Marine Hydrodynamics

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Lecture - 33
Navier-Stokes Equation of Motion

In this series of lectures in marine hydrodynamics, in the last lectures on waves, we have concentrated on flow fluid flow, where the fluid is a word assumed to be incompressible as well as in viscid. In fact earlier, we have seen the case of Euler equation or in the case of Bernoulli's equation also the flow, we had seen the fluids in viscid. In case of Euler equation of motion and in case of Bernoulli's equation motion, the flow fluid was considered conservative. The fluid was motion was assumed irrotational.

In today's lecture, we will go a little more general way; we will assume that as if if the fluid is viscous, then what will happen to the equation of motion. As you know that in case, there are 2 major equations satisfied by a flow fluid. One is the equation of continuity. The other is the equation of motion. Equation of continuity says that that is comes out comes from the conversation of mass, whereas the equation of motion comes from the law of conservation of momentum and again, which is also come comes from the Newton second law of motion.

So, before going to the data, let us discuss what are the various forces that act on a fluid element. Then, we will talk about various parameters, which are very influential in judging the the fluid characteristics. So, to do so, let us talk about the various forces acting on a fluid element.

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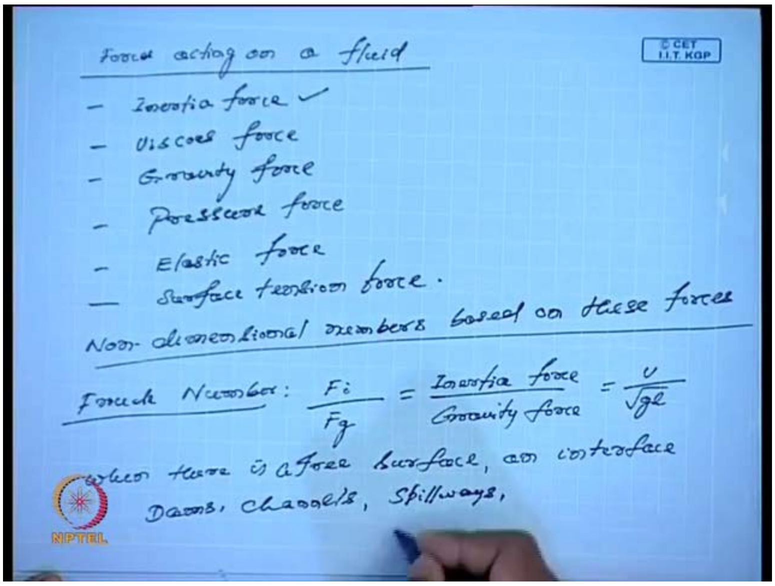

Now, look at the various forces acting on a fluid. When it becomes to mainly the inertia force, we have the viscous force. Then, we have the gravitational force. Then, with the gravity force, we have the pressure force. Basically, this acts on the surface of the body. Then, we have something called elastic force. Then, we have something called surface tension force.

So, these are some of the some of the major forces, which act on a fluid element and a fluid. Then, out of these, there are certain numbers that we call, which are the ratio of these forces. I call these numbers. There are some non dimensional numbers based on these forces because these non dimensional numbers, they signify the role of the fluid, character role of the particular type of motion and a particular type of fluids.

We have seen that in case of Euler equation, inertia force, viscous force was negligible, whereas there was a gravitational force. There was an inertia force. There was a pressure force, whereas elastic force was not present, surface tension force was not present and viscous force was not present.

Now, we will look at the non dimensional numbers based on these forces. So, first it comes to that is call 1 of the number, which is called the fraud number. It is basically if you look at the fraud number, basically it is the measure of the ratio of the inertia forces to that of the gravity force quality F i by F g, so that basically the inertia force force to the gravity force and you can say that this is nothing but u by root g l.

g is the as less than due to gravity that is the gravitational constant, when is l is a characteristic length and u is v is called the characteristics velocity component of velocity. Then, this fraud number has become very important when we deal with the flow, when there is a surface tension, when there is when there is a free surface flow associated having a free surface or even if an interface. There is a free surface. There is just an interface emit into miscible fluid.

This fraud number plays a very important role, particularly if we look at the motion of a ship inverter or any some other body. The fraud number plays a very significant role and more important are the problems as to assure to the dams channels and spillways. This fraud number plays a very important role because here the surface elevation plays significant role rather than and the gravitational flows the significant role. Then, even if there is the viscous force, this is what fraud number importance. Now, another number of this type is what we call the Reynolds number.

