Marine Hydrodynamics

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Lecture - 17
Schwarz-Christoffel Transformation

Welcome you to this lecture on marine hydrodynamics. In last few lectures we have spent time on various types of computer mapping. In fact, the last one being very important where we have talked about the application of Joukowsky transformation to ellipse from circle to using the Joukowsky transformation, we have used map circle to ellipse then we have mapped from circle to gambard aerofoil, symmetric aerofoil and half clue arac. Today, in fact yesterday also I have introduced, I have just meant introduction I have told that today I will emphasis on a new type of transformation, again a kind of confirming mapping that is the Schwarz Christoffel transformation. And this Schwarz Christoffel transformation helps us in the mapping a polygon particularly simply closed polygon to when the jet plane to a line on the zeta plane. So, only before going to the details of this Schwarz Christoffel transformation, let us talk a little about to what I mean a simple closed polygon and some of the characteristics. Although I have repeated it yesterday, let me repeat it again.

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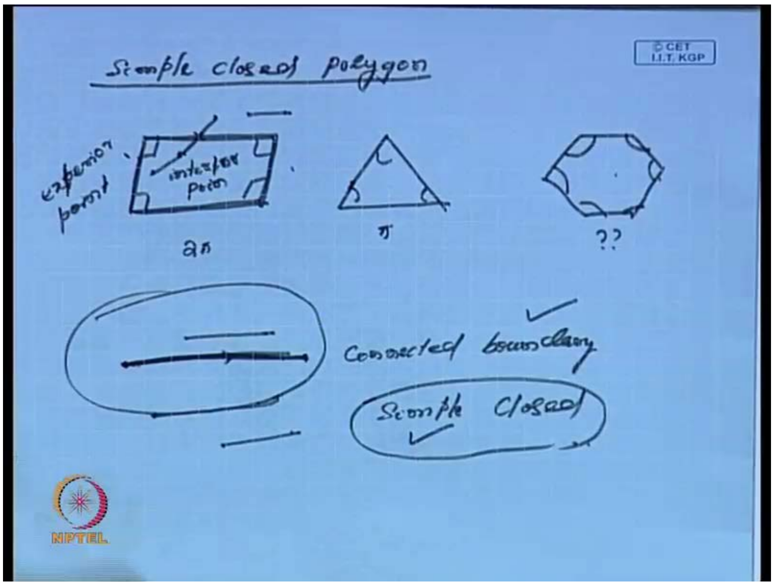

So, while I say let me start simple closed polygon. So, suppose let us look at these, this is at a panel, look at this is a triangle though we have four angles here, now total is 2 pi, and here we have 3 angles where sum of the 3 angles is pi. Now, suppose I look at another polygon; question comes what is the total angular of this is one aspect this will be the second point, this we know about this, we will come to this. Now the second thing is what I mean a, suppose I have 2 points on a. Let us look at suppose all point, what I mean above in all. Suppose, I have a point on this in a boundary and this is another point on the boundary, it is always possible to go from one point of the boundary to the other point of the boundary by following a path. And this is the path and which never sleep down boundary from this to this when you at arriving when you lead the path and this boundary is called a connected boundary. I call this boundary as a connected boundary.

The second point I want to say, in fact, this boundary what it does if I have or let me look at any boundary what it does, it will divide the whole plane, whole plane into 2. It partitions the whole plane into 2 parts. Because on this boundary wall is called the like this is a line in set up. If I look at the rectangle or this triangle or this, we have we call this as interior. Anything inside the boundary call interior, any point inside the boundary is a interior point. Then you have the boundary and we have the exterior on the boundary, exterior any point outside this boundary is called the exterior point. If I take point here as a point here it is exterior point, but it is an in one plane.

So, in plane we have three things one is an adjuster of the boundary, one is the interior and one is the boundary itself. So cussed cops now, can you go from one interior point to an exterior point without crossing the boundary it is not possible. So, from one interior point, if I have to go to exterior point; that means, we have to cross the boundary at some point at least at one point and in. So, this is what this is a characteristic of a simple closed polygon; one is the boundary, one point on other is you can always follow. You have to follow particular path; the second thing that from one point to another point. If want to go from one interior point to exterior, you have to cross the boundary at some point and the other things the boundary divides the points of the plane into 2 different regions; one is the exterior region, one is the interior region and the interior such interior point such that any 2 points can be join, can be joined by path which is again inside the boundary where as 2 exterior points. If you take any 2 points; if you join them, it has to be outside.

Then you can collect it by two exterior points outside the boundary. So, in that if they satisfy this, any circles then we call this as a simple closed, where simple closed polygon. So, if this understand on a polygon and sometimes what simply comes in case at any point on the plane is either because we are simple way is coming and 2 points in the plane is either both are internal points. If you have to join them or you can to join 2 points on this, on this boundary or we can always go from a point to another point on the exterior, and this is that is why it is called simple closed polygon. Now, this understands sometimes we call this simple collected for by collected system and with this understanding, if I have them. Now, let us look at clear cases on this boundary.

