Marine Hydrodynamics

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Lecture - 2
Law of Conservation of Mass - Continuity Equation

Today, we will start about to the 2nd lecture in the series on Marine Hydrodynamics. And yesterday we have given a brief introduction about marine hydrodynamics, it is importance and various areas, which will be required afterwards as a continuation of this course. With this background, now let us go to a further details about the various laws of nature, basically we will concentrate today on the equation of continuity whose physical significance is law of conservation of mass.

And in law while before going to the law of conservation of mass, I will again review what you have done yesterday about the material derivative (( )). We have done it in the context of a vector particularly in the context of velocity and acceleration, today let us look at in the context of a scalar function. Suppose, I have a when I look at a scalar function, I will consider that as if we are considering in the fluid.

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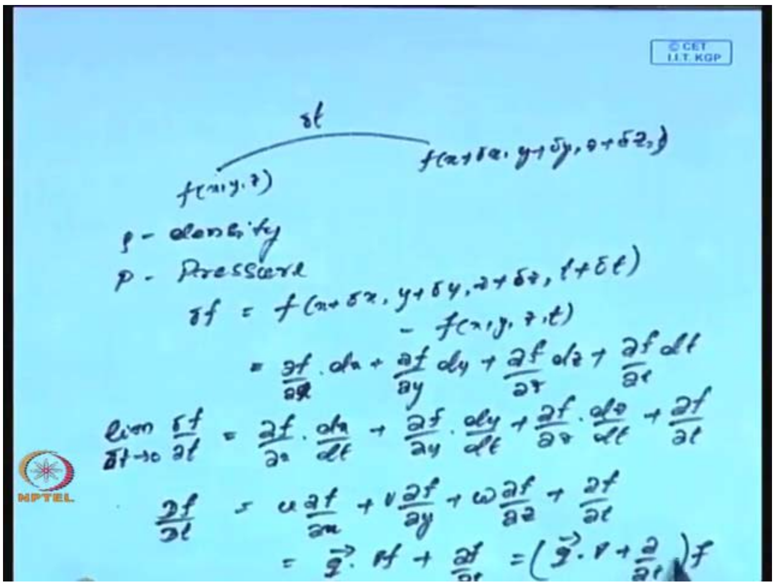

Some of the functions like density, fluid density or the pressure which are again a function of a x y z and t, because they all depend on the space as well as the time. Suppose, I take a point here P and the function that is f which is a function of x, y, z at time t and the same after time del t what will happen to this. These will be f of x plus del x, y plus del y because it is changing in a time and space z plus del z and del t.

So, if from here to here it moves, then what will happen my del f, how much is the change is observed. That is del f becomes f of x plus del x y plus del y z plus del z and t initially it was at time t minus f of x y z and t. If we do this and what we will get we will get this as, just assuming this time is small, so the changes in del x del y del z are all small. Then this gives us del f by del x into d x plus del f by del y d y plus del f by del z into d z plus del f by del t into d t.

So, if that is the case what will happen, then what will happen limit if you look at these in that sense that limit del t tends to 0. What will happen to del f by del t and this will give del f by del x into d x by d t plus del f by del y d y d t plus del f by del z into d z by d t plus del f by del t. And which is nothing but this is equal to u del f by del x plus v del f by del y plus w del f by del z plus del f by del t, which is again same as is equal to q bar to broad of f plus del f by del t which we can write q bar dot gamma plus del by del t into f.

So, this is what we had got in and this left side also it will happen that these will be give us that this is same as d f by d t, now this as I have last time I have talked about that this is called the total derivative. So, this is the locale derivative this is the conductive derivative. And now in practice in reality, this situation occur like when there is a change in the like in motion. When we have a change in the density from one place to another, because you can like look at examples like when there in a estuary.

There is a flow of water, the density changes with the space and time. Another example, I will give you what about stratification, the ocean water is a stratified and in this stratified fluid. The basically one of the factor which is very changes rapidly that is the density and that density of basically salinity. And the salinity changes with this both in space and time because we have at one point we may see the salinity is something, but if you go to another point you take an test and then we will see the salinity of the water is changing.

Similar, way you can also see that the pressure, another example I will say when there is a oil spill in the ocean. There are we come across on the top of the fresh water or the saline water, there is a layer of fluid which is, so that fluid flow from one place to another in the process which is change in time. The density also changes fluid density, because this is in if we consider non homogenous fluid or like the homogenous 1. So, due to change in pressure change in space and time, there can be a change in the density characteristics of the fluid.

And that is what this is a scalar function, so it is not always it is not necessary to that only we can talk about d by d t the total derivative. In case of a vector function we can also talk about the material derivative in case of a solar function provided, they have the similar characteristics and property of the fluid. Now, this background again yesterday I had talked about that how to convert using the gauss diversion theorem, I have told that we can always sense a surface integral to line integral. And this equation will help us, this theorem gauss diversion theorem will help us in deriving the equation of continuity.

