is very straight forward integrate it twice and then you will get the one acquire an integration constant, but if I do that now, what I will do? I will use that. What is the boundary condition? The mostly boundary condition will be because I have cylinder, which is of radius a.

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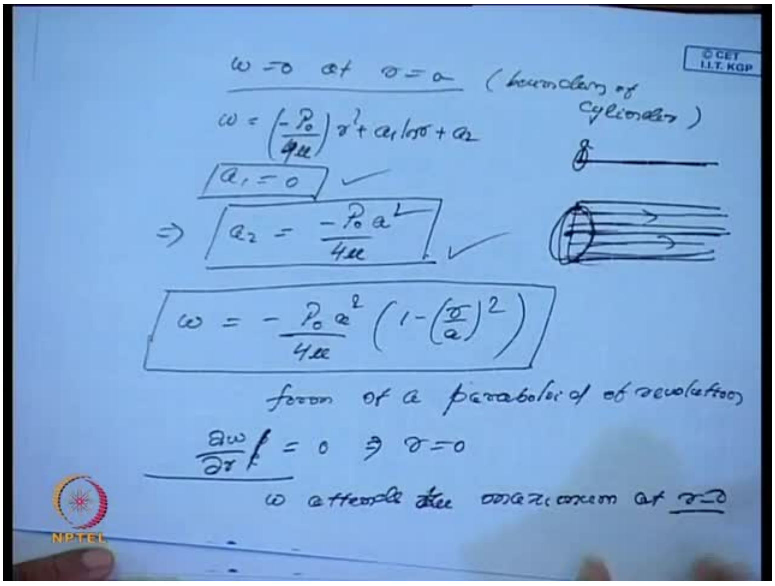

So, w is equal to 0 at r is equal to a, that is the boundary of the cylinder. This cylinder boundary there is no flow, no slip boundary condition is satisfied. Once say w is equal to 0 at r is equal to a. That again another condition it comes, what happen because may w has to be finite because I have r is equal to a and what is, in this cylinder along the generator of the cylinder, if this is a radius a and this what happen when w is actually minus p naught by 4 mu into r square plus a 1 log r plus a 2. What exactly happen, when r is equal to 0?

This flow would be unbounded, but I do not have any singularity, so simulate this flow type simulate in the flow. Since, I do not have a sore type simulate my flow is a uniform flow and it is fully developed flow. So, my a 1 has to be 0 because and I am not looking for a as inverse source in the flow. So, that gives me a 1 is 0. Again w is equal to 0 at r is equal to a and w is equal to 0 and r is equal to a gives me a 2 is p naught by mu 4 mu into a square. So, this my a 2.

So, if the two boundary condition, one boundary condition is that w is 0 at is r is equal to a, this be a 2 and assuming that the flow is bounded. There is no simulative in the flow that gives me a 1 as 0. If I substitute for this, then my w becomes minus p naught by 4 mu. I will take a a square common and I will have 1 minus r by a whole square. This main w, that is rotational flow general equation. This is which is the, this is the exactly the form of a parabola of revolution.

Again one can see that what will happen? Del w by del r at r is equal to 0 del w by del r sorry, del w by del r gives me 0 because in this part is a constant r. It will be 2 r this into r, so 0 at r as implies which imply r is equal to 0. So, that means this gives me that w attends in maximum at r is equal to 0 and r is equal to 0 is the center of the cylinder. So, as the center of the cylinder particularly along the generator, the flow speed can be maximum along this flow will be maximum. This looks like a parabola at a regulation, so here there is no flow here. There is no flow, what are these enter the flow speed is maximum. And then what will be the maximum speed in this case?

