Marine Hydrodynamic
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Lecture - 25
Linearised Long Wave Equation

Welcome you to this lecture on marine hydrodynamics. In our last couple of lectures, we have already talked about small amplitude wave theory, and then we have considered few special cases, few phenomena associated with this, certain physical calculation of our computation of some physical quantities, like the energy, pressure, how we calculate associated with wave; that is when it is providing. Basically our concern is now wave propagation in a water. Now as we have seen that in the, always we define or we model anything, any physical model we develop, always certain assumptions are required, for easy of computation.

And so we always idolize the problems, and also in the process of idealization, many times the problems are simplified, and this idealization varies from case to case, like when we are dealing with, which high non-linear waves in the ocean, and if you look at the, calculation of the wave forces and the cylinders, on particularly large of shore structures, then in fact our linearization, small amplitude wave theories sometimes may not hold good. In such situation what you do, we always look into the finite amplitude wave theory, and consider that is basically we talk about the stroke's finite amplitude wave theory, and then you consider, first order, second order, third order, and the higher order wave theories.

Of course this individual order of the theory, there also is a realization, another case of hydrolization, from the general non-linear wave equations, and the boundary conditions. In a same manner, suppose we are looking into problems related to a storm, tide, tsunami. Then we have a different kind of situation there all together, and these are all kinds of many situations, and we analyze them as long waves, and for the very basic reason of analysis, again we linearise it, with certain approximation, and permutation. So this. In today's lecture, what we will do, we will talk about the linearized long wave equations, particularly we have seen that, out of the three types of the wave; the deport waves, solubility waves, and intermediate waves of intermediate depth. You have seen the small amplitude shallow water, refers to the case, that we are c is a root g h, and in that case a square lambda is less than 1 by 20. So in fact in the past people have develop theory, where directly we can get, that a wave equation, particularly which satisfy. The way satisfy. The wave equation with c is a root g h. So let us see today how the, long wave theory is involved, particularly under the assumption, of the small amplitude wave theory. So this back ground let us start, have a look at the all the equation motion.

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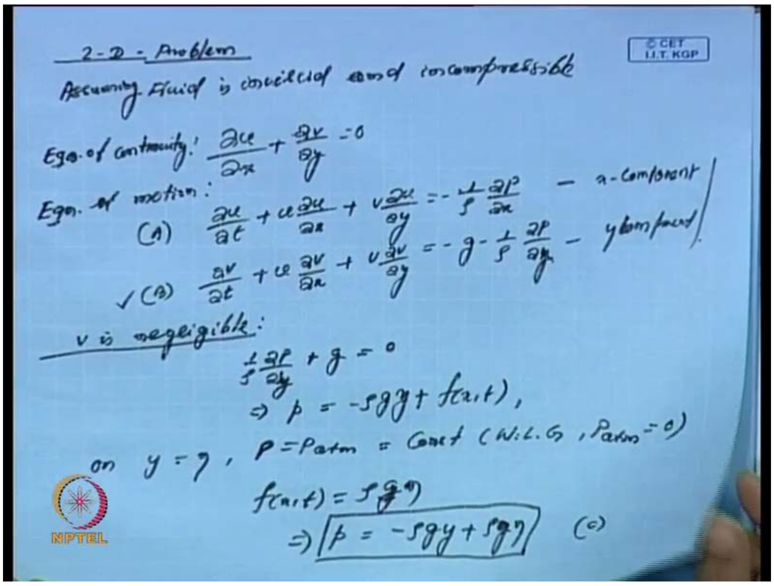

We have the first level, we will consider that as if we are calling with the two dimensional. I will consider two dimensional problem, 2 d means two dimensional under them, I will have. In two dimensional, suppose I say what is a fluid, I say which fluid is in viscid, and in compressive, it is in viscid, and incompressible. We have the continuity equation del u by del x plus del v by del y is zero, this is called some continuity equation. And we have the equation of motion; that is the equation of continuity. And we have the equation of motion, and the component wise they are nothing but del u by del t. So since I am considering two dimensional equation, is in x coordinate, x component, x component, and this is the y component, equation number. Then we have seen some equation in the derivation of the, small amplitude long wave theory. We have seen that v is a negligible, vertical velocity v is negligible; that means the vertical velocity, of the water particle is assumed to be negligible. So if my v is negligible, then what will happen to. From equation two, this I call it b, this I call it as a.