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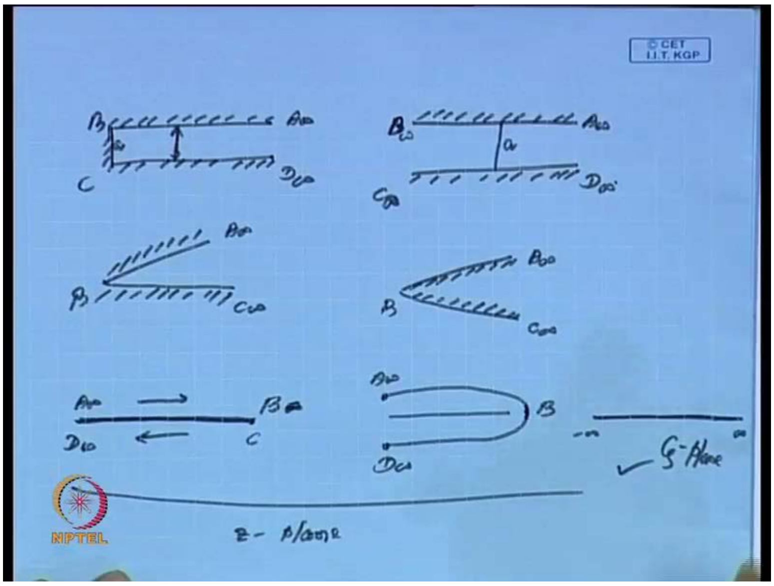

Suppose I have, if you look at this; suppose this is on the boundary, this is on the surface, when we say A infinity, this is called B. This is C, this is D. I will call it infinity. So, there are 4 vertices, 2 vertices are infinity and 2 are at some point finite point and this is this side is closed. So, there are 3 sides which are closed; one side it is open up to infinity. Now if I just say that 2 straight lines; suppose this is one boundary this is another boundary, and suppose this is called A infinity to B infinity and this is C infinity to D infinity. This is a rectangle; this is a kind of rectangle whose vertices are positive infinity. This is negative infinity and they both are certain distance apart. Suppose A here also you can say suppose there distance apart is A, but in this case this side is closed, this side is extended up to infinity. On the other hand, I will go to third case. Let us look at a triangle. This triangle, I say that A infinity this point is at infinity and I have B and this is suppose C infinity. So, suppose A infinity and C infinity they are at infinite and B is a fixed point, then you call this and in a similar value we can also look at this.

Suppose, I look at the exterior part, suppose this side is closed so that means, I look at this open region. Again there also A infinity B and C infinity. Now I look at another, suppose look at I have A infinity to this is B infinity and then or may be this is a fixed point, then I suppose again this line goes to C to D infinity. That means, this is a line which is coming this way and going this. Where what is happening A infinity and D infinity are conceded and again B and C are conceding. This is a line which is conceding 2 lines which are conceding. Then I may look at another point.

Suppose, I have line if I say this is A point infinity, that is only one point B and this is another point D infinity. So, we have 2 vertices and this is this point B is one point, so these are some of the simple polygons. Now with this understanding about polygons I will just go to what I exactly if from this polygon can we plot this two lines. Suppose, these are available in the z plane this is the z plane, can you map this all these things to line in the zeta plane that is on the real axis of the zeta plane. This is minus infinity to zeta plane zeta is A then a complex plane. So, you can do so. Let us see how we can go with this and that transformation is what the Joukowsky transformation. We talked about now this is what the Joukowsky transformation types.

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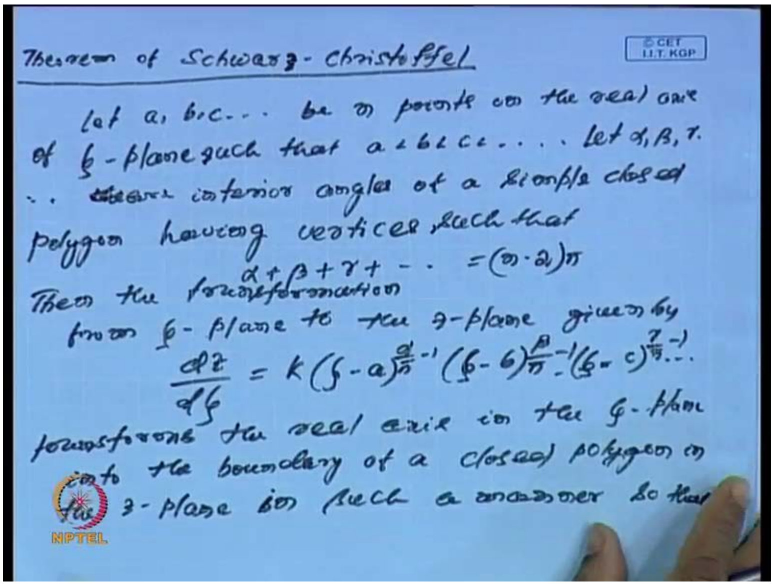