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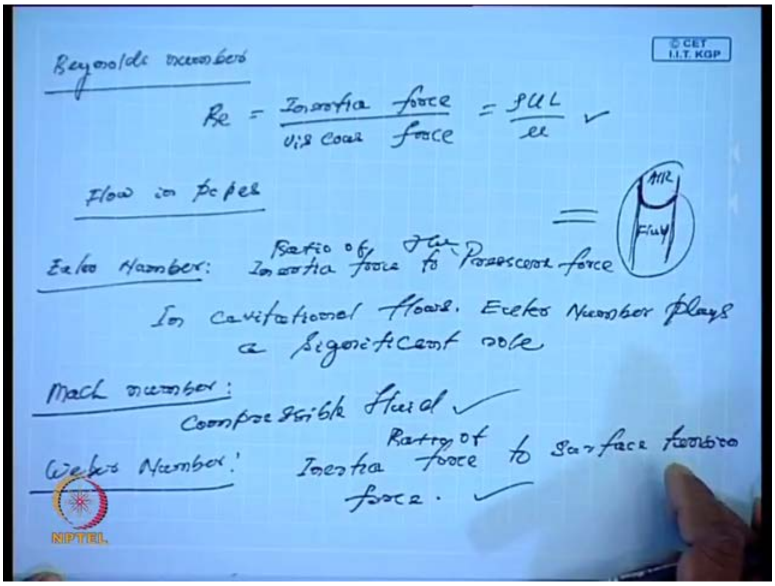

This is because these are as I mentioned, this forces plays very significant role in character characterizing the flows. Then, if we look at the Reynolds number, it is again the ratio of 2 numbers that is the inertia force. I denoted by rise of the measure of the inertia force to the viscous force. Often, we write it as rho u L by mu. Mu is coefficient of viscosity, rho is the density, u is the characteristic fluid fluid velocity and L is the characteristic length.

So, this is Reynolds number. This number in fact when it can vary between 0 and 1, it can be 0, very small or it it can be very large. Particularly, when you think of flow in pipes, Reynolds number plays very significant role. On the other hand, the pressure chains along the pipe are small. So, the pressure force does not make much difference. In the same way, if you look at surface tension forces because in case of pipe flows, surface tension forces are negligible. They are not considered because there is no there is not a sequence surface where are second fluid where the surface tension effect will be feasible.

This is because there is only 1 single fluid unlike if you consider a viscous or a tube, you have always the rise is because there is air. Here, there is a fluid. So, what happened? This is because of the difference in pressure on both sides. There is a rise in the fluid velocity along the 2 sides of the wall, which is not case in case of a pipe flow.

Similarly, so here, inertia force plays a significant role. In addition to the viscous force, compare to the inertia to the viscous force. Similarly, the buoyancy, buoyant forces are also there. Almost, they are also not equally important. Similarly, if we will look at and as I mentioned that this depending on, further inertia force is large. The viscous force is large. Again, we characterize the flow whether it is a laminar flow whether it is a turbulent flow and whether it is a highly viscous flow.

So, various types of flow can depend based on this Reynolds number. Now, there is another number, which is called an Euler number. Again, ruler number is depending as the ratio of 2 forces. It is the ratio of inertia force to the pressure force. This Euler force is important particularly when there is gravitation. We all know that there is a region when you have seen that in case of the formation of cavity, when the pressure becomes negative, the cavity formation takes place and near in gravitational flows flows. This Euler number becomes very important, plays a significant role. It is a ratio of the ratio of the inertia force to pressure force in gravitational flows. The Euler number plays the significant role.

Similarly, we have some of the other as there is something called Mach number, particularly when the fluid is becomes income compressibility. This is more important for compressible fluid. This occurs at high speed flow or motion of an object like jet airplanes throw a finite passes through subsonic region subsonic speed. This Mach number plays as a significant role.

Then, we have this Weber number. Then, this Weber number is it is again a ratio of the inertia force to the surface tension force, ratio of inertia force to surface tension force and particularly in physical model testing for a channel or for a harbor. These forces play a very important role. This is Weber number here the rather viscosity is not that important compared to the surface tension effect.

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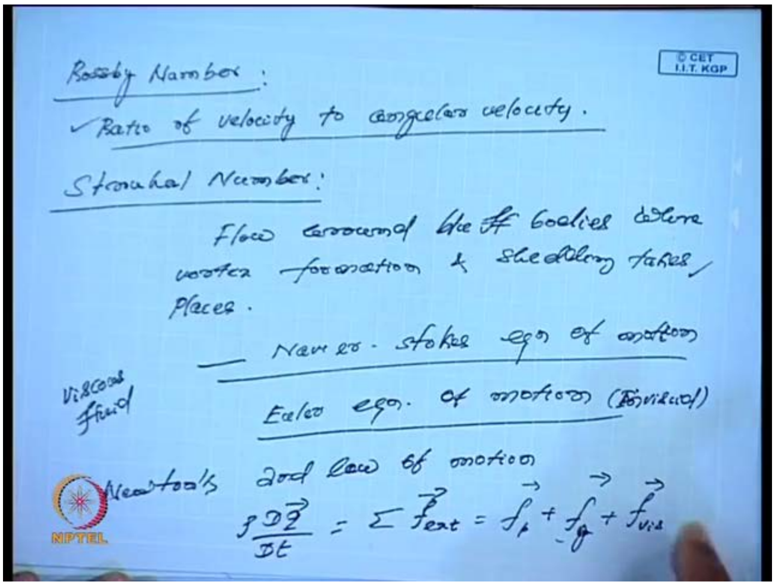