From b what we will get, we see the left side all terms will be zero, and that will give us one by rho del p by del x plus g is equal to zero, which implies p is equal to minus rho g y plus f of x t, where f of x t is an obituary function of f, because derivative is independent of x y, it has to be a function of x and t. Sorry now this is del p by. I am sorry, this is y, this is y. So the process what will happen, and on the free surface on y is equal to eta; that is on the free surface p is equal to p atmosphere, and this is our assumption. So if you p atmosphere is assumed may be atmosphere is constant, which can be taken without loss of gravity, p atmosphere can be taken as zero, as you have done in case of small amplitude wave theory. So in that process what happen, that gives me from this, this assumption we get; f of x t will be rho g eta, which implies p is equal to minus rho g y plus rho g eta, this is the hydrodynamic pressure. This is the pressure, on that we are writing from equation b.

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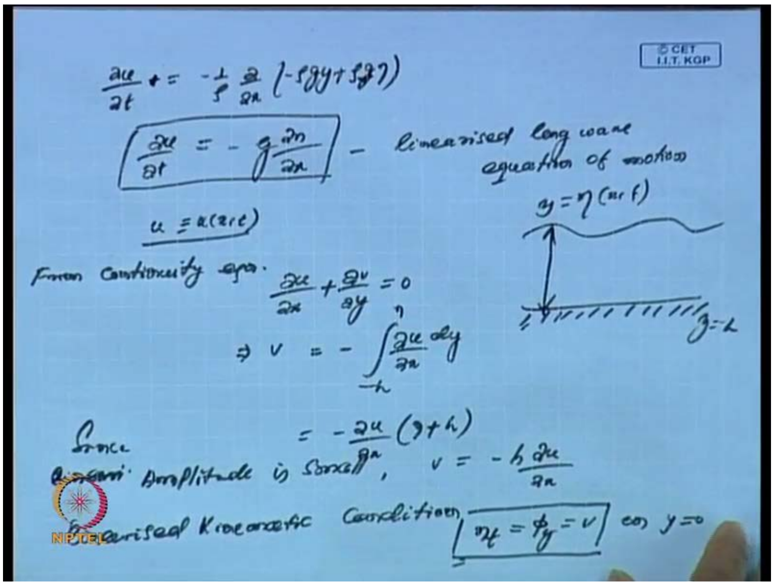

Now if substitute for this, this is the first equation, this is c. If I substitute for c in a, then what we get, and again consider the only the linear terms. So you have what del u by del t, because second term will equal to zero, third term is a term v, is involved so is negligible, and on the right side minus one by rho, so that will give me, minus one by rho del by del x p is minus rho g y plus rho g eta, that gives me minus g into del eta by del x. So that gives me del u by del t minus g into del eta by del x. So in the process, so this is the linearized long wave equation of motion. Now again if we look back, what I see that we have. From here another thing we will see that. This is where what it says that, the eta is function of x and t. If eta is a function of x and t, so del by del t, this right side is a function of x and t, so that means del u by del t is a function of x and t. So I can say u is equal to u x t; that means the original component of velocity, is only depending on the x and t variables. Now I look at the another wave; that is del u by del t, now let us look at the equation of continuity.

From continuity equation, we have del u by del x plus del v by del y is equal to zero, and that gives me, when v is minus integral minus h to eta, because of the water surface, is y is equal to eta, and my bottom is, I have considered finite water depth. This is y is equal to minus h, and this is y is equal to eta. So this is my y variation, so v is minus h to eta del u by del x, into d y. Since you have seen del u, u is a function of x and t, which is independent of y, so that will give us minus del u by del x to eta plus h. And in the process, what happens when I say, I look at as I say small amplitude, I look at the linearized problems, assuming amplitude is small. Since amplitude is small, so my v becomes minus h into del u by del x, and again from the linearized kinematic condition, fron the linearized kinematic condition we have we have eta t, is nothing but phi y and that is v, this is on y is equal to zero; that is the linearized kinematic condition. Now if I substitute it for this eta t is, so what type will get it, and this is a.

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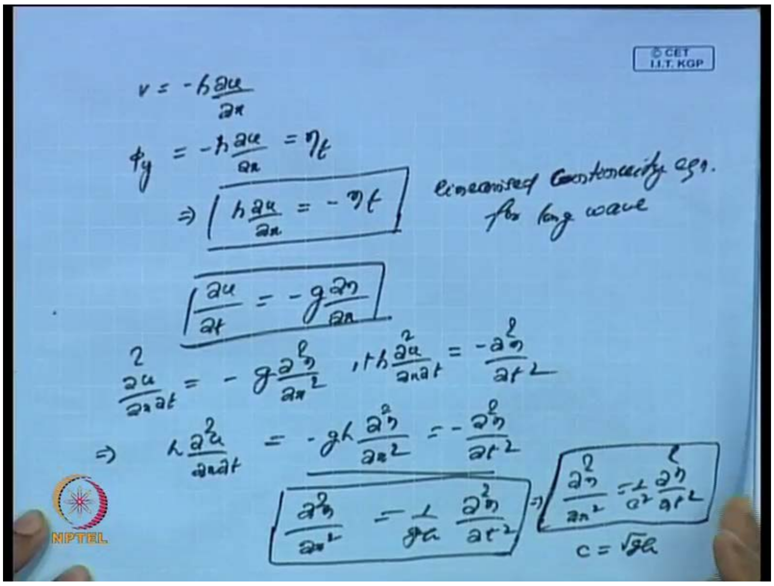