So, let me say if you wrote the theorem of Joukowsky theorem of Schwarz Christoffel. If theorem says that a b c b n points on the n points on the real axis of zeta plane I have 2 planes, I have come to that, such that a is less then b is less then c is less then the elegant such points. Let alpha beta gamma that the are the interior angles of a simple closed polygon having n vertices such that alpha plus beta plus gamma plus all this angles interior angles together is n minus 2 into pi. Then the transformation, then the transformation from zeta plane to z plane given by d z by d zeta is equal to K is equal to the K zeta minus a to the power alpha by pi minus 1 zeta minus b to the power beta by pi minus 1 up to again solve gata minus c to the power gamma by pi minus 1 dot dot. So, this is what it says transform the real axis in the zeta plane into the boundary of the closed polygon in the z plane boundary of this in such a way in such a manner.

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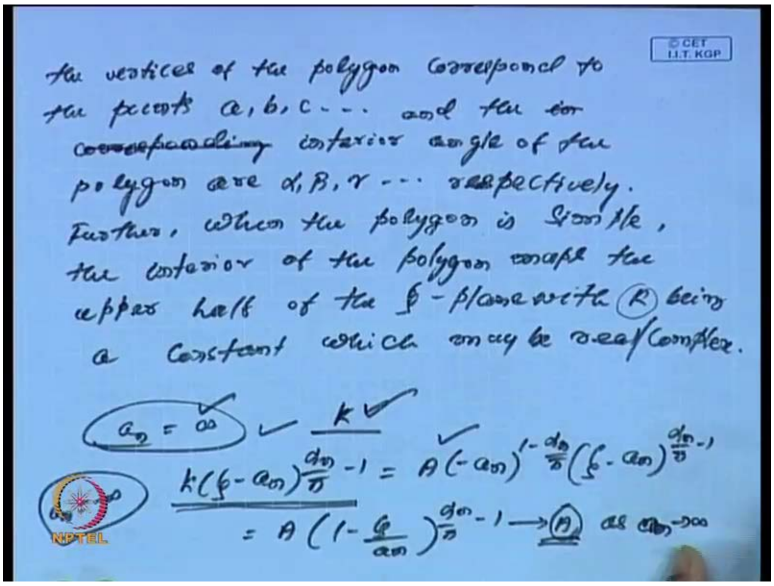

So that the vertices of the polygon correspondence to the points a b c and the interior angles on the corresponding inter and the interior angle from the polygon are alpha beta gamma respectively for the when the polygon is simple, the interior of the polygon is mapped of the polygon maps the up upper half of the z zeta plane and here in the transformation with K being a constant which mean by the real off on the real or complex. So, this is what the Christoffel transformation; all that glue slide on the things are very clear. Because basically we are defining one transformation, one function d z by d zeta and wishes is what will happen to the point is, because all this points are consider a finite point. Suppose any of this point a n is infinity or if n a of this point is infinity, then what will happen; that means, if another point is at infinity then this point will not apply in the transformation.

Like suppose to, so that I can always adjust my K if this is one of the terms, I can adjust my k so that this point I can make it. Let us look at K into zeta minus a n to the power of let me call this as alpha n by pi minus 1, and to be that I can also K. If I choose properly, that I will call it K as A into minus a n to the power of because K n is a complex number; 1 minus alpha n by pi and then the term associated with K is zeta minus a n into alpha n by pi minus 1. Then that will give me a into 1 minus zeta by a n to the power alpha n by pi minus 1. And once this is there, then when and this will tend to A as may a n to attend to infinity. So, because of that what will happen this term the n th term when a n is infinity? We can see it will adjust by constant K because this is a constant so that this will not contribute, this will not contribute. So, because it is in that way, it is another complex constant. So, this constant can be at zeta. This constant is which can be chosen properly.

This constant K which is not known which has to be chosen depending on the characteristics of the transformation we are doing or depending on the kind of polygon that is using. So, we can always adjust this particularly k. So that similarly if we say that suppose, a is in minus infinity you can say a n is minus infinity. You can also say that that are will not contribute to the. So, any point. So, only the finite point will contribute if n's are vertices, the vertices which are at only finite point then contribute to the transformation, while other points will not contribute the points at infinity, what it says the infinite will not contribute to the transformation. This is what this helps in a many way to solve simplify the problem to large extent.