Now, then we have something called rossby number. Rossby number is an important number when you consider the geophysical fluid force. Basically, it is a velocity which is associated with the velocity to ratio of velocity to ratio of velocity to angular velocity; particularly most of the geophysical fluid problems fluid flow problems. These are more important in oceanography or atmospheric science. They are more important to deal with problems associated to oceanography or atmospheric science is rossby number, particularly where the centrifugal and the centripetal forces are more important in this case.

Then, we have struhal number. It is important basically, when flow around bluff bodies, particularly where vortex shedding takes place, vortex formation and shedding takes place. These are some of the important non dimensional numbers. There are other numbers depending on when the flow is in a magnetic field or sometimes we call it when we call it viscous elastic fluid or even if sometimes we will call about talk about elastic fluid.

So, in such cases, there are certain other numbers which are non dimensional numbers. They play a significant role, but we will not go to those details where this basic background is. This is because of course, here are emphasize numbers, which are important to wherein environment. So, these are the things, which emphasize with this understanding about the various various forces and the various non dimensional numbers.

Let us look at that. We will we will derive the Navier stokes equation of motion. In fact, the Navier stokes equation of motion is again as I say that it is a generalization of the Euler equation of motion, Euler equation is a generalization. Here, viscosity is neglected in viscid fluid, whereas in this case, Navier stokes equation of motion, which is suitable for viscous fluid. In fact here also, when the fluid is incompressible what happens? So, from all, we all know the from Newton second law from Newton second law of motion, we have rho D q bar D t bar. That we have already seen that is equal to sigma f ext.

So, it is a combination of the external forces that is acting. This is the rate of change of linear momentum is equal to the total force. As we have seen that there are several forces, which act on a body in case of the Navier, stokes equation. Here, we will emphasize the 3 forces that is one is the pressure force, one is the body force and that body force is sometimes called the gravitational force and plus another is the viscous force.

So, these are the 3 forces. In fact, in case, one is the pressure force or surface force. Surface pressure force is the gravitational force of the body force. This is the viscous force. This is important in case of Navier stoke equation, but in case of Euler equation, viscous force will leave contours in all be negligible.

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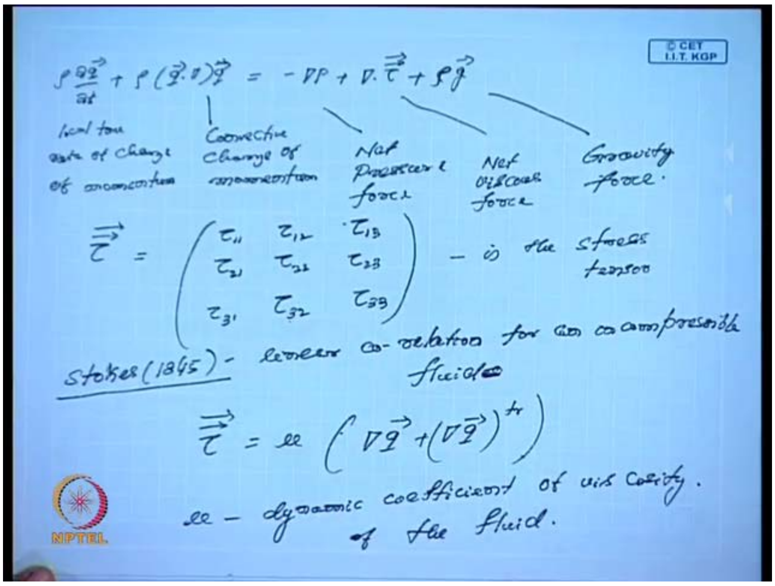

So, now to do this, if I simplify further this, then what will happen that rho del q bar by del t plus rho into q bar dot del this is the left side. The right side becomes minus grad p plus rho double bar plus rho g bar. If I simplify look at the individual terms, this is called the local time rate of change of where time rate of change of momentum. This is this term that refer to the convective change of momentum. Then, this term is the net pressure force. This is the net viscous force. Finally, this is the net gravitational force, which is called the body force gravity force or gravitational force. So, these are the various forces acting on it.

Now, again what is tau double bar? Tau double bar is basically called the stress tensor. I will define what exactly it is. So, it is since it is just stress tensile, it has the 9 components, pi 1, 1, pi 2, 1, tau 3, 1, tau 1, 2, 1, 3, these are called the stress tensor. In fact, it has stokes is the stress tensor. In fact, it was stokes in 19, 1845, he gives a very postulated a linear correlation between provided a linear correlation for an incompressible fluid, an incompressible fluid. What were Stokes's postulates? Stokes's postulate was tau double bar equal to mu times q bar plus delta q bar.