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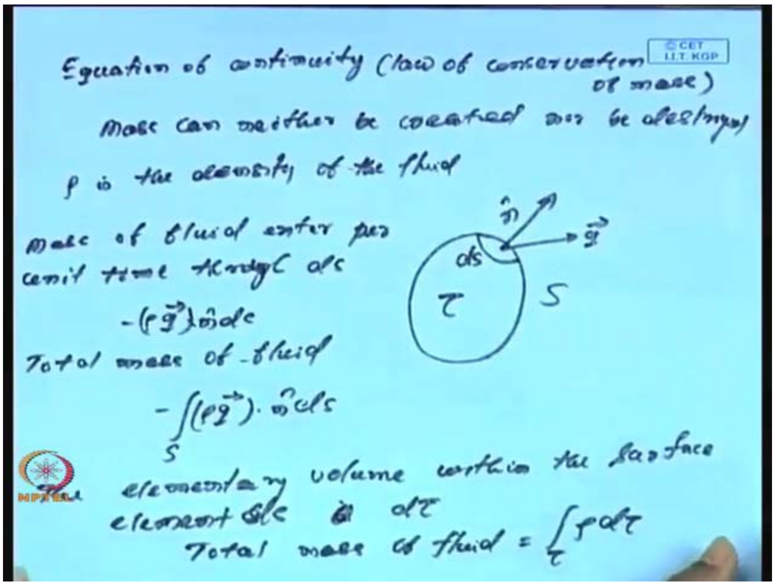

And when it comes to laws of equations of continuity, I will call it as, physically it is means equation of continuity in other words we call it as physical significance is law of conservation of mass law of conservation of mass. So, we says the law of conservation of mass so it says the law of conservation of mass says that creed mass can neither be created mass can neither can be created nor be destroyed it cannot be created nor be destroyed.

Particularly, within a specific volume; that means, the amount of fluid that will enter to a particular region, it will be a same amount of fluid has to will go out from the region.

Now so, let us derive this mathematically what does it says let us consider a to do that let us consider heavily fluid surface, and let me consider small this is the surface in the fluid a closed surface. Now, let us say the volume of the in this region the amount of fluid volume of the fluid is tau, and let us consider any elementary surface on this d s.

And on this elementary surface, let me say that velocity of the fluid particle is q be the velocity of the particle. And let me say that n is the out at the normal on the surface, then in that case then and let me say rho is the density of the fluid density of the fluid. So, if rho is the density, then we have already had defined n at q bar, s is the elementary mass. Then what will happen mass of fluid that will be what will happen to the mass of fluid That will enter per unit time, through this cross section through the surface element through d s. The amount of fluid that will enter through this surface elementary d s, will be minus rho q bar into dot n hat d s in outer direction it will enter through the fluid. And then already I have mentioned that here rho is the density q is the velocity, then what will happen to the total mass of the fluid.

So, we have the total volume is s, and then the total mass of fluid that will enter through this will be minus s rho q bar n hat d s, that gives a total mass of fluid that will enter through the fluid. Now, if you look at the dimension let us say that the elementary volume of the fluid, the elementary volume within the surface element s element d s. If it is a d tau, then what will happen to the total mass of the fluid, and we have the total mass of fluid because our total volume is tau which will give us integral about tau rho d tau.

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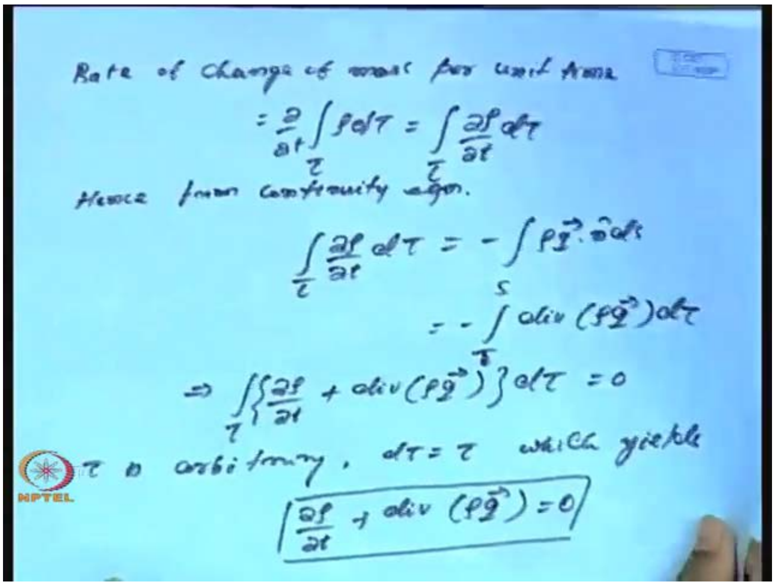

If the total mass is rho d tau then what is the rate of change of mass per unit time the rate of change of mass per unit time will be equal to del by del t integral of tau rho d tau. That is nothing but integral over tau because tau is a continuous fluid media, so we can always call it del rho by del t d tau. Now, from conservation of mass, hence as I have said that mass can be neither from continuity equation which that is the conservation of mass which says that mass can be neither created nor be destroyed.