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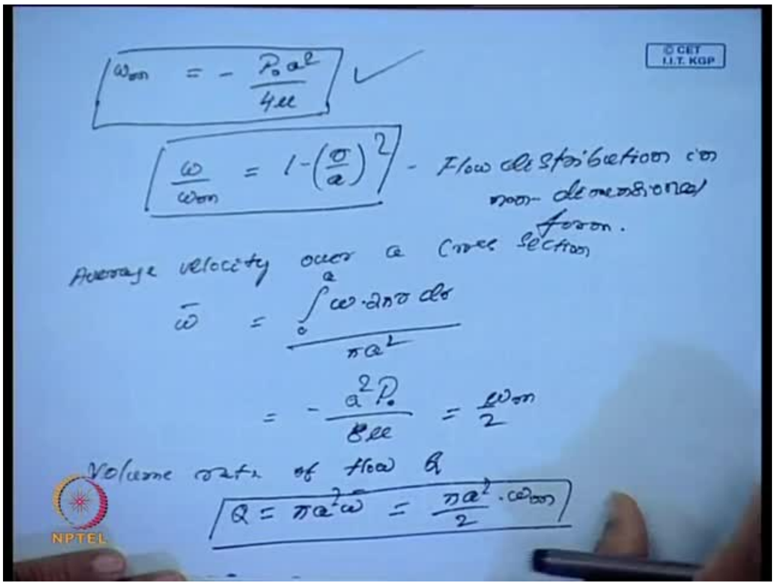

Then the process in w m at r is equal to 0, so it will be p m will be a minus p naught a square by 4 mu this becomes the maximum speed at the center of the cylinder. Hence, if we put in the non dimensional form, then w by w m gives me a square by 1 minus r by a square. This is the non dimensional from of the flow distribution, non dimensional flow. Now, if I look at the average flow speed average velocity over a cross section, that I will call it w bar and that is nothing but 0 to a, because a is the radius w into 2 pi r d r by 2 pi.

Pi a square, pi a square is the cross section of the cylinder and that gives me minus a square p naught by 4 mu 8 mu and that is nothing but 1 by 2 value can 1 by 2 times w m.

That is a w m by 2, so average velocity over any cross section is given by this. Now, then if I look at the volume rate of flow, now that is equal q, which is equal to pi a square w bar. And that is equal to pi a square by 2 into w m that is q this is the volume rate of flow. Now, if I assume that what is my w m p naught is basically d p by d z.

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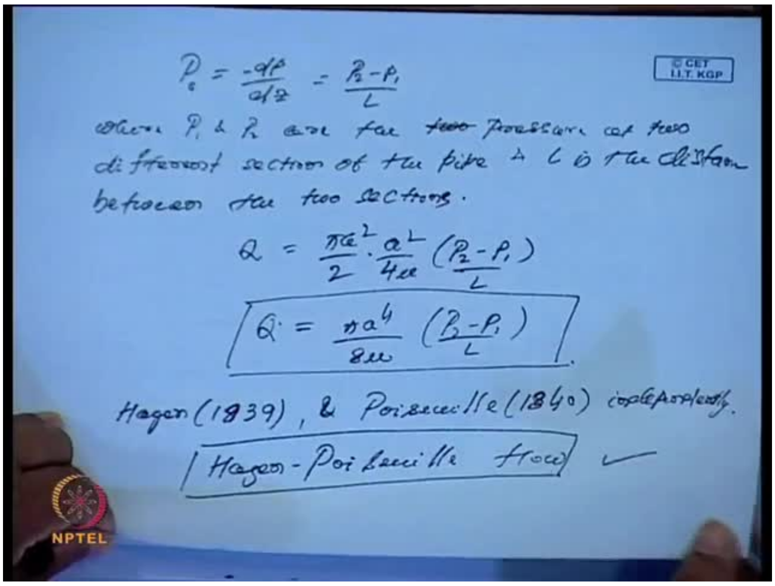

P naught is d p by d z d p by d z is minus p naught. So, and this I call it as p 2 minus p 1 by l p 2 is the at any point is the difference of pressure at along the generator at two different points by p 2. And then and l is the distance between that two sections, where on the pressure on the pressure at 2 different location two different section of the pipe l is the distance between them, between that two sections. If this is the case, then what happen, then in q can be written in terms of p 1 p 2, that is pi a square by 2.

This is into a square by 4 mu into p 2 minus p 1 by l and that is nothing but pi a square sorry, w m i a square w. Then this will give me pi a square by 2 pi a square by 2 into w m and w m is… So, this is pi a square by 8 mu into p 2 minus p 1 by l that is a q. This is what the flow is. This is what the volume rate of flow and this is relation. In fact this relation was given by Hagen in 1939 and then by Poiseceille in sorry, this was a 1839 and that then by Poiseceille, 1840 independently.

And then this lesson will determine the.. Often, this relation is used to determine the flow of viscosity in a fluid and this is this flow is called Hagen Poiseceille flow in the literature. So, in the flow in a circular pipe is repeat twice the Hagen Poiseceille flow.