It is the transverse of this. This is what stoke postulates word and where mu is the dynamic coefficient of viscosity dynamic coefficient of viscosity of the fluid. This stress tensor has it has 3 normal components that is tau 1, 1, tau 2, 2, tau 3, 3 and the 6 tangential component that is tau 1, 2, tau 2, 1, tau 3, 1, tau 1, 3, tau 3, 2, tau 2, 3. So, it has basically 6 tangential components.

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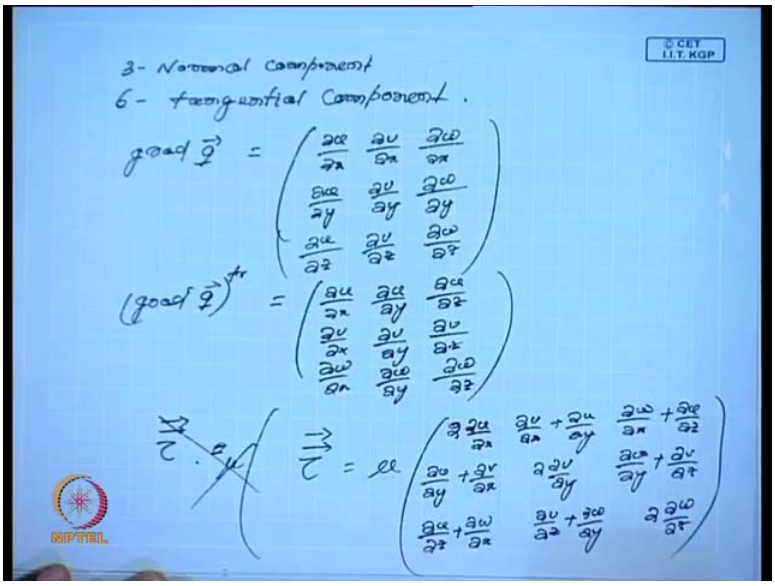

This stress tensor it has 3 normal components and 6 tangential components. If I simplified this stress tensor again, what will happen? Now, we know that grad of q bar, if I look at this, then this gives me u del u by del x del u by del x del w by del x del del u by del y del u by del y del w by del y, which is del u by del z del v by del z del w by del z. You look at the transpose of this. What will happen to the gradient term q bar transpose?

If you look at this, this gives you again del u by del x del v by del x del w by del x and then del u by del y del u by del z. This will remain as del v by del y. Then, it will be del v by del z del w by del y y del w by del z. So, if I have this, then what will happen to my tau? Here, the stress tensor itself is tau double bar. That will give me if I combine this 1 is mu times tau double bar mu times, if I add these 2 things that means, so I will get 2 del u by del x. then, this becomes del v by del x plus del u by del y in the third components del w by del x plus del u by del z.

Similarly, here you will have del u by del y plus del v by del x. Then, you have 2 del v by del y. The third component will be del w by del y plus del v by del z. Then, the third component, it will be del u by del z plus del w by del x. then, we have del w by del z plus del w by del y. It is 2 times del w by del z. So, these are the normal components. These are the these are the tangential components. So, of the stress tensor, you just compare them. You can go for that.

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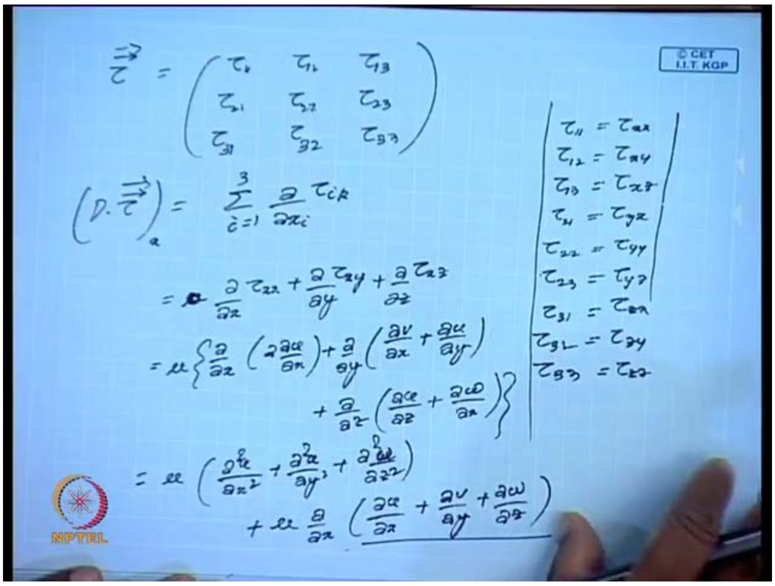

If I say from this, we can easily find out because the stress tensor is tau double bar. if I put component twice tau 1, 1, tau 1, 2, tau 1, 3, tau 2, 1, tau 2, 2, 2, 3 tau 3, 1, tau 3, 2, tau 3, 3. Then, I can give the individual component of basically the 3 normal stress components. We can compare with the previous extension for tau. I can get the 3 normal components and the 6 tangential component of the stress tensor.