So now v, so now velocity v minus h into w by del x, and which is equal to, and this is nothing but that means phi y minus h del by del x, and phi by is eta t, so that gives me eta

t. So this is what I get, through from this cycle easily get; h del square u by del x equal to minus eta t, and in fact this is the linearised equation of, linearised continuity equation for long wave. Now, what I will do, I will do the two things, I have the continuity equation, and have the equation of motion. So the equation of motion is w by del t, is minus g del eta by del x.

So if I look at this two equations, so what will happen, by eliminate if I take this del square e by del t square del x del t minus g del square eta by del x square, and again from the first one I will have h del square. So this will give me del square e by del x del t. So this is minus del square eta minus. So you have minus del t del t. So if my i take h into w by del x h into del square u y del x del t minus this, and this is minus h into del square u by del x del t or this is equal to plus h into eta t theta t will be a minus del square eta by del t square.

So, del square u h del square u, so which implies h del square u by del x del t minus g h del square eta by del x square and h del square u by del x del t is same as minus del square eta by del t square. So if I just look at this two terms that gives me del square eta by del x square is equal to one by g h del square eta by del t square. In fact this is the one dimensional linearized, so which can be written as del square eta by del x square is equal to 1 by c square eta by del t square, and s is root g h. In fact this is what the linearized long wave equation in one dimension. So we started with a flow, with a fluid which is two dimensional in nature, and we have arrived that, the long wave equation this is one dimensional in nature. And we have seen that, in fact a in the earlier case, we have seen that this del square eta by del x square 1 by c square del square eta, this is basically the one.

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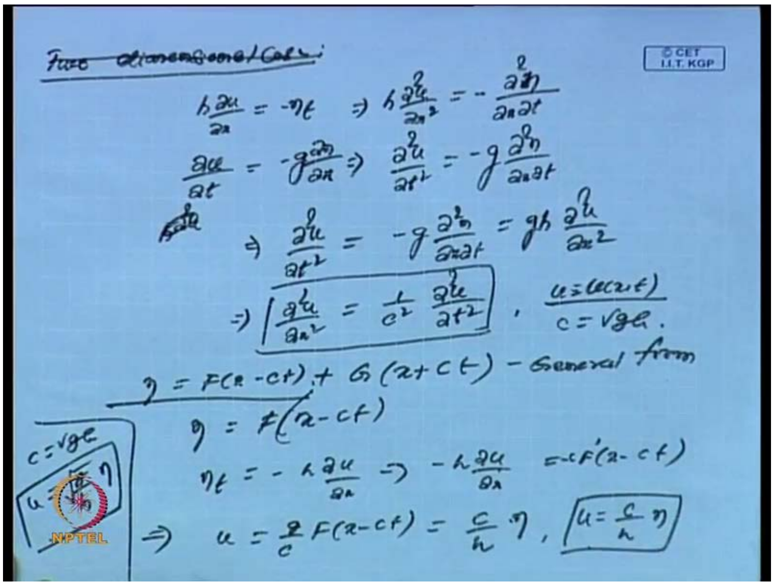

Now, what will happen, if I consider the two dimensional case. In case of two dimensional. Before going to that two dimensional case let me see another thing. So I have already seen that, if I again analyze this eliminate, because I have h del u by del x is equal to minus eta t and I have w by del t minus g 2 del eta by del x. So again what I will do, I will just say that; suppose what will happen h del square u by, this implies so what I will do. So this implies if I combine this two, I say del square u by del t square, is equal to minus g into del square eta by del x del t, and this is equal to same as minus g del square eta by g h 2 square u by del x square, and in simplifying this, we will get it del square u by del x square 1 by c square.

So again we are seeing here that in this case also, where u is the function of u x t; u also satisfy the wave equation, and here c is equal to root g h. Now this understanding, if will move a little more. Suppose I say that I have been given a wave, if eta is, if I start with like eta is a function of f of x and t x minus c t. If eta is f of x minus c t, because we know that (( )) form of eta is f of x plus c t x minus c t plus g of x plus c t. Now if I just asking that eta is f of, is the general form. So if eta is x minus c t, then what will happen, that I can always find that my u. We have eta t, you have eta t minus h sorry minus h into del u by del x, and which implies minus h into del u by del x is equal to eta t f of x minus c t.