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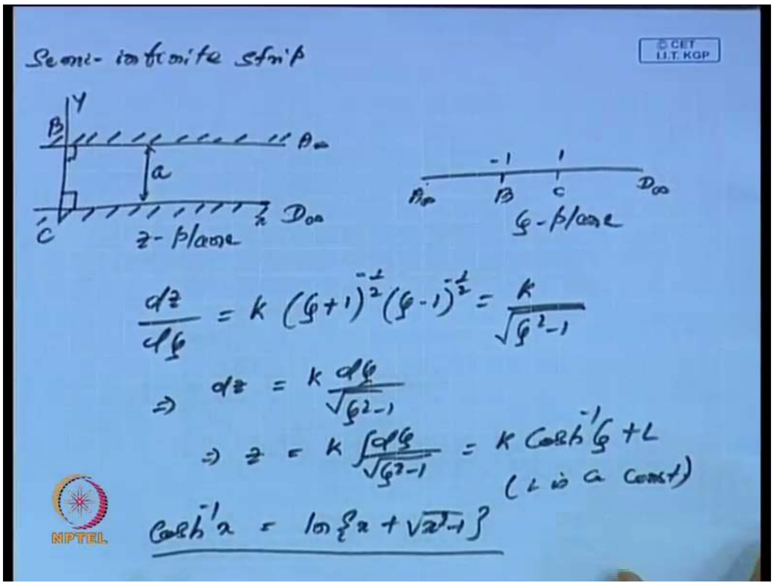

Now let us map a semi infinite stream will have a better clear understanding on this once we work out here through so infinite strip. Suppose this is a y z plane. In the z plane, let me call this then we call one; what is at infinity this is fixed at B this is C and this is y axis and this is by D infinity this is the x axis really this is the x axis. Now I will take because this is zeta plane. In the zeta plane what I will do what I have to map this 2 line when it is zeta plane. So, as usual say that A infinity be at this point and I choose my B infinite and this distance is let this distance be to the 2 lines is A, then B I call it minus 1; that means, another point that C point I call it as one and D infinity is at infinity. So, this stream flow A infinity B C D B infinity in the z plane is now we will map this to the zeta plane A infinity to D, A infinity B C D then what will be main transformation as usual as I have just mention that the 2 vertices are infinity they will not come to the transformation and again.

So, I have the 4 vertices here at in this plane zeta is minus infinity minus 1 one and infinity. So, this vertices are this 2 ends that will not come to the transformation hence by source (( )) theorem the only points that will be contribute to the transformation in the points B and C then I can also write the transformation will give integer zeta d z by D zeta will be K times zeta plus 1 where this point as A is a to the power of and then this angle is what, this is alpha by pi and this angle is pi by 2. So, this is 1 by 2 pi by 2 minus 1 solution will be minus half and again this point is again this angle is pi by 2 where these are the vertices angel made by b and c and this will be again zeta minus 1 to the power alpha by beta where pi is pi by 2 minus 1 that will be again half and the and that will be, and this gives me K by zeta square minus 1 root of.

So, this is the transformation while you will map this plane to this plane. Now what exactly it will give me, it will give me d z will be K where d z will be K d zeta by Geta square minus 1. If I simplify this I will get z is equal to K d zeta by integral d zeta by zeta square minus 1 and that will give me K cos hyperbolic inverse zeta plus del, del is a constant. Now I want to know what is by l then I will complete in know where you know one of the characteristics of cos hyperbolic inverse x that is equal to log of x plus x square minus 1, if I look at this characteristic because that simplifies this whole exercise.

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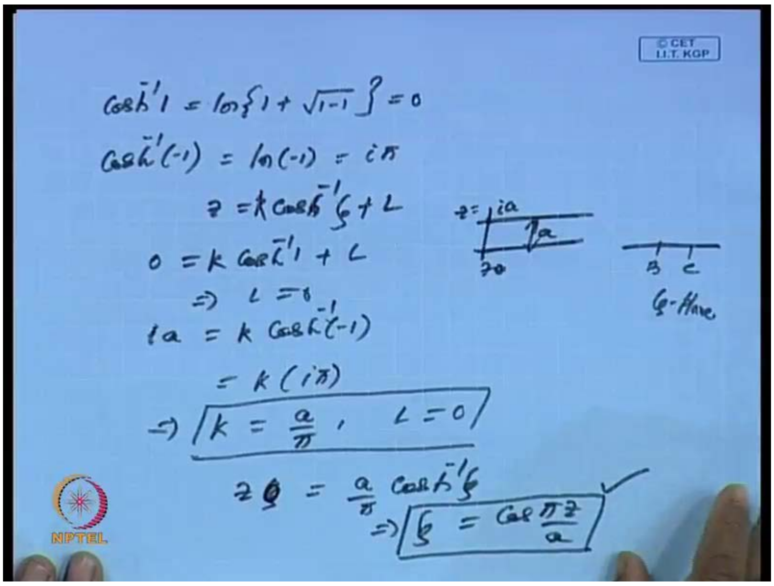

Now I have already wrote my that itself says cos hyperbolic inverse 1 is equal to log of one plus 1 minus 1, but basically log on log 1 is 0. And again what will happen to cos hyperbolic inverse minus 1, because on the other 2 points and then that will give you log of minus 1 and that will give a pi, that means. So, now, I have z is equal to cos hyperbolic inverse cos hyperbolic inverse zeta plus L. So, if I put 0 what will happen to because I have 2 unknowns here, K type you have 2 unknowns K and a if I put cos hyperbolic inverse one because I have 2 points zeta at minus 1 and 1. So, if I put z is equal to at this point. You will refer to z is equal to 0 because I have the, if I have look at this thing this point is z is equal to 0. And this point is nothing but the z plane is point z is equal to the y direction this is I a, because this is I and this is I a, because this is I this is a. So, this point will be in the y x this is I a.