So, if by doing so, then what will happen? If I simplify further, I will take the divergent of tau double bar. What will happen? We can write it as a summation of i is equal to 1 to 3, del by del xi tau i k. Suppose that I look at the x component. I substitute for this 1, 1. Sometimes, we call it basically tau xx tau 1, 2 is tau x y tau 1, 3 is tau xz.

Similarly, we can have tau 2, 1 is tau y x, tau 2, 2 is tau yy and tau 2 3 is tau y z. Similarly, we can proceed ahead and the tau 3, 1, tau 3, 2 and tau 3, 3, this will be main. Of course, tau zx, tau z y, this is tau zz. So, this, this and this, they are the x components. The normal component and the rest are the tangential component, sometimes this is used as this notation or this notation. So, if we look at the normal component, the x component of the stress tensor. Then, this will be divergent of this. Then, it will give you tau i k and substitute for tau i k. It is just nothing but in terms of xx and for an i k. Then, what will happen? x component means k is 1.

So, in that case, we can easily see and again substitute for the individual expression for from stokes postulate. Then, what we will get? This will give us mu times, I simplify little. This will give us del by del x tau xx plus del by del y tau x y plus del by del z tau xz. This will be further simplified. This is because I substitute it in terms of stokes theorem, stokes postulate rather that will give me mu times del by del x tau xx is 2 del u by del x plus del by del y tau x y. It is nothing but del v by del x plus del u by del y plus del by del z. This is tau xz. That is del w del u by del z plus del w by del x.

If I simplify further because I will take mu. I will take 1 component from here out of 2. So, this will give me del square u by del x square. From here, I will take this component plus del square u by del y square plus del square w by del, del square u by del z square and plus. Then, I will have mu times. I can write it as a del by del x del u by del x plus del v by del y plus del w by del z. So, this gives me because this this quantity is considered divergent of q. If the fluid is incompressible, this part is 0. Once this is 0; then it will give me the condition for incompressible fluid. Thus, this is x component of the stress tensor.

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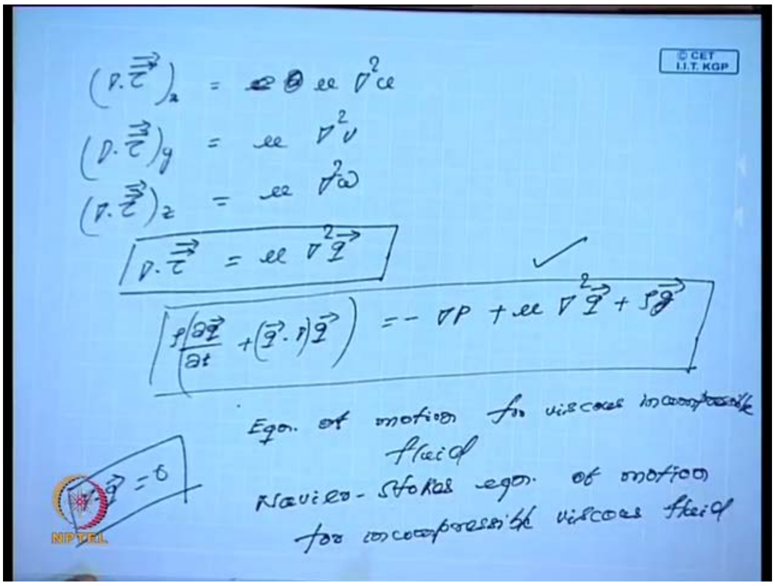

The divergent of the x component of the stress tensor gives, rather I will call it del square u. What u? u is the x component of the velocity vector. In a similar manner, if I proceed with y component, I will get it mu del square v del dot the z component. I will get it mu square w. If I combine this, so if I combine this, then I will get del dot tau double bar equal to that will give me mu del square q bar. So, this becomes my forces that is the viscous force.

So, if I substitute for this in the equation of motion, what I proposed in the beginning, so that will give me rho del q bar by del t plus q bar dot del into q bar is equal to minus del p plus mu del square q bar plus rho g bar. This becomes the equation of motion for a viscous incompressible fluid, a viscous incompressible fluid. Sometimes, we call these as Navier stokes equation of motion for incompressible viscous fluid. Since, the fluid is incompressible, we have divergent of q bar is 0 that is the equation of continuity. This is the equation of motion.

So, this is the equation of continuity. This is equation of motion. This is often called as the Navier stokes equation, a motion for incompressible viscous fluid. Here, the additional term is this, whereas if this term is absent, then we call this as the Euler equation.