So this is eta is a proper eta t minus h into del eta by del x, so it will give minus c times f dash x minus c t, and del if we integrate with respect to t ,then we get u into that with respect to x when you get u is equal to. So my u will be, it can easily see that my u would be g by c, and we have seen that u is g by c f of x minus c t. There is del u by del x this gives del u by del x you get c by x, and then you can see. So and that is what is equal to g by c, and it is same as, because c is a root g h, and that will give us c by h into eta, so which gives me u is equal to c by h into eta, this can be easily applied, as we always see from here, taking one derivative here, and that will be, because this. Then another result only. Again we have seen that c is a root g h, once c is a root g h, then we have u can comes, c is, this will be g c is a root g h so g by h into eta. So this is another relation, so it says that, what it says that the speed of propagation, is inversely. Speed of propagation is also inversely proportional to the water depth, so that is what another of the lesson.

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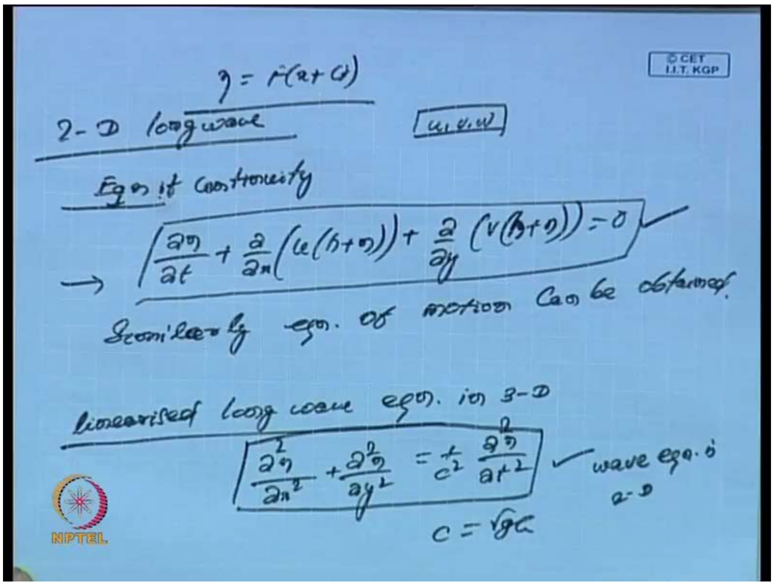

In the similar manner we can also, get some of the results in case of sorry in case of, when you consider physical to g is equal to or eta is equal to f of x plus e t, and that solution. Now question comes what will happen in case of a two dimensional, two d long wave. In case of two dimensional case, you can always see. In case of two dimensional long wave, you can see that the equation of continuity will be simple. I will not go into the detail, but I follow the simple procedure, where all the two horizontal component of the velocity, there will be two horizontal component of the velocity and one vertical component. So if u v are the horizontal component of velocity and v is the, sorry u v w are the component of a velocity, and let the horizontal component of velocity be u and v, w is the vertical component of velocity, assume w is negligible compared to u and v. So equation of a continuity in the assumption of the linearized long wave, as the equation of continuity, will give us, it will give us del by del x del eta by del t plus del by del x u into h plus eta plus del by del y v into h plus eta and that is zero. On the other hand, if you look at equation of motion.

And if you look at the equation of motion, then we will have, again we can get from the equation of motion; that is similarly, you can rewrite the equation of motion, can be obtained and n s, and it can be shown that, the linearized long wave equation of, linear long wave equation, in two dimensional, or three dimensional fluid domain would be del square eta by del x square plus del square eta by del y square 1 by c square del square eta by del t square, and here again c is root g h which will be, so that will (( )), and this little I am not going here I am just given the equation of continuity, as I mentioned here I will get two equations for the equation of motion, and then one is u and v two relations, and that we eliminate u v and eta then we, once we eliminate u v we will get the linearized longer equation of in 3 d, here the u v equation is 2 d is two dimensional. So when we have the, we have seen that when the fluid domain is 3 dimensional, the wave propagation is 2 dimensional, where as when the fluid domain is 3 2 dimensional, then in that case the wave propagation is one dimensional. Now, again I will just think of another case, where we call a case of variable topography.