This understanding on this when it is a circular tube in. Now, let us look at what will happen if the tube is annular in nature?

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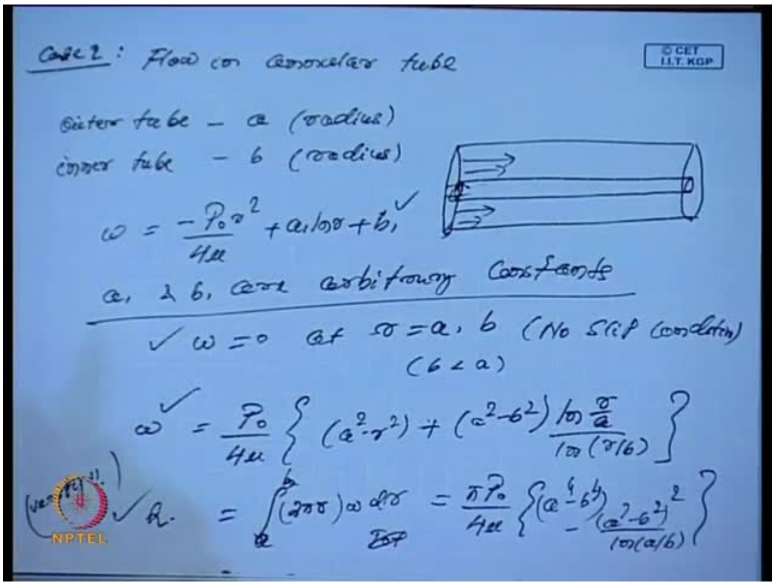

Flow in a case two if you just mere extension of this case one, flow in an annular tube. That means I have 2 tubes, the outer tubes and the inner tubes. The flow is a basically this is as solid tube and this flow between these two tube. So, if that is the case, that means I will have like at two tubes. In the outer tube is of next one outer tube for outer cylinder is of radius a and inner cylinder in a tube is of radius b. In this case again the flow is assume to be fully developed flow and in the process will have again. This we assume that on the same assumption when w will be minus p naught by 4 mu r square plus a 1 log r plus b 1, where a 1 and b 1 are arbitrary constant.

These constants as to determined using the condition is no slip condition on the wall boundary and my wall boundary reached w is equal to 0 at r is equal to a and b. This is no slip boundary condition, this is a no slip condition and if w is zero at r is equal to a and b. I have two unknowns say a 1 and b 1 and here of course, b is the inner cylinder and b is the radius of the inner cylinder. So, in the process b is less than a, so so in a process if that is the case, then by substituting by using the condition that w is 0 at r is equal to a and b I can obtain a 1 and b 1. Then we can easily see that I can write it, can be easily derive that w is equal to p naught by 4 mu a square minus r square.

I am not going to detail because it is simple plus a square minus b square and log r by a by log r by b. Once I know w, then I can easily get q that is 0 to 0 to a 2 pi r into w d r because here is a. So, along 0 to a 2 will have basically a to b because that is the fluid region. I can call it a to b and then this will give me the total flow volume of fluid. That is a going through this, and that will give me as pi p naught by 4 mu. And by this can by which at to a fourth minus b fourth minus a square minus b square divided by square by l n a by b.

This is the q will get parallel data flow and this is the relation, these two relations and we have obtained directly from using this from these using two boundary condition. This is the details are left as an exercise to verify this, can be verified verifying this this is the case of flow in a annular tube. Now, this this two cases will consider fluid flowing in a tubes of a circular cross section. Now, if I look at a flow in tubes, of non circular, of a non circular cross section.