In fact, the approach we have seen the derivation of Euler equation. If the rate of change of linear momentum is the total force, the same approach same Euler in approach can be used. When we calculate the total force, there is a another additional term that is the viscous term is comes into picture. These viscous terms come from that is based on the assumption of stokes basically based on stokes postulate. It is it is one of the simple derivations.

You can find this same thing in various textbooks in a very elaborate manner by by considering the forces acting on the fluid element and by by considering in the Cartesian coordinate system. This is one of the most viscid and simplified manner of looking at looking into the equation of motion. Now, I will further simplify this. By considering that, let us have g that is a constant. If the gravitational force is constant and rho is constant, if I assign in most of the cases, we have seen the fluid density is constant.

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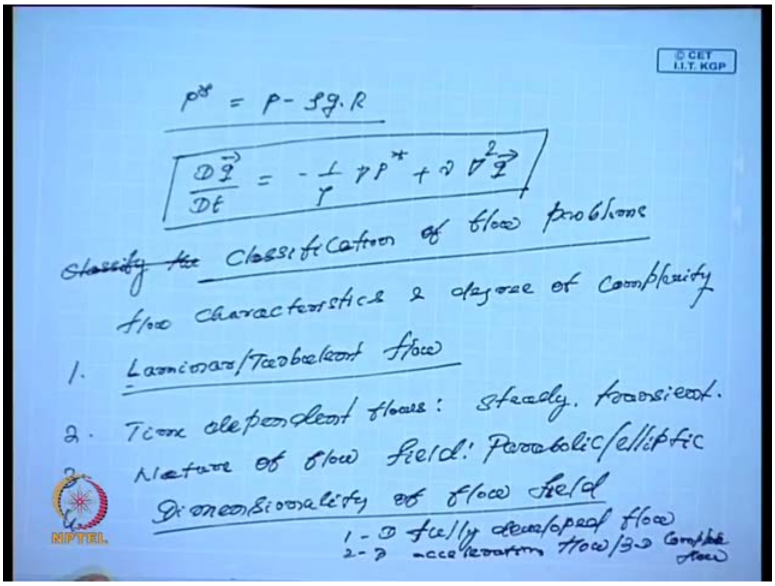

So, in such situation, I can always write p star equal to p minus rho g R. Then, R is the position method. So, then you can, it can be written as an d q bar by equation of motion can be written as D q bar by D t is equal to minus 1 by rho grad p star and plus new del square q bar. So, this is also another formula writing the same thing. When the fluid motion is as same, a fluid is assumed to be incompressible by particularly when the flow is of constant density.

Now, before going further, let us classify. We will classify various as I have classified the rather I call it classification of flow problems of flow problems. If you look at the classification of the flow problems, here are a few things comes into mind. This classification is always based on the flow characteristics and degree of complexity degree of complexity of the fluid.

So, one of the things is that whether the flow is laminar or turbulent flow is a laminar or turbulent, in case of a laminar flow, the fluid characteristics follows like the fluid particles follow a particular path along deviation when the fluid flows. In case of a turbulent flow, the fluid, there motion is random or very reflect in nature.

One of the typical examples of the laminar flow is like it is a fluid particle trajectory. It is always there. You can predict it in case of a laminar flow like when a group of soldiers. They are marching on a particular on a bridge or may be on a field. On the other hand, if you look at the turbulent flow, you can compare the same thing like a group of pilgrims.

Those were moving towards a religious side, either before visit or after visit or may be particularly after coming out from the pilgrims.

When they come out from a religious side, they just move in a very random manner. A same flow same can be compared with the turbulent flow. On the other hand, this is like a as I mentioned that group of soldiers. They are marching on a bridge or may be in a field. So, this is the, from this, it can be clear how irregular the turbulent flow particle can be and how regular the laminar flow can be.

Now, the second example is time dependent and transient motion, second type of flow time dependent flows that is one is steady. The other I can call it as a transient flow. Then, we can have the flow field nature of the flow field. It can be parabolic or elliptic. The flow field can be parabolic or elliptic in nature.

Then, we have dimensionality of the flow field dimensionality of the flow field. We have dimensionality of flow field. Here, I have 3. It can be 1 d fully developed flow 1 dimensional fully developed flow. It can be 2 dimensional accelerating flows, 2 dimensional accelerating flows or it can be 3 d completed flow or it can be 3 dimensional complete flow 3 d complete flow. So, when it comes to 2 dimensional, these are the 3. Then, we have some of the some of the other things, other characteristics.