So, we can have the amount of fluid that is del rho by del t d tau will be same as minus integral over s rho q bar dot n hat d s. And we will again apply to this the stokes divergent theorem or the gauss diversion theorem from that will give us minus tau divergent of tau rho q bar d tau. So, now, if I add this to which implies integral over tau del rho by del t plus divergent of rho q bar d tau is 0, which again can be rewritten as since tau is elementary, tau is arbitrary, tau is an arbitrary volume. We can always say d tau is tau, this gives once we say d tau is a tau, so this is this elementary volume is same as this, which yields; that means, this part as to be 0, because del rho by del t plus divergent of rho q bar is 0. And this is what is called the equation of continuity, else give a simplification of these, this further can be written as this equation.

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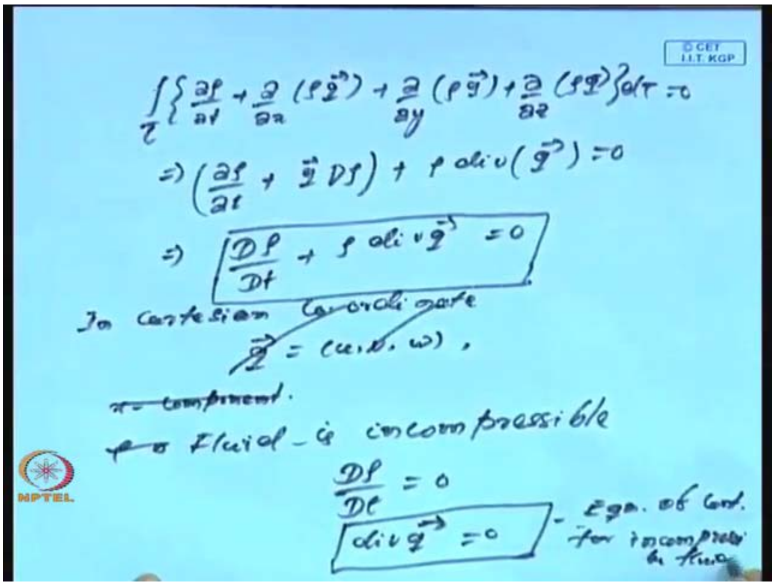

And just expand it in Cartesian coordinate sorry del by del x rho q bar plus del by del y rho q bar plus del by del z, you know which is same as del rho by del t. You can also write q bar you can always write q bar dot del rho plus rho divergent of q bar is 0, which is same as write as d rho by d t plus rho into divergent of q bar is 0. And this now in Cartesian coordinate again, we will write it in Cartesian coordinate we can say that if we because you have q bar is equal to u v w.

Then we can always say that the three components x components will be or rather we will put it in this way, I will come to this a little later better. So, now, this one we call this as the continuity equation, if this is the continuity equation then what will happen if rho is t rho if the fluid is incompressible. if the fluid is incompressible Then we have d rho by d t equal to 0, once d rho by d t is 0 then we have divergent of q bar that is 0. So, that this is what the equation of continuity for continuity for incompressible fluid. So, this becomes the equation of continuity for incompressible fluid.

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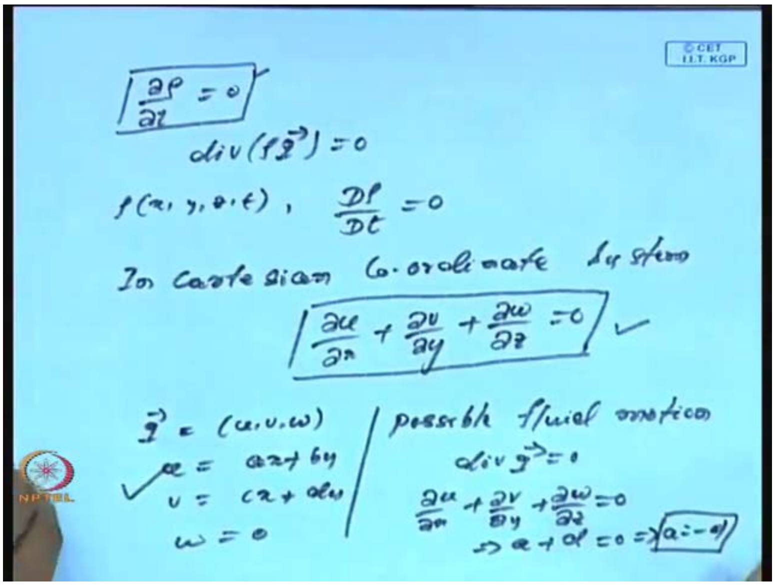

Now, if we look at this another point of view suppose the convey the another way of looking at, if del by del t del rho by del t is 0 means the fluid changes in the density is independent of time. Then we can have divergent of, but we have already seen divergent of rho q bar is equal to 0, but only when, but this does not mean these does not mean this is stereo type. It is the independent of the time, but it does not mean that the fluid is incompressible.

So, for incompressible fluid because rho is a function of x y z and t for incompressible fluid it has to have d rho by d t is 0. It is not necessary that d rho by d t is 0, so for a compressible fluid we have once for an incompressible fluid we have divergent q is 0. Where, as for the compressible fluid we have d rho by d t plus rho into divergent q is 0. Now so, in Cartesian coordinate we can write at for the incompressible fluid in Cartesian coordinate system.