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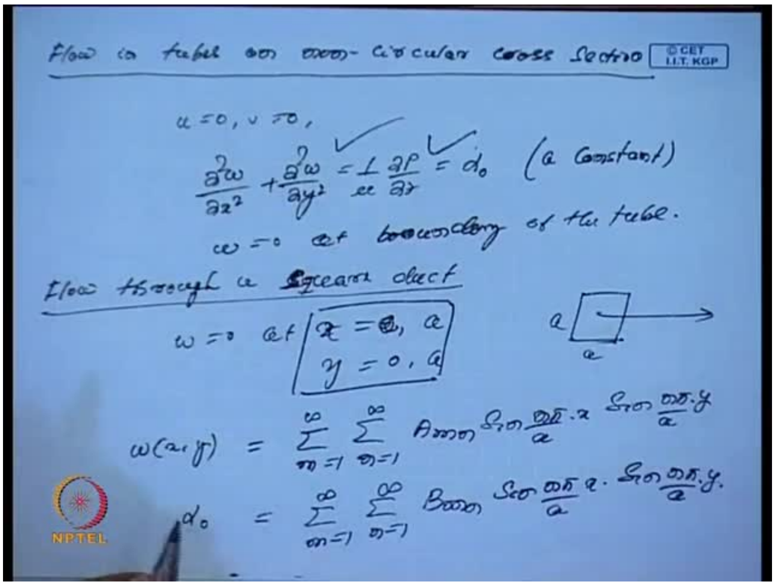

Again I am looking at a fully developed flow, I will have the same thing because a if I recall that I have two dimensional flow, here three dimensional fluid domain and I have a the generator is along the z axis. May have u is equal to o v is equal to 0, then I will again another same as assumptions on the for the fully developed flow. I will have del square w del x square plus del square w by del y square is equal to 1 by mu del p by del z, which is again a constant and and I call this as alpha naught, call it as a alpha naught a constant.

Then as a case of a and again I will have closely boundary condition w is equal to 0. At body boundary, at the boundary of the tube, let me consider two cases. The first case I will consider the case of a flow through a circular square duct. That means I have a tube which cross section is like a square instead of a circular, it is like square of side a. So, if that is the case, I will have to look at this flow this equation has to satisfied the body boundary condition.

That means w is equal to 0 at z is equal to a naught z is equal to a x is because this is the cross section z is along direction of the flow z is along the generator. This is the section of the front section of these cylinder and then at this is at x is equal to a 0 and a and y is equal to 0 and a. So, these are the flow condition will have and then if you test to satisfy this, this is the covering equation of alpha z is a constant and my domain is a z is along in a.

Now, it is like a solving a Laplace equation, Poiseceille equation. I have write to this boundary condition if I look at the solution from we can easily see that if I write w x y regular sigma m will be 1 to infinity sigma n equal to 1 to infinity A m n sine n pi by a into x into sine n pi by a into y. If I put w in these form, then I can easily see that w is equal to 0. At x is equal to 0 w equal to 0 at x is equal to y is equal to 0.

Further I have at x is equal to a, this is this is a sine in pi. That this is also 0 and at y is equal to a also this becomes n pi sine n pi, that is a 0. So, this is the most general form of the solution. Similarly, I can have for this expanse sine series expansion for alpha naught is a constant in terms of a similar way sigma m is equal to 1 to infinity sigma n is equal to 1 to infinity b m n, sine n pi by a into x sine n pi by a into y. Then this alpha naught is a constant, then I can get B m n. Then my, if so if I can get a m n and B m n. Then I will easily no for exactly is happening to the w so if I go to the. B m n derivation, then B m n can be easily find out the B m n is 4 alpha.

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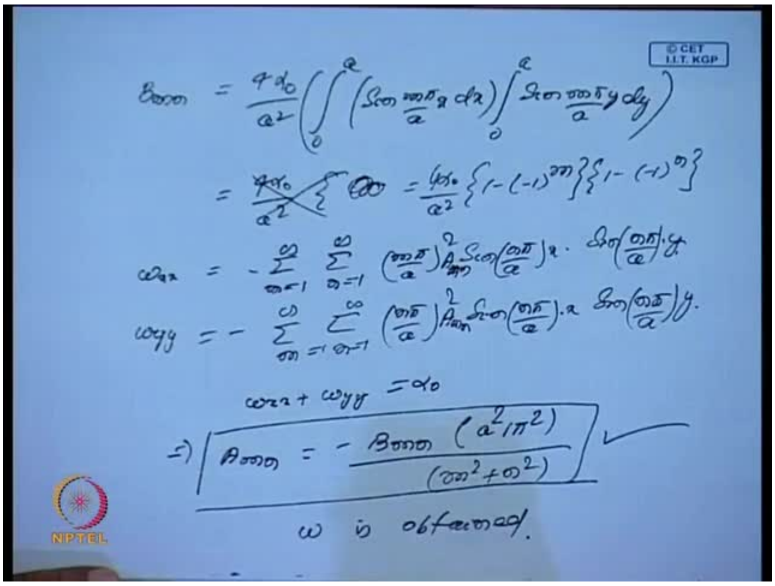