So, if I put this z is equal to 0. So, that makes my z is equal to 0 gives K cos hyperbolic inverse zeta is referring to the in the zeta plane this is this is might be B C, the corresponding if z is 0, here the this point you referring to C, and C is nothing but 1. So, this is cos hyperbolic inverse 1 plus L at cos hyperbolic inverse is 0. So, it gives my L is 0. Now second thing we have we will go back to this point this point is I a if it is I a then this is K times cos hyperbolic inverse. By this point here refers to the point B here and B is nothing but minus 1. And this is K cos hyperbolic inverse minus 1 is i pi. So, which gives me my K is equal to a by pi and L is equal to 0. Once I know K and I then my transformation zeta is equal to a by pi into cos hyperbolic inverse zeta, z is equal to which implies my zeta is equal to cos of pi, pi z by a. This is my transformation. Now I will go to another example. This is a very simple example we have seen that by this transformation from a same infinity steep we can map in the z plane to a line in the zeta plane.

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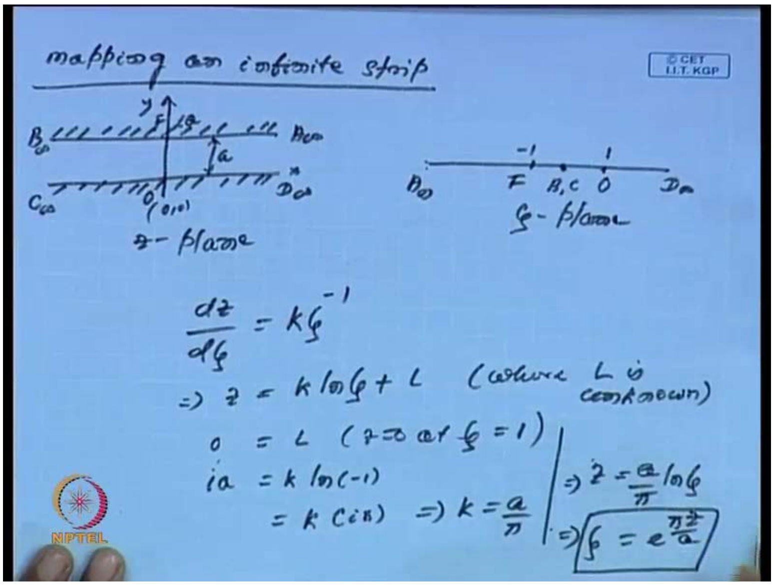

Now let us see, let us take another example. Because I will just substitute here examples to make it clear, but this mapping an infinite strip, because semi infinite strip in the beginning, now let us see to have a infinite strip if I have a infinite strip then how it will look like. So, and let me say the in this distance between the 2 strips is A. So, this point will be let me call this A infinity, this is B infinity, this point is called as C infinity, this is D infinity. Let me say this is the boundary of this and this is pi by x axis and this is x axis. So, this distance is a. So, this point is I a and this point is 0 0. Now this is the z plane, I will map this by zeta plane in the zeta plane.

So, A infinity I will call this here and then I have a point I have 2 points this point let me call it as a x and this point is call it as O. So, now, if A infinity is here, then I have to somewhere B and B infinity and C infinity, then we call this points as origin as B, C. Let me similar these both I put it as a common point at origin, because I am this is a infinity to infinity when it seem to infinity and this is again infinity to infinity. So, I call this as B C and then my a pi call it as minus 1 this is by F and then I have a point over this O, it will call as 1 and then as usual I have D infinity if I choose this according to this.

So, I am able to cover each point on this line with one of the points in this plane. Now let us see what will happen to the in this line A infinity at the contribution at the A infinity and B infinity will be 0. So, what will happen? Now the angle here is also 0, angle here is also 0. If this 2 angles are 0, now you have 4 vertices, 1 2 3 4. So, angle here, but these points here are refer to as origin this point is also refer to the same point origin. So, because of that and the angle is 0. So, then in the process, my d z by d zeta will be K time because of the points this point will be this point will not continue and the points now B infinity C infinity will refer to the point C and B and C this is the origin and here the angle it makes angle 0. So, this will be K zeta to the power minus 1 in the process.

Because the both are refer to the same point here. If K zeta to the point 1 then what will happen to my z where is equal to K log eta plus some constant a, which implies where as a L is unknown, I am willing to find L what will happen to zeta is equal to zeta if zeta is equal to one this is the point here and zeta is equal to one refers to 0 to z is 0 in scale and zeta is equal to 1 l n 1 is 0. So, this is L is equal to 0. Now, if I put this is by considering I have taken z is equal to 0 at zeta is equal to 1. Well suppose I take z is equal to I a, this point. Because this point will be I a and this point I am referring to minus 1 is equal to K l n minus 1 and l n minus 1 is K minus 1 is I by it implies K is equal to a by pi. And which implies this implies my z will be l is 0, K z by pi. So, z is equal to a by pi l n zeta, which implies zeta is equal to e to the power pi z by a. Now what will happen if I consider a particular flow?