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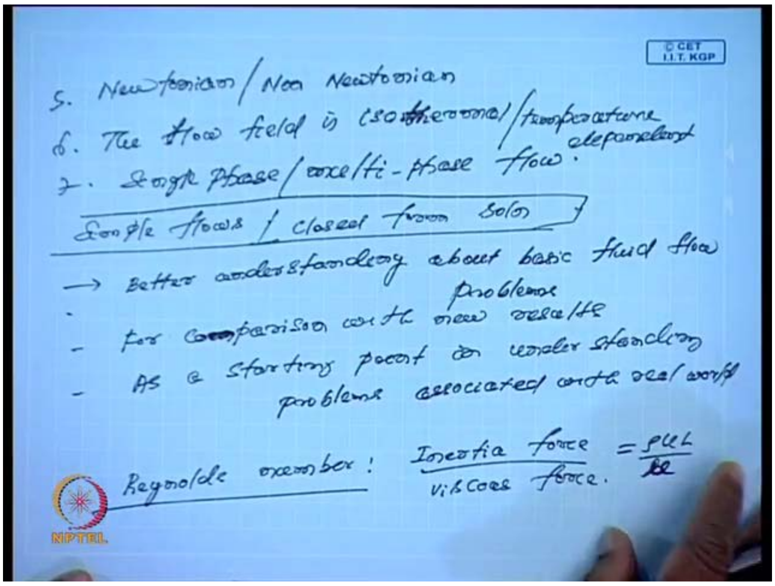

The characteristics are the flow can be Newtonian or non Newtonian. The flow field can be isothermal. The flow field is isothermal or temperature dependent. Another type of flow can be, it can be single. The flow is single phase or a multi phase fluid. So, these are some of the various characteristics. When you look at the degrees of complexity, somewhere it is types of flows, they can be, and then the question comes that are you going to discuss all this kinds of flows. In principal, I said no. In fact, it is not always possible to discuss all these things in a series of a particular course, rather what we do emphasize will consider some of the flows, which can be discussed.

These are very simple flows. We will only discuss some of the simple flows. If we will discuss some of the simple flows, then the question comes whether this kind of simple flows, where we can get a closed form solution. Then, the obvious question comes if I will get some of the simple flows. I get, I can get a closed analytic form of solution or may be some of the flows, which will have semi analytic solution or may be approximate solution, then why it is important to analyze all the simple flows.

Again, this has already been solved. This gives us. These are the following some of the regions have, but when we analyze a simple flow, it is a better understanding of the flow characteristics, understanding about fluid dynamics about basic fluid flow problem and for better comparison between unknown regions with new solutions new results.

The third thing I will say as a starting point in understanding problems associated with the real world. So, because these are some of the reasons for often in a classroom simple flow characteristics have simple very simple problems have discussed. Now, with this understanding, let us look at what is, if I look at the the Reynolds number particularly as I mentioned, this depends on 2 ratios. It is the 2 quantities. One is basically this is the ratio of the inertia force to the viscous force. This is also often calling it as rho u L by nu rho u L by mu.

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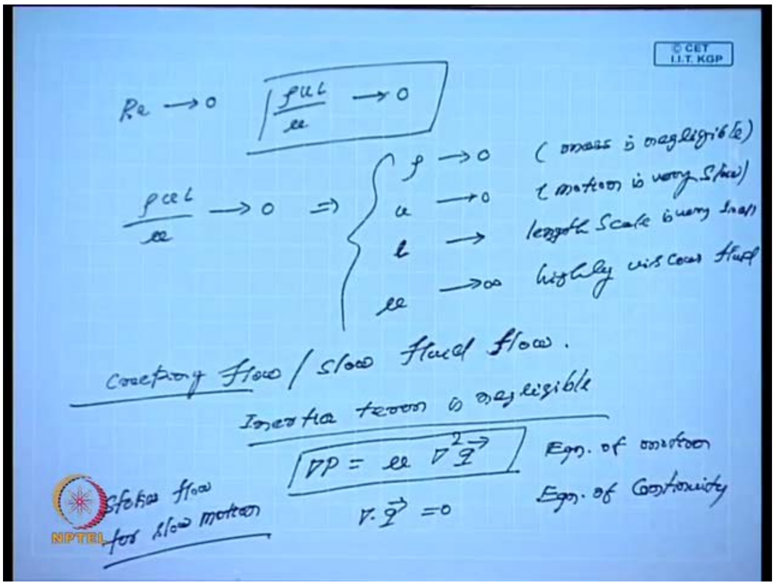

Now, what will happen? This Reynolds number, if R e tends to 0, let us see what happens if R e tends to 0? That means we have we have rho u L by mu will tend to 0. So, these are the possible things that when it will tend to 0, this factors that 4 possibility.

So, rho u L by mu into 0 is implies, I can consider 4 possible things. Either the mass is rho is tending to 0. I may have u tend to 0. I may have a situation when l is tending to 0. I may have a situation when mu tends to infinity. What physically does it mean? So, here when rho tends to 0, I say 0 mass or mass is negligible. Mass is negligible. It can be very slow motion u tends to 0 means motion is very slow. Motion is very slow. When I say l tends to 0, the length scale is very small.