These have been telling for alpha naught by 4 pi by a square integral 0 to a sine m pi by a into x d x and again integral 0 to a sine m pi by a into y d y. This is what exactly will get and these gives me 4 alpha naught by a square and that is 1 minus sorry, 4 alpha naught by a square 1 minus minus 1 to the power m into 1 minus minus 1 to the power m. This is my B m n and hence the B m n is known. Then, this has to produce, this expansion of the fourier series expansion of the power w.

And alpha naught in the equation Poiseceille equation, that gives me may because this is a I have w. If I look at what is w x x, that will give me sigma m is equal to 1 to infinity, this sigma n is equal to 1 to infinity. Then I have n pi by a m pi by a square, then I will write minus sign n pi by a x into sine n pi by a into I. Similarly, wyy minus sigma m equal to 1 to infinity sigma n is equal to 1 to infinity n pi by a square sine.

This will give if I substitute for a in a Poiseceille equation and Hagen equate to the alpha naught where B m n are known. There is a w axis a m n, then I will get easily by substituting for these in a wxx plus in wyy is equal to alpha naught r, alpha naught r already given. I will get A m n, this minus B m n into a square by pi square divided by m square plus n square and this is may A m n in where B m n is already obtained.

Once A m n is this A m n's and B m n's are obtained, we have obtained what is called w? This w is obtained because that all depends in we know Amn we know Bmn. Hence, we not develop, hence w is obtained then what will happen to problem, the mean flow?

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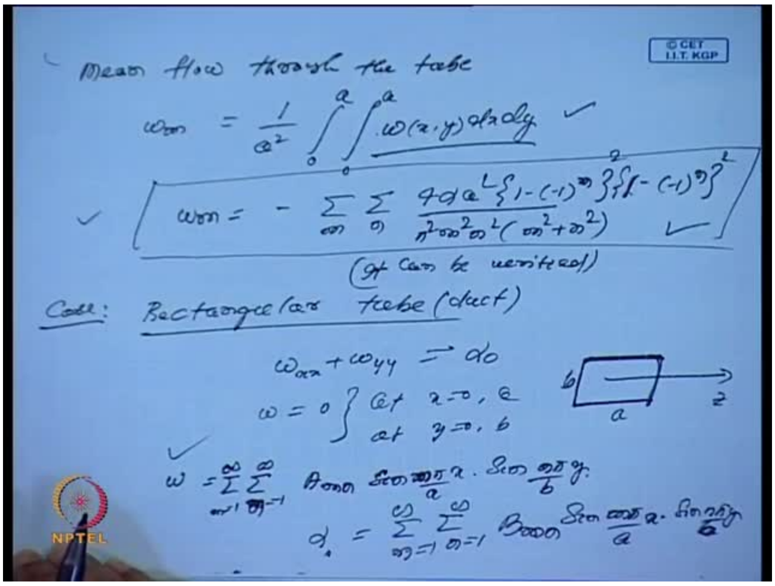

Through the tube the mean flow through the tube becomes that is w m. Again that is 1 by a square 0 to a again 0 to a w x y d x d y and the w is nothing but q. Because it has to other components as 0 and that gives me minus sigma m sigma n for alpha a square by m square n square into m square plus n square into pi to the power by m n. So, let say m square n square into pi square B m n by alpha naught is, so that will give me into 1 minus minus 1 to the power m into 1 minus minus 1 to the power n. This is the square, this is the square, this is what I get.

This is mean w and this is gives the mean flow in the tube. So, what is about I was a looking for? So, the once I know w first is what w I get the mean flow. That is mean w this can be verified, this can be verified I will write to going to this, so we have seen that not only the case of a circular tube, in case of a squared duct also we can get. What is the, how the flow looks like and what is the solution associated, what is the flow distortion in the flow of in the tube? Now, let us case another case will take up a think of like a of rectangular tube.