We always write it del u by del x plus del v by del y plus del w by del z equal to 0 and this is the equation of continuity for an incompressible fluid. So, in this case because this is very important, and we will be using this because in hydrodynamics our major emphasis will be on incompressible fluid. So, most of the time when an incompressible fluid will always the Cartesian coordinate, we will always we will always refer to this equation as the equation of continuity. And in fact, it helps us in solving many problems in marine hydrodynamics, then now I will go with an example.

Suppose I have been given that suppose for a particular fluid, suppose the velocity factor is given by q bar is equal to u v w, and which are given by were as u is equal to a x plus b y v is equal to c x plus d y and my w is equal to 0. So, what should be the criteria on a and a b c d, so that this will represent a as a possible fluid motion. Particularly, in this case this is independent of a rho, so you can always say that if divergent q will be 0.

Then from this equation from because we have been u, so we have del u by del x plus del v by del y plus del w by del z equal to 0 gives us. It gives us del u by del x that will give you a and del v by del y that will give you d and w is 0. So, a plus d 0 is implies a is equal to minus d, and this is the criteria for the velocity vector given by this. To ensure that there is a fluid motion is possible, so a has to be minus d in this case for a possible fluid motion.

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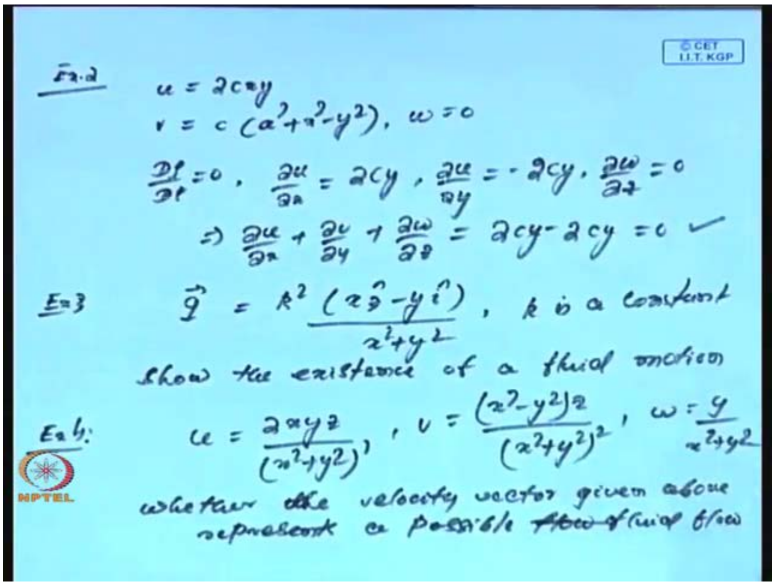

Now, I will go to another example suppose, I will say suppose u is equal to two c x y and my v is equal to c into a square plus x square minus y square and my w is 0. If this is the case here also u v w are independent this is my example 2. So, in this case also u v w are they are all independent of rho. So, it can be easily seen that d rho by d t is 0, then we have we have del u by del x is equal to 2 c y del u by del y minus 2 c y and you have del w by del z where w is 0.

This is 0 which implies, but del u by del x plus del v by del y plus del w by del z is equal to 2 c y minus 2 c y that is 0. So, since this is 0, so we can always say that the velocity field that represented by u is equal to 2 c x y, v is equal to 2 c into a square plus x square minus y square and w is equal to 0 represents the flow of an incompressible fluid. Now, with this example I will give some of the example for you to work it out at room, in your home that I will just give you a homework.

Suppose, my q bar is given by k square x j hat minus y i hat divided by x square plus y square where k is a constant. Here, it is independent of z component, so the velocity, so w is since it is not mentioned about z components. We can always say w is equal to 0, in this case, so the motion of, so the existence of for these velocity of the q, so the existence of a fluid motion (no audio from 26:59 to 27:08).

So; that means, again we will show that del u by del x del v by del y plus del w by del z is 0, another example I will say example 3, I will just this give you as a homework and you can try this. Suppose, u is equal to 2 x y z divided by x square plus y square square, and v is equal to x square minus y square into z divided by x square plus y square square, and w is equal to y by x square plus y square. So, the velocity vector the vector fluid, if q bar is q given by this u by w by this then also we can show that, so whether a possible. whether.

So, whether there is a velocity vector given by these velocity vector given above representation a possible fluid flow possible fluid flow. It is a possible fluid flow. So, it can be easily checked it, because again you are using because you have to again say that whether the divergent of q is 0; that means, del u by del x plus del v by del y plus del w by del z whether it is 0. Thus we are now clear about the equation of continuity, it provides us the knowledge about whether there will be a possible motion or not.

If a once we are sure then I will say because this is all in Cartesian coordinate what happen, and particularly when there is we do not we have not talked about there rotationality of the fluid particles. Now, in this context I will talk about vorticity vector.