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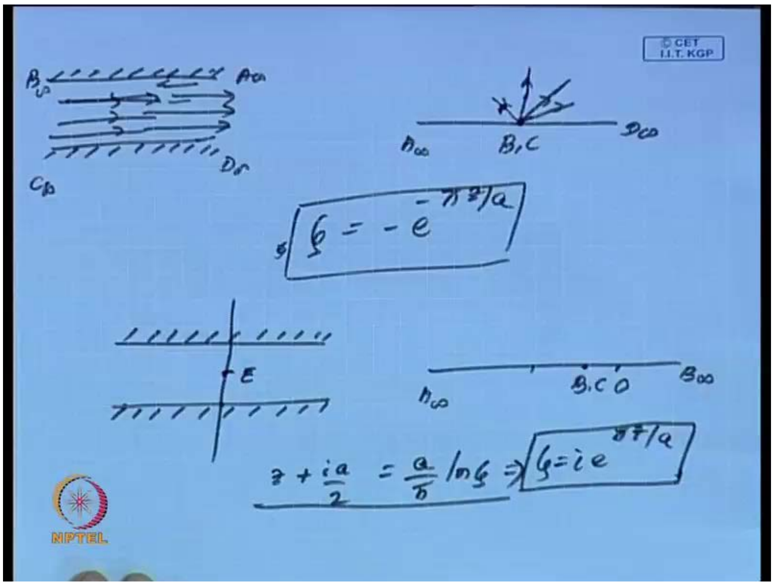

So, suppose I have considered this as a channel. Let us work out on a example I consider this as a channel A infinity B infinity this is my C infinity and this is my D infinity. And here in the zeta plane, I have A infinity this is my D infinity this point suppose this is a B C, this is origin. Suppose, you have a uniform flow in a channel, we consider as a uniform flow in a channel, there is a uniform flow in a channel with parallel sides. And the lines of equal velocity or perpendicular to the sides the inside flow, it can be regard as a due to a source of infinity. So, when I say you a uniform flow as it needs a source infinity and that is. And again this fluid is again moving to this side, and this is there is a resist sink infinity here. The same thing if what will happen here? So, if I put it in this way. So, I can only also say that as it energy fluid is flowing in both from this point. Like this is also it is behaving like this is a uniform flow, but here it will like a source from which the fluid is going out.

So, on the other hand, if we say, if I say the direction is this the opposite direction then I can also from here I can always find that instead of a source it can be like a sink and in that case, my zeta can be minus e to the power minus pi z by a. So, it is a flow in a channel this is like source I am able to represent it by a source. Now I will take suppose, if what will happen? If in the same channel, I consider suppose e, I consider the origin that is mid way between B and C, mid way between the channels either origin it will be mid way between the channels. So then I have A infinity, I have B infinity then I have B C here or I will have a point I have B C here then that is the origin. So, if I take this origin is here, then what will happen, my z plus i a by 2 this will be a by pi l n zeta instead of o origin, here if I see the origin somewhere here it is the origin then this origin will be this is the presentation will be. And in that process also you can see then if you do that, then what will happen to my zeta? Simply the zeta will be I times a to the power pi z by a. So, this is another example. Now, let us take a different example, one more example, because basically 2 with few examples.

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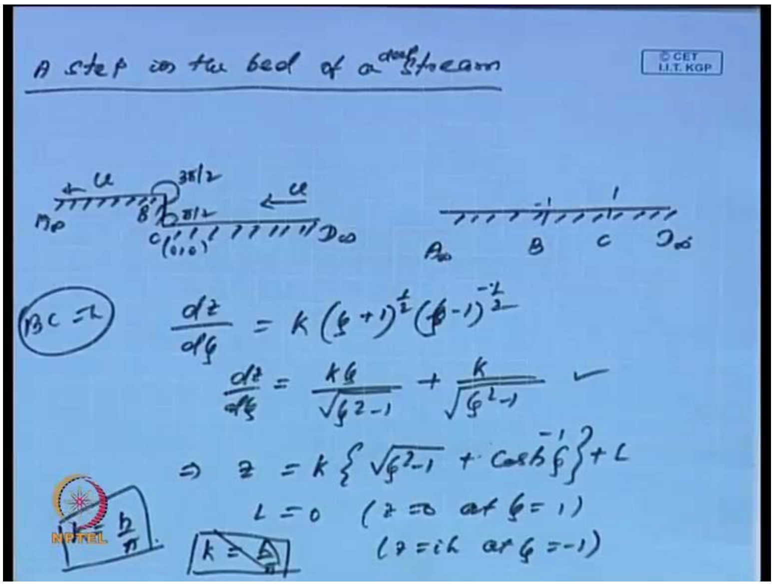

So, be able to clarify that how this term transformation is what is happening to this transformation? A step, a step in the bed of a deep stream; in fact, this transformation I will let this will be used to discuss the flow part in a channel of where as a depth or a channel varies cross section and map it,because the solution to analysis the solution because it is one of the easiest way. Suppose I say I have a channel where there a change sudden change applicable at this point I call it as A infinity this is B this point is C this is called D infinity is open it is a stream. Then if I represent this by this line then you can also find when we say that suppose the fluid is flowing and velocity u here. And here this angle will be pi by 2 this angle will be 3 pi by 2 and the fluid will be flowing in this way suppose this is the way that this is the direction which the fluid is flowing then.