Although the analysis similar here, also we have one of the same assumption. What we have done in case of square tube, we have wxx plus wyy. It is equal to alpha naught alpha naught is a same as we discussed and in then we have w is equal to. At x is equal to 0 and a because and this is equal to at y is equal to 0 and y is equal to b where a and b at the cross section. Because, this is the front face, this of length a, breadth b, this is the tube and tube is along the z axis. This is the x axis, x is equal to 0, x is equal to a.

This is y is equal to 0 y is equal to b and then this is along this is basically the front face of the cylinder. So, in that case also will get in case of a, rectangular tube or a duct, what will happen here? Again will get a w, in this case general expression for w will be sigma m is equal to 1 to infinity A m n. A m n, this is called sine sine n pi by a into x into sine n pi by B m pi by a into x into a sine n pi by b into y, is the general form.

Again alpha since alpha naught is a constant we can use the same alpha naught as sigma m is equal to one to infinity sigma n is equal to one to infinity B m n, then we can take the same sine m pi by a into x into sine n pi by a b into 1. If you substitute for alpha naught and w and 1 is the same because this functions are orthogonal function and we can get a m n in terms of B m n as in this case will have, B m n as a 4 alpha 4 alpha by m n pi square into 1 minus minus 1 to the power m into sorry, 1 minus minus 1 to the power n.

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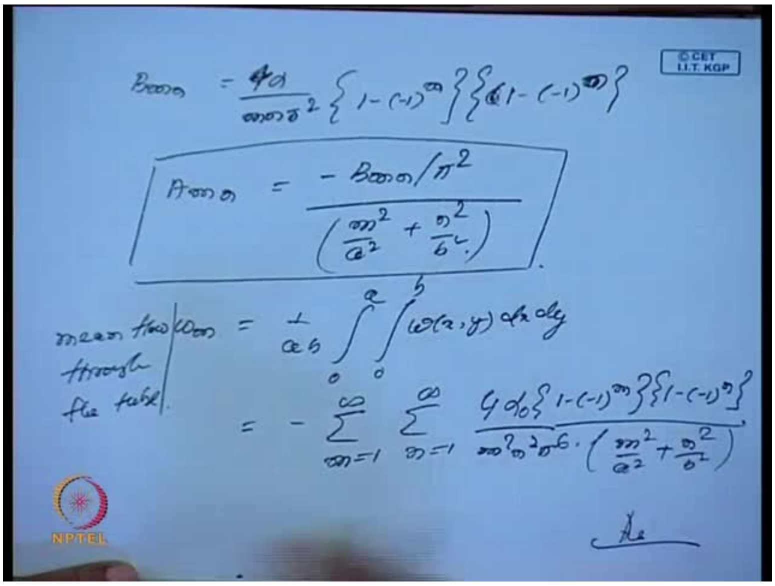

This is my B m n and again may A m n will be minus B m n by pi square divided by m square by a square plus n square by B square, this should be what? If I substitute for A m n and B m n, then may w l obtained once I obtain the w may w m will be… That means the the mean flow through the tube. This is basically the mean flow through the tube that is w m. This w m will be 1 a b 0 to a into 0 to b w x y by d x d y and that what exactly, will give me minus sigma m is equal to 1 to infinity sigma n is equal to 1 to infinity.

This is 4 alpha 4 alpha naught by m square n square into pi to the power 6 into 1 minus minus 1 to the power m 1 minus minus 1 to the power n and that is by this is m square by a square plus n square by b square. This will be what we are looking for, that is the mean flow. So, so once the w is known in any other quantity of interest we can always find the Wallicier stress and thus coefficient on the wall boundary as usual. So, this is again another example of the flow, where we have talked about fully another example of a fully developed flow, where we are able to whether it is a problem of a rectangular duct.

On the other hand in the previous example, we consider the case of a square duct and in both the things both the case we have, the results are it is a little different than the case of a circular duct. On these are the true cases non circular ducts. Now, I will just another simple example, because these few examples will play very significant role in the understanding a various flow.