When I say mu tends to 0 that means I will look at highly viscous fluid. So, these are the question comes when R e is tending to 0 and in fact, so these are sometimes associated with, sometimes we call these as creeping flows. When R e tends to 0, the associated flow we call it creeping flow or we call it slow fluid flow. The fluid motion is very slow. Here, in a natural way, the inertia term is because R e tends to 0. So, if it is a ratio of 2 forces that means the inertia term. The ratio of 2 forces inertia force to viscous force inertia term is negligible and viscous force is dominating compact to the viscous force.

Then, what will happen in such a situation? The inertia force will be negligible to the compact to the viscous force. Then, if I look at my Navier stoke equation of motion or incompressible fluid flow, I will have grad of p into mu to del square q bar because the inertia term is negligible.

So, this will be the typical equation or a creeping flow motion. Again, we have divergent of q bar equal to 0 because this is the equation of motion. This will give me the equation of continuity because we have fluid is incompressible. The fluid is incompressible and flow is. This is sometimes we call these as stokes flow for slow motion. Now, if I will do one thing, I have grad p is mu del square q bar, what will happen? if I just say that that means…

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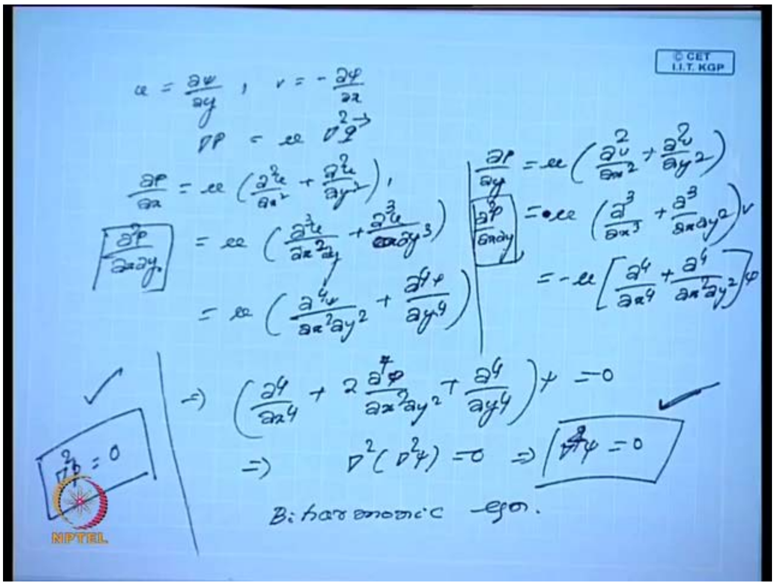

If I introduce, you introduce the stream functions psi. So, u will be del psi by del y and i will p will be the component of velocity will by minus del psi by del x. then, what will happen? We have grad p is mu into del square q bar. So, then this gives me del p by del x. What will happen to mu del square u by del x square plus del square u by del y square? If I look at this, simplify this and I substitute for u is equal to, so what will happen to this?

So, if I say del square p by del x del y that will give me mu del square del cube u by del x del x cube plus del cube u by del cube by del y. This will be del x square del y plus del del y cube del y cube. Then, again I have my q e is equal to psi y. So, I will put it mu. I can call it del fourth psi by del x square del y square plus del fourth psi by del y fourth.

Similarly, if I look at the other component that means del p by del y from this that will give me mu del square v by del x square plus del square by del y square. Then, that is that will also give me del square p by del x del y. This will give me mu del x del y. So, it will give me del q by del x cube plus del cube by del x del y square and p is nothing but minus del psi. Let me call this v. This is again, I will call it v is minus del psi by del x. That will give me minus mu times del fourth by del x fourth plus del fourth by del x square del y square and 2 psi. So, if I said del square p by del x del y c r, so let us see.

Both are same. The left sides are same. Then, the right sides will be same that will give me del fourth by del x fourth plus 2 del fourth psi del fourth by del x square and del y square plus del fourth by del y fourth psi will be 0. I can write it as del square of del square psi 0 that is same as call it as del fourth psi that is 0.

So, that means when the motion is so when the Reynolds number is small, then the stream comes in psi will satisfy the biharmonic equation. It can be same way if you eliminate p eliminate, rather from the equation of motion, if I will eliminate rather q, it can be seen that p also satisfies the condition del square p 0. That means the pressure satisfies the Laplace equation have the stream functions satisfy biharmonic equation.

So, these are the 2 equations satisfied and that is further. When the motion is slow or we say for creeping flow motion, the Laplace equation, the pressure satisfy the Laplace equation, whereas the stream functions satisfy the biharmonic equation. This is one of the very simplest solutions associated with the viscous fluid flow, particularly this is voluble particular when the fluid motion is slow and we call it creeping motion. Sometimes, we call it stokes flow.

With this, we will stop here today. In the next class, we will come to simpler solution associated with Navier stoke equation. We will analyze several cases; very simple cases particularly as application of the Navier stoke equation. We for go into little more complex problems. These simple problems will be illustrated nicely and the various applications of the Navier stoke equation. I will stop here.

Thank you.