And then we say. So, this is again a simple polygon, because any point this can act as a boundary and. So, similarly I call this A infinity, I will take this point as A B as minus 1 and this distance is called as B as minus 1, I have C is 1, then I have D infinity and then we say the fluid speed is hence in this these are the points you represent then what will happen to my d z by d zeta will be K. And that will give us B z 3 pi by 2 angle and B is minus 1. So, this is zeta plus 1 it is alpha by pi minus 1. So; that means, alpha is 3 pi by 2 pi by 2 3 by 2 minus 1 is half and this is this point is pi by 2. So, this will be zeta minus 1 to the power of minus half and the contribution from this end and this end will not come into picture. So, d z by d zeta is this so which you can write it as K zeta by zeta square minus 1 plus K by zeta square minus 1 root over.

So, this is the simplification, this is by d z by d zeta if you simplify this which implies a integrate d z will be K root to the power of eta square minus 1 plus this part is called hyperbolic inverse zeta again a constant of integration. So, if as seen that point the z plane the point C has a 0 0 and this distance is depth h. So, my b c in this distance is h, and then what will happen at z is equal to. So, if I put z is equal to 0 z is equal to 0, 0 refers to the 0.1 zeta is 1 zeta is 1 minus this is 0 this 0 this will be. So, that will give me a large 0 as seen z is equal to 0 zeta is equal to 1. In a similar manner if I take L is i h, this will be z is equal to i h at zeta is equal to minus 1 then that will give me K, you can easily find out K is equal to h by pi. This will give me K is h by pi. So, this will refer to this point.

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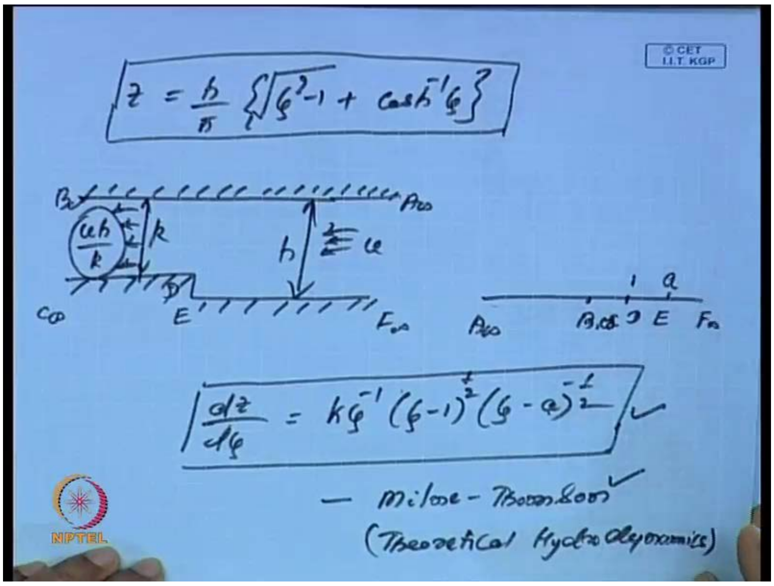

So, that is the case then what I will get hence by z will be h by pi into eta square minus 1 root of a plus cos hyperbolic inverse eta. So, this is my transformation. In the same manner suppose we say my last example was also last to suppose I take it, suppose I have a channel I will not just replacement on this channel, if this is a channel. So, I have A infinity, B infinity, may have C infinity this is I will call it as A D this is point is E, this point is F infinity. Let me say that this depth is h and there is a flow fluid which is flowing at the speed u and this is when we say this K and again the same fluid is flowing here. So, here the speed will be u h by K, a continuity equation.

This will be the speed here point is the fluid will be flow when what will happen here? I can map it on this line A infinity I will have B C because this 2 I like the previous case I will call this as a single point and this point and then I have D this point I call this as one this is B C this is D and E is call it as A and then I have an another point that is F infinity. So, then if I look at the transformation by A apply Schwarz Christoffel theorem then I can have d z by d zeta is equal to K zeta inverse, I can easily see that zeta minus 1 top the power of half like the if I look at the previous 2 cases, zeta minus A to the power minus half. So, here I have to know what is K what is and then if I easily.