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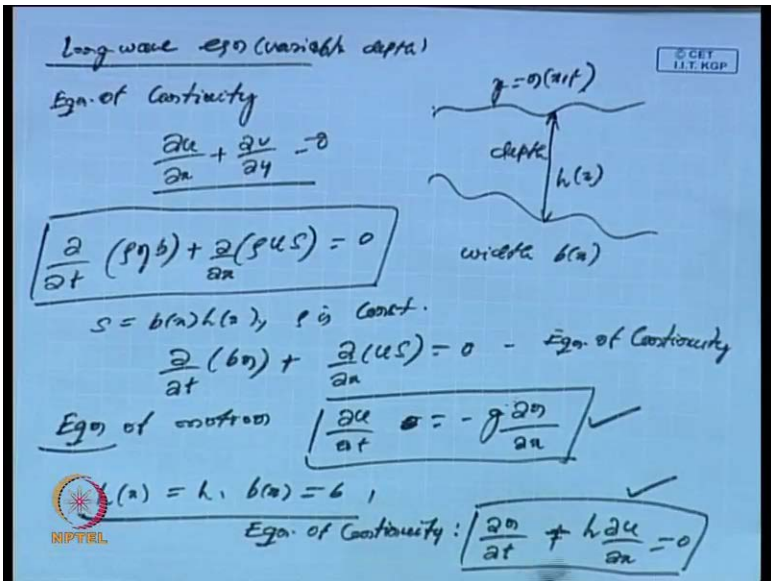

Suppose I say that I have a p h, which has variable in x and I say eta is equal to eta x t. Then in 2 d we have the equation of motion equation of continuity; variable depth, long wave equation, variable depth. So equation of continuity, will remain the same del u by del x plus del v by del y is zero, and it can be simplified. sorry We have seen that in case of 1 dimensional this is this. Now if I am looking at a ocean of a depth h, and I just look at the width of ocean as the. If I look at the width as b x, this is my depth, this is my width, and then this is that depth, then what will happen to the equation of continuity. Then in that case the equation of continuity will look like del b by del t rho eta b plus del by del x, this comes from the conservation of rho u s is equal to zero, and where s is nothing but b x into h x; that means rho is constant, then we will have del by del t into b eta plus del by del x into u s zero.

This is the conservation of mass, in case of variable depth and variable width. Now again we have the equation of motion, we have the equation of motion that is del u by del t plus minus g del eta by del x. This is equation motion, in case of only, and if you combine this, so then what will happen in this equation; s is h x u x b is. So if I say h is equal to h x is equal to h and b x is equal to b this is the motional equation case, then equation of motion will remain the same, where as the equation of continuity will reduce to equation of continuity becomes del eta by del t plus del eta by del t, this is equal to plus b will go plus h times del u by del x zero. This is what we have seen in case of a one dimensional, this is the case of a one dimensional long wave equation of continuity, and if you combine this, sorry this is. So this is the equation of continuity, for variable depth and breadth, but when h x is h and b x is b that means again, the same equation of continuity.

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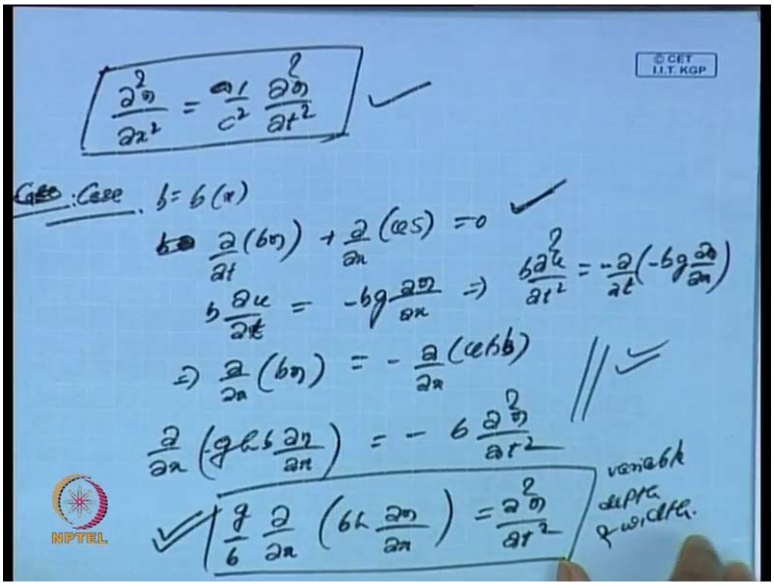

And if you combine this; that means if you have channel of width b and depth h, which is a constant ,then we can see that same del square eta by del x square is equal to del square 1 by c square del square eta by del t square. So this is again same equation is satisfied, if you are thinking of channel of constant width and constant depth. So it is a. So from one dimensional to, we are always looking at a problem, where the channel width is constant, then also same one dimensional equation is satisfied. Now I will further generalize this, now our next lesson I will have a look at it, another case I will take. In this case, and in this case I will say that; suppose h x is a, suppose I consider. We have already seen that in case of channel of, when b is equal to b x.