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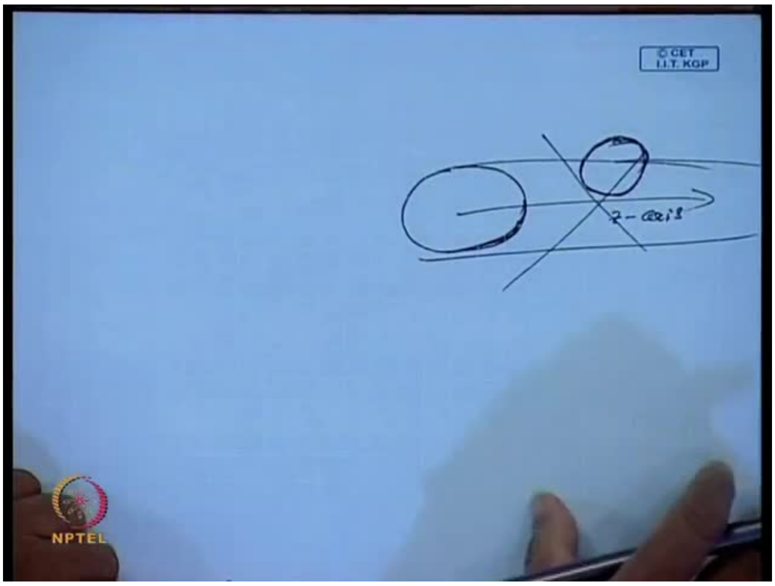

I will just look at a already I have done the case of a circular duct. Now, what will happen if I have elliptic duct? If I have ellipse, then I will like a case of circle, if this is the cross section of the ellipse and again the generator is along z axis sorry, I will take it as a.

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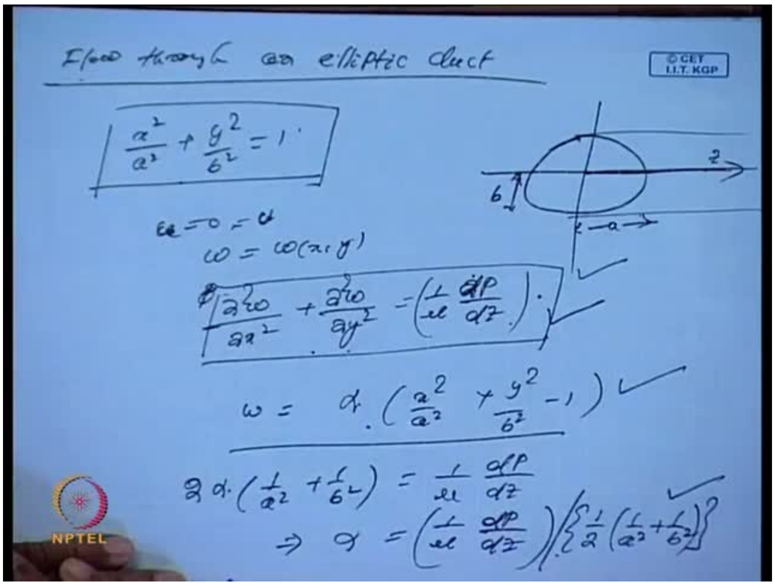

Now, it is a consider the flow through an elliptical cross signal, flow through an elliptic duct. The problems, in case of ellipse we have the equation of the ellipse x square plus by a square plus y square by b square, this is equal to 1. I will look at the cross section of an ellipse and this is in fact this is a major axis and this is the minor axis. That is b and we have a flow is along the generator. This is along the z axis, then I know that if you look at the same, assume the same fully developed flow in a two dimensional case on where u is equal to 0 equal to v.

We have w is equal to w x y and will have p is again a function of p z and will have del square w by del x square plus del square w by del y square is equal to 1 by mu d p by d z, which is constant and this constant I can call it as a alpha. This I call this as, so this is a del square w by del x square this becomes by Navil stress equation and I have my boundary using these. So, the boundary is x square by a square plus y square by b square is one this becomes may body boundary. Then what will happen here?

So, that means here this is the equation of motion through this fluid is flowing, this equation as to satisfy subject to the boundary condition x square by a square plus y square by b square is 1. If I look at may w, what is the solution of this case? Where this is this equation has to satisfy subject to this boundary condition. Let me just look at w like this, alpha into x square by a square plus y square by b square minus 1. If I have to take this, then what will happen?

If these w has to satisfy this alpha is for unknown constant, if these has to satisfy because if I take this form, then what will happen? When x is equal to a y is equal to b, then this will satisfy the equation of the ellipse. Then if w is this whether if this has to satisfy these equation, then what I will get? I will get two types, two alpha times my x will be del square w by 2 alpha times 1 by a square plus 1 by b square. If I substitute for this w here, so this will give me 2 alpha types 1 by a square plus 1 by b square is 1 by mu into d p by d z.