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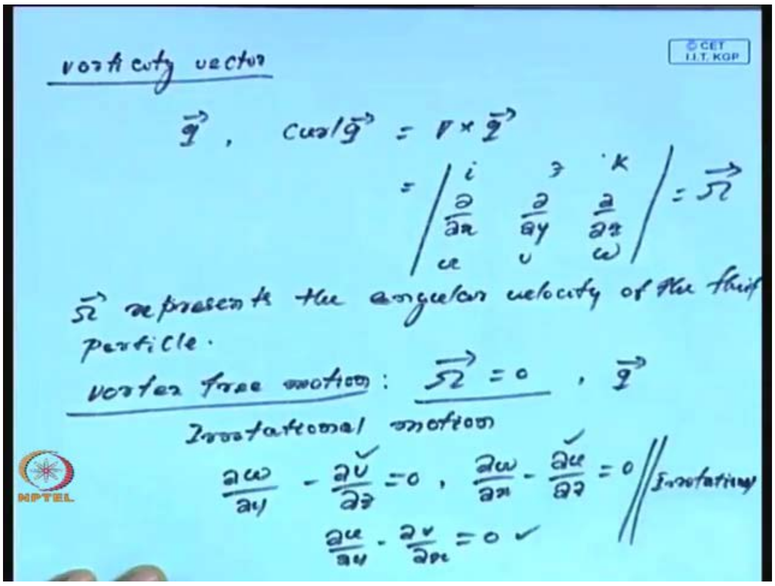

Vector vorticty vector, if q bar is the velocity vector of a fluid particle then what will happen, if q is the velocity vector of fluid particle then what will happen curl of q bar. Curl of q bar will be del cross q bar and that also we can write it as i j k then you have del by del x del by del y del by del z that is u v w. And this I call it as omega bar and I call this vector at the vorticity vector, this represents this vorticity vector omega bar represents the velocity of a represents the angular velocity of the fluid particle of the fluid particle.

It is kinematic property of the flow, and at each point it gives the angular velocity of the fluid particle. Now, once omega bar is this, this is a vector and it is the angular velocity of the fluid particle provides. Then with this we can go to what is vortex free motion rather we will call it vortex free motion, in case of a vortex free motion; that means, if omega bar is equal to 0; that means, fluid particle will not rotate and we call this as. So, if omega bar is 0 then we call this a vortex free motion, sometimes we call this as a irrotational motion same we call it as irrotational motion.

So, in case vortex free motion if a component wise if you will separate them, because it has 3 components for i j and k, basically the x y z components. So, we will get del y del w by del y minus del v by del z equal to 0, then we have del w by del x minus del u by del z this is equal to 0. And another is that is your del u by del y minus del v by del u by del y minus del v by del x 0.

So, these are components wise if these three quantity is as 0 for a particular velocity vector q bar, our q bar is the velocity vector omega bar is the angler velocity. And what we call the vorticity vector, then if omega bar is 0, than this 3 quantities are 0 and we say the flow as irrotational flow is irrotational or we sometimes we call is at vortex free motion.

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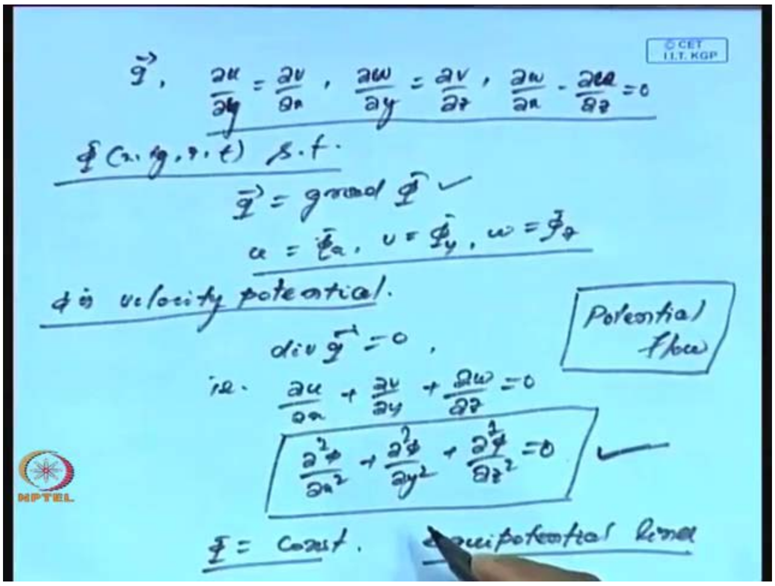

Now, in this case of irrotational flow, once we have the irrotational flow we have seen what will happen, because we have already seen for a velocity vector q bar. If del u by del x is del u by del y if del v by del x and del v by del v by del; that means, we have del w by del y equal to del v by del z. And then we have del w by del x minus del w by del u by del z this is 0. So, can you find a vector can you find a scalar rather phi which is a function of x y z and t.

Such that if I say phi is the scalar function, such that q is equal to q bar is equal to gradient of phi. If I can find a function such that q is equal to grade phi and what will happen my u bar will be phi x, v will be phi y and than my w will be phi z. So, we can if I get u v w as phi x phi y and phi z and q bar is grade phi, than we can see that it will easily satisfy the 3 condition. Because, if you substitute for u v w from here, in these expression then we can easily say that this satisfy the this equations sorry this is not 0 this is same as this.