If b is equal to b x, then we have del eta by del t then I have b into del u by del t rather b del by del t b eta, so equation of a continuity del u by del t b eta plus del by del x into u s is u s is equal to zero; that is the continuity equation, and when b is equal to b x, and we have the equation of motion that is del u by del by del x; that is, this is the equation of continuity, and equation of motion is, del by del x; that is del u by del t is equal to minus g del eta by del x, and which can be written as; if I say b this as b. So that means, I can also write it as del by del x u b del u by del t b del u by del t is minus b g del eta by del x, and I have the… This is from the equation of and here del by del t, I have b eta b eta minus del by del x u h s, this is from here, some juggleries we have to do. and then we have seen that, then what will happen del by del x b eta so u h s that means u h b, so I will say that del by del x, if I take the del by del x now del square u b b is eta is (()) so that will give me b del square u by del t square.

So if I take from here I will get b del square u by del t square, and that will give me minus del by del t minus b g del eta by del x. And again so that is minus b del by del t so. So this is b g del eta by del x so that means; del by del x of so del by del x of minus h g b so h is constant so if I take this that del by del x of h g b. the series. I have b del square u by del t square, so I can easily get a del by del x g h minus g h in to b h b into del eta by del x. Then this is minus b del square eta by del t square, and again this will give me finally g by b, I can easily get it with little juggleries del by del x b h del eta del x is equal to del square eta by del t square. So this is what happens, when h is not constant, and b is not constant. Now this is what variable depth and width; this is one equation. Now again this steps on as to follow, little and then one will arrive usually this, I just a leave few steps here.

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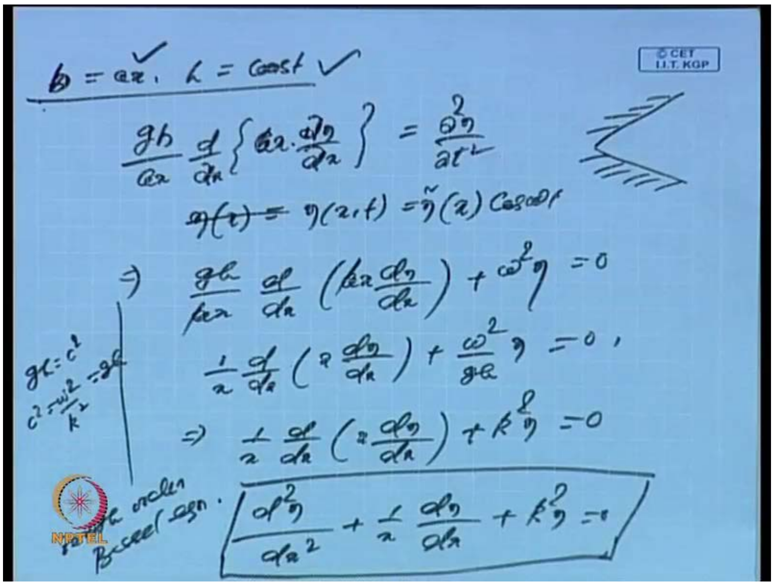

Then one can easily get in particular when, h is equal to a x, b is equal to a x and h is equal to constant, and this will give me; that is h is constant, so this will be g h by, because by b, b is a x into d by d x b h into b is a x h is constant already h has gone, so a x into d eta by d x equal to del square eta, and if I have, is equal del square eta by del t square. So this is, if i x eta x is equal to, sorry eta x t equal to eta x into cos omega t, then this equation will give me g h by a x into d by d x a x d eta by d x plus omega square eta is equal to zero, and that gives same as a a get cancelled, that gives me one by x, g h is c square and my, g h is c square so it shown by x d by d x into x d eta by d x plus omega square by g h is c square by g h let me put it zero, and then omega square is, we have g h is to c square, c square is g h, and we have c square is nothing but omega square by k square, and c square is omega square by k square; that is g h. So that means omega square by g h is not a k square, so its gives me 1 by x d by d x x into d eta by d x plus k square eta a zero.

If I further simplify this, that will give me, that d square eta d x square plus one by x d eta by d x plus k square eta equal to zero. So this is the case, this is the equation of long wave. And for a channel, where the width is varying, here the death is constant, and with h varying like this; e channel in h. And it can be seen, that this is a kind of first or zero third of this equation. We have this is zero third of the Bessel equation, and you know the solution of this equation is very interesting; that gives me eta x t, because the solution of this is c naught j naught, or other call it c 1 j naught k x plus c 2 c 1 c 2 y naught k x into cos omega t, this is the general form of eta. And again as we know that the property of five naught, as just at near x is equal to at x is equal to zero, y know naught will tend to infinity, and however we are looking for eta, which solution is bounded at the, where x is equal to zero, so that it shows that, it will say that c 2 has to be zero. So the process we get eta equal to c 0 c 1 j naught k x into cos omega t.