So, that gives me a alpha which implies a alpha is o1ne by mu d p by d z into divided by 1 by 2, 1 by a square plus 1 by b square. In fact this answer is very simple, so that means this w subject to this alpha will satisfy by this equation and also the body boundary condition. So, here I know what exactly my w is, the flow directions. Then if I once I know, so my alpha is known should in terms of the result gradient on the viscosity, then I know my flow velocity and then I can get what is my mean velocity?

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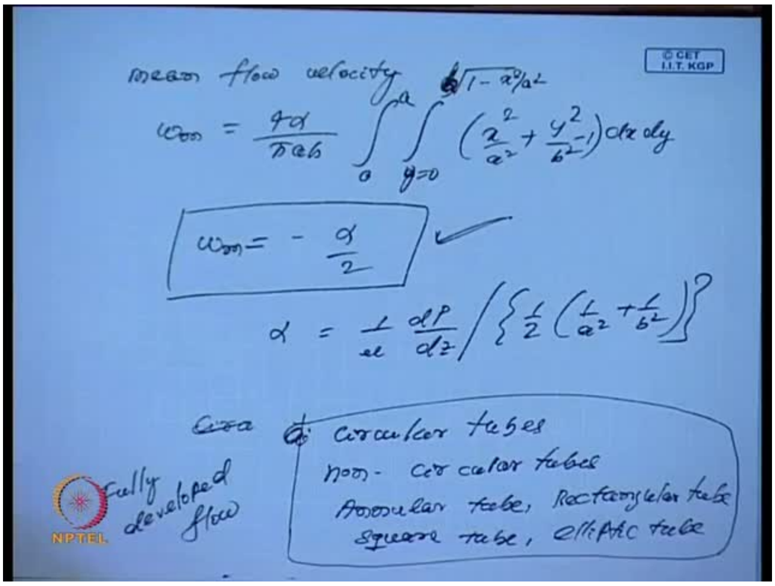

Means by mean flow velocity in this case also I will get it what is called w m? That will give me 4 alpha by pi a b 0 to a 0 to b. That is usually it is not zero b it is a y is equal it is y is b and b is again in terms of a x 1 minus x square by a square. This is b times if I put in terms of and this is y is equal to 0, this is b times and this is x square by a square plus y square by b square minus 1 d x d y. This is my and if I simplify this and that is nothing but if I do that, I will get it minus alpha by 2. That is main w m, that means flow velocity is just a half of the, half of what alpha is and alpha is nothing but 1 by mu d p by d z divided by 1 by 2 1 by a square plus 1 by b square.

This is a case of a where mean velocity in a case of a elliptic cylinder. This alpha minus is there because this quantity d p by d z always although it is constant and it is a previous case we have seen that as the flow proceed d p by d z will be a negative quantity. Because the flow is moving from one into the other, so the flow temperature decreases. So, in that process because of that it comes in minus sign. Otherwise, so this is the, our in velocity of the flow and so we have to consider large number of cases. Started with a basically of circular cylinder, we say thus circular ducts. We have consider circular ducts, circular tubes and the case of non circular tubes, flow in within annular annular tube rectangular tube.

Flow in a square duct and also the case of a elliptic tube. These are the solution for these class of program particularly for the viscous in compressible flow. We have analyze how the flow pattern will be in particularly when we have a fully developed flow. In fact these these results will be very useful in analyze more complex for problems, in the where can I will not able to get a close from solution approximate solution. We obtained and this is known results will be of immense value in such situation to bench mark the result of the approximate solution that we obtain by a direct.

We have seen that in all these cases, they all satisfy the Poisecielle equation is nothing but a, but in a as the partial differential equation Poisecielle equation. All the cases we are able to get very close form and I will take solutions and these are all some of, because of the we are able to manage to get the full solution because of the symmetric at a restrict of the flow on the boundary. More complex problems it will be difficult to obtain close from solution, but we have to go for after some solution.

These results will act as a bench mark result and all these case we have consider the flow is a steady. So, in the process we have made this has a time depend time independent problem. We could not handle it easily and the next class. I will talk about transient program, where we will consider a flow as on steady. So, we will talk about steady fully developed flow in the next class.

Thank you.