So, this will be my transformation. So, if you this is a little complex and to it when K a and because if I solve this the whole algebra is a little complex and I refer to you to have a look at the Milne Thomson book of Milne Thomson Theoretical hydrodynamics. This book by on theoretical hydrodynamics Milne Thomson you can again simply this because this is a little algebraically already complex. So, I am not going to further detail here, but I have made it very clear that this can be and this constants can be obtain a K and I n if you integrate it will get interior constant and may this what will give you the results are associated with this transformation and what you seen from this that by using where taken again many have seen here that typo of a polygon if we have a polygon then where you seen that the total sever a simple closed polygon.

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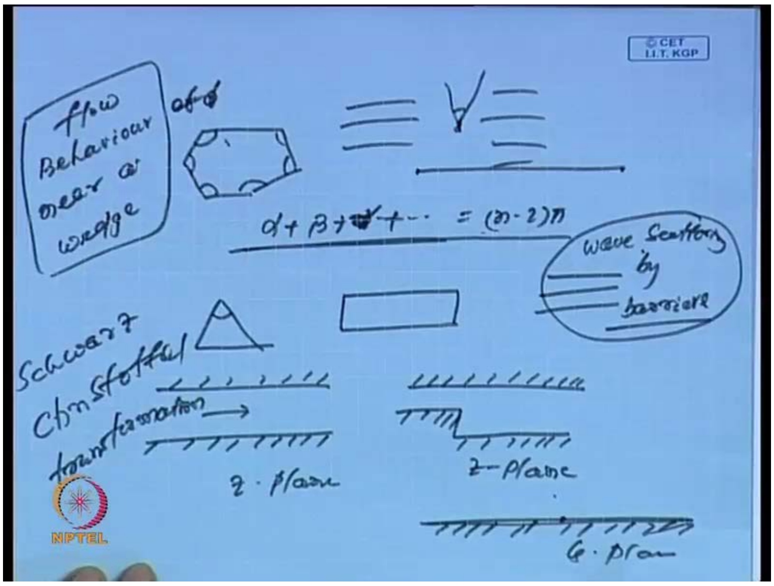

We have seen that over alpha plus beta plus gamma plus gamma if I have n such what is rational then total is n minus 2 into pi; that means, I am assuming the all the avail together is giving me together that is n minus 2 into the power of pi. And in that case I am able to transform each point on the polygon to a line on a real axis on the zeta plane this is on the zeta plane. So, you have seen that in case. In fact, if you have triangle or you have a rectangle this criteria whole square and again you have seen that in case of sub channels beside this is either open channel or a closed channel. Or you may have certain number of change in the certain change in the channel depth either depth or width, because we can accordingly map it to a point in the z plane to a line in the zeta plane. In all the cases we have able to see and that we can always map it by using this transformation what is the Schwarz Christoffel transformation**.** In fact**,** the earlier days pi using the transformation last class of Proline streams, due to changes in water depth in breadth of the channel.

And also we know to c any pro length where handle and also flow faster several flow inside polygon flow outside polygons with the because as I mention that the polygon the inside the boundary any point in this boundary will be have a point on this zeta plane on this upper side of the boundary. So, that happens for a flow the fluid flow in a channel and even if in the obvious channel or even if in a cannel can always delay you can get it to a flow just considering the flow on the line. And you have seen where as the example where I have seen that always a uniform flow can be represent to the point source for a sink on this zeta plane. So, that why many pro lengths are always using this Schwarz Christoffel transformation. In fact, in potential flow we have again seen that the wised type behavior is often we are able to analysis the behavior of flow behavior, flow behavior near a wedge type boundary, flow behavior near a wedge. Suppose when this is a kind of waves.

So, if we have a fluid is flowing then what happen, where this kind of tip and even if where we have seen and this theorem has even further applications in hydrodynamics particularly coastal engineering in analyzing flow when we have a wave, wave scattering length by barrier wave scattering by barriers. Particularly to understand the flow behavior at the tip of the barrier we have rectangular barriers. So, many barriers to understand the flow behaviors at the tip this was (( )) theorem helps in a big way. Infact this has been used in many papers related to coastal area particularly to when on a rise wave scattering.

So, this is our another important result with this we will stop today about the Joukowsky transformation and now we have talked about lot about this transformation and we have seen the various application of Joukowsky transformation in application areas of basically complex when we are able to deal with pro length associated with you ideal fluids. Particularly when we hear a we are able to define a complex potential pi w; that means, you have per potential flow pro lengths this complex potential placed in a significant role and particularly we have seen that the role of Joukowsky transformation, Schwarz Christoffel transformation and other confirm on mapping.

So, this our today the series large number of lectures I think I have given seven or eight lectures on this conformal mapping this. I will stop here and next time we will go for we will talk about wave particularly we will talk about. So, when there is a cylinder we have talked about in a fit cylinder I will talk about moving cylinder, and then we will talk about added mass talked about added mass and energy associated with it kinetic energy, potential energy is associated and whether 3 more things we will talk about this potential problem. Before we go to the wave carry understand the wave of water waves. This I will be able to went today.

Thank you.