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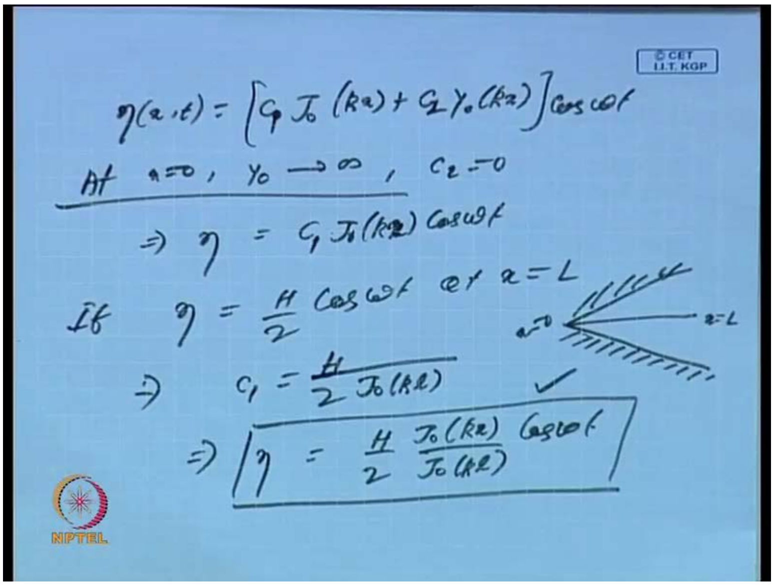

And again, if I say, suppose I say eta is equal to h by 2 cos omega t, but x is equal to l and that is; that gives me c 1 as h by 2 cos omega t c 1 will give me c 1 is. If it has attacks is equal then my c 1 will be h by 2 cos omega t, h sorry h by 2 into j naught k l, because cos omega t at x is equal to l this will be c 1 j naught k l, c 1 j naught k l is h by 2, so c 1 h by 2 j naught k l, and which implies by eta, if eta is this at x is equal to l then that will. So my c 1 is this 1, c 1 is this my eta will be h by 2 j naught k x by j naught k l into cos omega t, this is what the. If I am looking at a, this will be the general form of this lesson. And this is alike, there is wave which will, here my channel will look like this. This is at x is equal to l, and what x is equal to zero, this is the analyze like this, and this is the wave, eta will look like, and it can seen that, this is a spherical, it is a cylindrical waves, particularly for center wave, which decades at. Now in the same way, I only not go today much into the details of, more about the waste potaline again summarize few more simple cases. So, let me just say few things.

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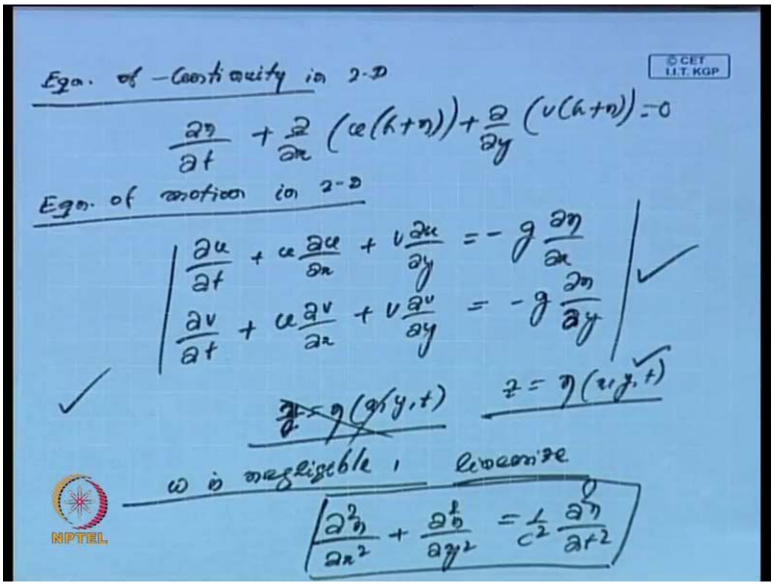