So, if this is place then what will happen; that means, since we can find a phi were q is equal to grad phi, and it satisfy these equation then we can say that this phi is call the velocity potential. So, what we have observe phi is the phi is the velocity potential, now what we have observed when the flow irrotational; that means, when the flow is irrotational we can always find we can find a phi such that q is a grad phi.

And again let us see that what will happen to divergent of q, we have divergent q is 0 that is we have del u by del x plus del v by del y plus del w by del z this is equal to 0. And if we put e is equal to phi x, v is equal to phi y and w is equal to phi z, then we will get del square phi by del x square plus del square phi by del y square plus del square phi by del z square is 0.

So, what we have seen here that when the flow is irrotational when the flow irrotational, there exist a velocity potential phi, such that q is equal to grad phi and that phi satisfy our Laplace equation. And if I say phi is equals constant, then for each constant I only get a surface like Accor, and I will call this equipotentials lines. I will get a line for each constant, I will call them as equipotential lines. So, phi is equal to constant were each phi will get and will call as equipotential lines and the corresponding flow.

Sometimes, we call this flow as potential flow so; that means, when the fluid in compressible when the fluid is in compressive as well as the motion is irrotational, then we will have a potential, will as flow we call it as potential flow. And in such a situation we will have that the potential for a potential flow, there exist a velocity potential phi which satisfy the Laplace equation this is what. So, this is a is a kind of irrotational flow, this is the same as a continuity equation. Now, if this and always you can see that weather the flow is this is for a three dimensional flow, this potential function exists phi will exits.

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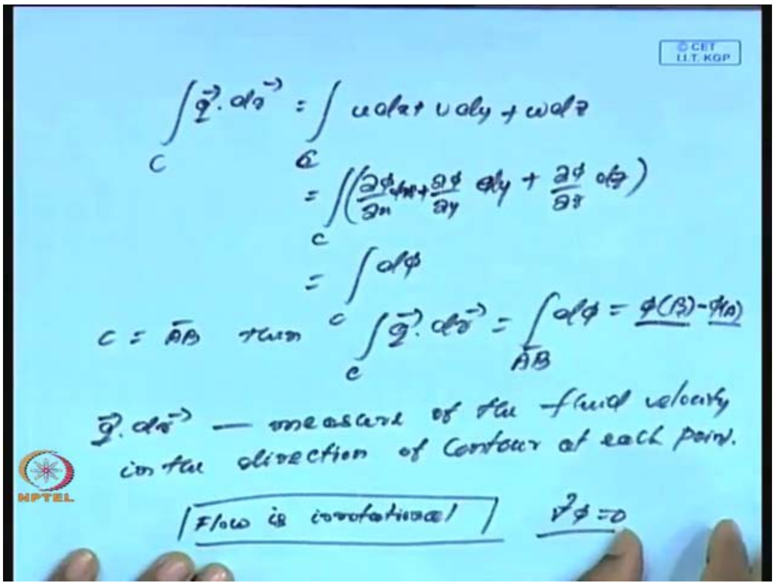

Now, what is the physical meaning of phi, if I look at the physical meaning of phi, because if I look at this like if I considered the integral c my q dot d bar d r bar. What does gives me? This gives me integral over c u d x plus v d y plus w d z and this gives me integral over c. If I say the flow irritational for a rotational flow, this will del phi by del x plus del phi by del y plus sorry d x plus del phi by del y d y plus del phi by del z d z. And this is nothing but d phi integral our c that is nothing but d phi.

And this if I say when c is equal to A B, when integral over c q bar dot d r bar integral over A B A to B basically, and is d phi that gives me phi A phi B minus phi A. And what does this gives? And now that q bar dot d r bar represents, this is a measure of the fluid velocity in the direction of counter at each point. So, the velocity potential phi velocity of potential phi and here we see that this is these are the two ends points B and A, so it is the independent of the path, because it is just phi B minus phi A.

It does not dependent on the path has a velocity potential it is related with the product of the velocity vector, and the length along the path between the two distinct point A and B. This is what I understand, so can I say that we will come to this that the necessary condition now for the velocity potential phi to exists it that the flow has to be irrotational. Unless flow irrotational only for a irrotational flow we can have del square phi is 0 and basically the velocity potential phi exit in this case.

And this when the flow is irrotational we always call this is a vortex free motion only. So, weather the flow is a two dimensional or three dimensional, we will always have the velocity potential means suppose I say that.