Here one seen that, equation of continuity in two dimensions, and that is what it is; del eta by del t plus del by del x u into h plus eta plus del by del y v into h plus eta equal to zero, we have already seen this. Now I just leave this as homework to do, I can prove it, as told earlier that we will discuss in (( )) about here, and now giving it as a homework. This is the equation of continuity, what about the equation motion, in two d, two dimensionally equation motion, and that will give me; that is. We will have, already what we have already here, that we have del u del t plus u del u by del x plus v del u by del y is minus g del eta by del x and we have del v by del t plus u del v by del x plus v del v by del y this is equal to minus g into del eta by del y, and again if you linearise this. So if we linearise this, that what it will give. So on linearization I just, through this can easily shown that, there are two things which is satisfied. I have to say, and here we as seen y is equal to eta (( )) z is equal to eta which is position of x y l t, so these are the some of the hints, for the two dimensions. sorry x y t and in that case if I show motion, will be this two case, and as now continuity will be this two, and this again, where as we, in this case we assume w is vertical, w is negligible, because where I have seen w is the vertical component of velocity, on this negligible, where you where the horizontal component of velocity, in which.

And from this, we can say that if you apply this small amplitude, linearise it. Once you linearise, then from the equation continuity and equation of motion, we will get del square eta by del x square plus del square eta by del t square del the y square, if I. This is the most rather simple diazole dam, I would like to mention here. So although initially I told that, this will be easily port, but I think that I am little, hence will help you to arrive at this. And this is very easy, because again, just like the case of one dimensional case, you can do this. So in fact, this one d and two d waves, and as I mentioned that in case of, for a two dimensional fluid we have a one dimension wave equation, here if I get three dimensional fluid, I have a two dimensional wave equation, and where in this case of, off course eta is a proposition x y and t.

So this two equations, this two long wave equations, one dimensional, and two dimensional long wave equations, plays a significant role, in several analyzing, several problems particular several problem associated to the wave proposition, along a post. Particular when you look at cases like tsunami, planetary waves, particularly where the wave length is, this is soluble to this for the long waves, or even if the analysis of strong kind of situation. Even if in many cases this linearised long wave equation is considered, and this gives quite, either I will say, the (( )) is quite satisfactory, it is a, although we have taken a small amplitudes, but particularly when we have seen that, travel time chart Tsunami travel time chart; that is prepared in various country, often the linearised equation of the long wave is taken and talk about, because the formula they have taken.

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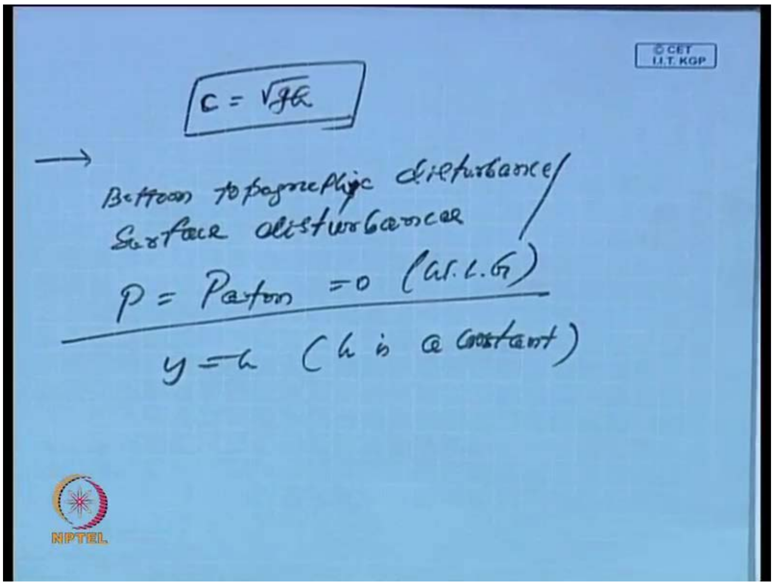

In many cases c is to root g h, and off course. So this large class of problem of force engineering, can be handled, under this assumption of long wave theory, where the wave amplitude is assume to be very small, and will be consider few more cases, of bottom variation, how the variation of bottom topography, on the surface disturbances, bottom topography as a disturbance in the bottom, or disturbance on the surface, bottom topography or surface disturbance, topography disturbance or surface disturbance, it helps how these are accumulated, into the long wave equation, or the equation of, while solving the equation, because in all these two cases, in both the cases, if we assume p is p atmosphere, and which is taken as zero, without loss of generality. Question is that when, p is not, suppose there is a wind, there is a storm, in such situation how the, because in such situation atmospheric pressure is not constant. Further in many cases you have taken y is equal to h; that is h is equal to constant, h is a constant.

So but in reality that it's not sure as we have seen in the previous case, how the width variation has changed the nature of the problem and how the wave proportional has changed. So these things few, at least few cases, while consider in detail in the next few classes, before going to close this wave promotion chapter, or basically the discussion of one wave motion, in this I will stop today.

Thank you