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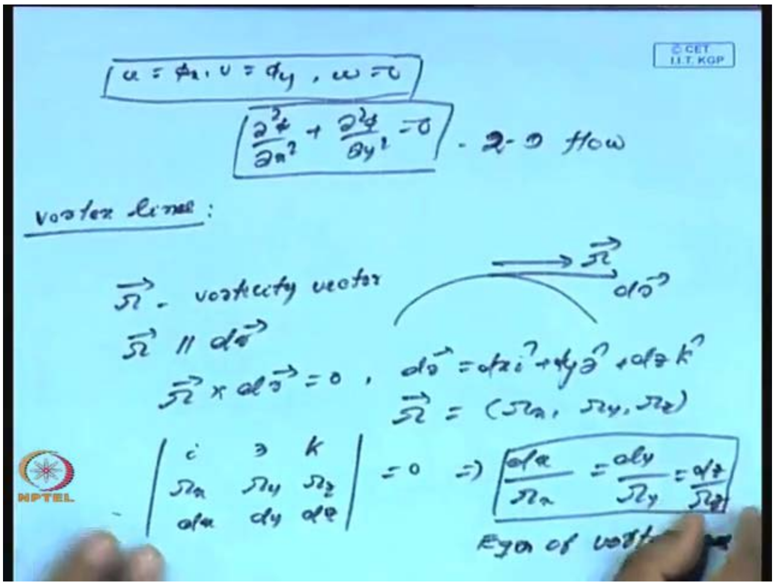

If I say u is equal to phi x, v is equal to phi y and w is equal to 0 and I can always say that this is the two dimensional flow. And in that case also I have a velocity potential, in that case the corresponding equation, continuity equation will give me del square phi by del x square plus del square phi by del y square is equal to 0, that I for a two dimensional flow two dimensional flow. Now, in this we have understood by now what is a velocity potential? what is a vortex free motion? what is irrotational motion?

Now, I will go to what is a vortex line also we have come across what is vortex lines? Suppose, I have a flow and if omega bar is the vorticity vector and let so if omega bar is the vorticity vector above this is the omega bar the viscosity vector. And then I draw a tangent at this flow that is d r bar, what will happen if omega bar is parallel to d r bar; that means, the vorticity vector is in the direction of the tangents.

If omega bar is parallel to d r bar it is same as telling omega bar cross d r bar is 0, this is a cross product where d r where that d r is can be given by x i cap or d x i cap or d y j cap plus d z k cap. And already we know omega has a component, omega bar has a component omega x omega y and omega z. If that is the case then what will happen omega bar d r bar, if that is 0, we can also find easily from this; that means, i j k you have omega x omega y omega z, these are the components of the vorticity vector.

In the direction of x y z directions d x d y d z are the component of the (()) d r. So, if this is equal to 0 which implies we can easily seen that d x by omega x, d y by omega y is equal to d z by omega z. So, this is this will give us the equation of the water vortex line this gives us the equation of the vortex lines. So, when the; that means, again when it comes that when the flow is irrotational these vortex lines will not exists because for, so I can conclude that because the omega bar will be 0.

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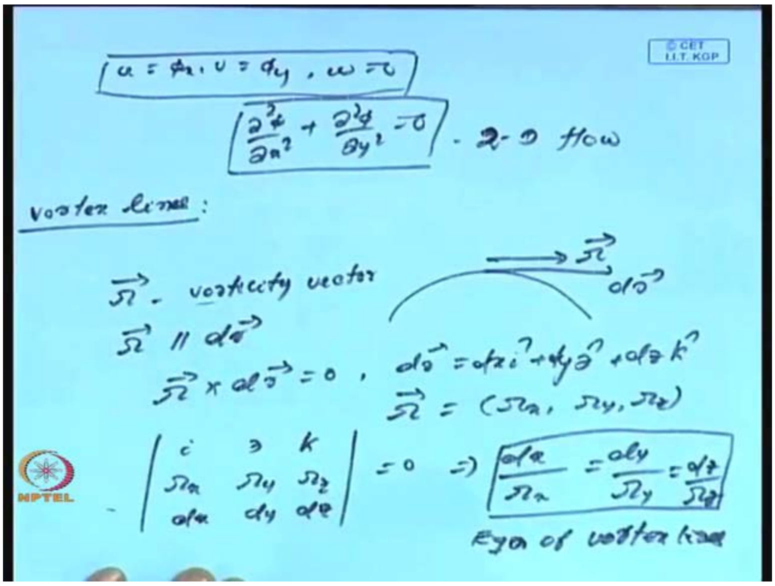

In case of irrotational flow because vorticity vector will be not there it will be 0, so in that case we will not have any vortex lines.

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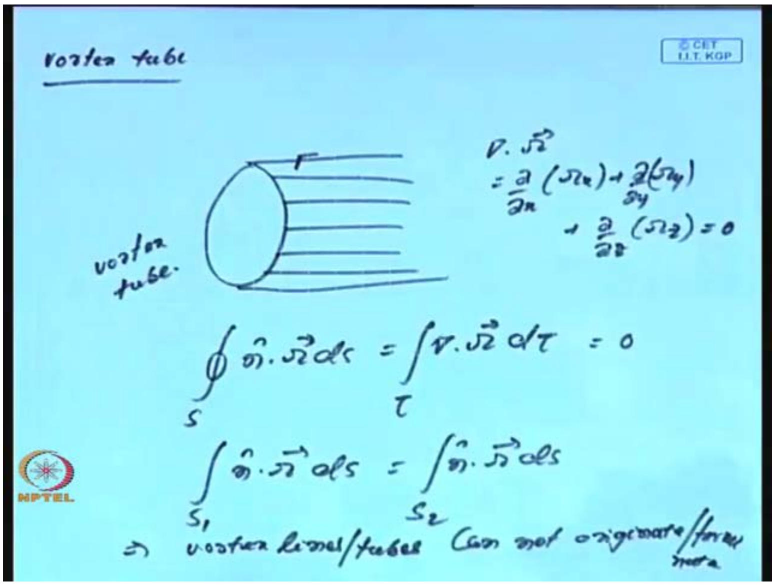

Now, we have already talked about then what will happen to vortex line, suppose what about vortex tube. In the fluid let us take up any closed curve gamma and at each point we draw vortex lines at each point we will draw line. And this (()) path change is a closed curve in the fluid and at each point we are, so the tube will be obtained will obtained, this is called a vortex tube.

Then from here I will make one conclusion from here, suppose I have any close surface s, what will happen to n hat dot omega bar d s. If I apply the gauss diversions theorem this will give me integral over tau delta omega d tau. And this can be this can be if u substitute for omega, because omega bar is the vorticity vector and then del dot omega bar that will give us 0, this will give 0. Once this is 0, because this is from this to this we get by gauss diversions theorem and this part it can be easily seen that.

If you say that del dot omega bar that will give you del by del x omega x plus del by del y omega y plus del by del z omega z, and that will be because this will be equal to 0, it can be easily checked by using the components of this. So, that is shows us and this is 0, so what does it gives us? That means, if I take two surfaces, two curves, two close surfaces and then I just look at the calculate the n hat dot omega bar d s at this point, and take it at another surface calculate the n hat dot omega bar d s.

Then both are same because this is 0, so this both can be same which means that the total strength because that will show us the total strength our vortex tube emerging from s 1, the same of emerging out of s 1 will be the same as the strength of the vortex emerging out of s 2. And that is we suggest that we suggest that the vortex tubes vortex line a vortex tube which implies vortex line cannot a originate from cannot originate what are minute at any at internal points in a fluid.

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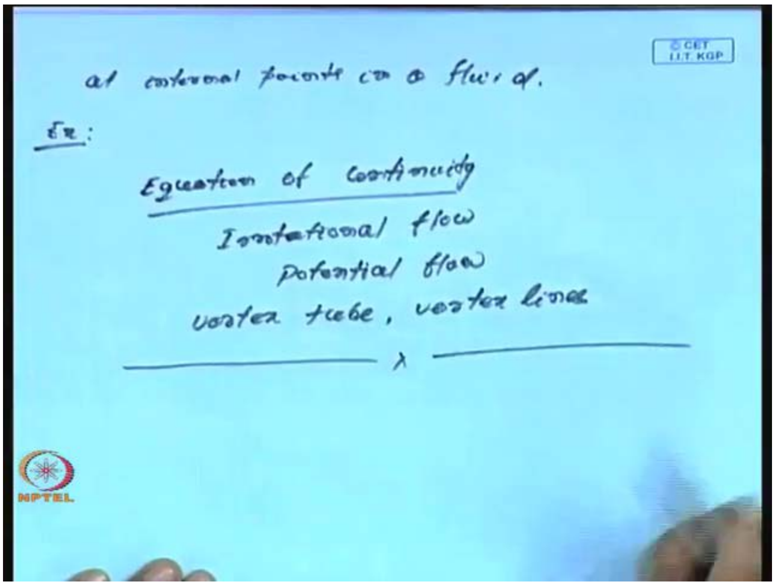

Again only for closed curves or terminate at a boundary surface, for example, if we look at the smoke rings. In case of a smoke rings the vortex lines they from close curves on the other hand in case of while pole the vortex lines, the terminate at a boundary in the fluid. If we look at the hurricane another example is that in case of hurricane the eye of hurricane we know, that it is always makes a circular motion.

In that in that case also only the hurricane dissipate, only when if hit is from the sea when it approaches the towards the land and approach a certain boundary. So, then it dissipate the energy dissipate dissipated and then we do not see any a vortex, the lose there identity the strength they lose their strength. So, this understanding approves to vortex lines and vortex tubes, now we will go to we will talk a little about stream lines and path lines.

So, here what we have talked by now in this lecture, I have already talked about the equation of continuity by now I will just summarize. I have talked about equation of continuity we have talked about equation of continuity, then we have talked about some of the characteristic motion that is irrotational irrotational flow and then the flow is irrotational. And the fluid incompressible we call this as a potential flow, I will just summarize all this things in this lecture.

And then we have talked about vortex lines, so we have talked about angular velocity of the fluid particle, q is the velocity vector. Then we have the we have the vortex vorticity vector and then we have a from vorticity vector, we have talked about vortex tube and vortex lines, so this is the this is all about in this lecture. In the next lecture we will come to stream lines, path lines and then we will talk about stream function try to connect the stream function with the stream lines.

And then we will go to we have velocity potential in velocity potential, we have the velocity potential and then we have the stream function we will try to relate both the stream function and the velocity potentials. And afterwards we will go to work out few examples on stream lines and to find the flow direction of a particular fluid, like that that will be interesting and I will stop this lecture with this.

Thank you.