Marine Hydrodynamics
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Lecture - 12
Conformal Mapping and Joukowsky Transformation

The series of lectures in Marine Hydrodynamics, today we have the 12th lecture and we will talk about Conformal Mapping and some of it is application in fluid flow problems, particularly in two dimensional flow problems. As, we have seen in our previous classes that large number of problems can be handled by using the theory of complex function and it is various characteristics, there is one of the one of the very important transformation, but it is called the conformal mapping.

Here basically the angle is preserved in this mapping, and let us see what exactly it is we can work out few examples to understand, what exactly the conformal mapping is and then you apply the same concept to see how the conformal mapping can be used to solve fluid flow problems. In this case what, because it is a the name itself is called a conformal mapping means; that means we must having two planes in the complex plane, and in this two plane; that means, as I have told in my last class that when we have problem.

It may be difficult to solve in a particular plane, but we can transforms this function to another plane, then the problems will be simplified by a suitable transformation will transform, and then the will solve the problem in the new plane. And after solving the problem in the new plane in a simplified manner, again we will come back to by our original plane and then the problems becomes much simpler and easy, so let us see through an example what this conformal mapping is…

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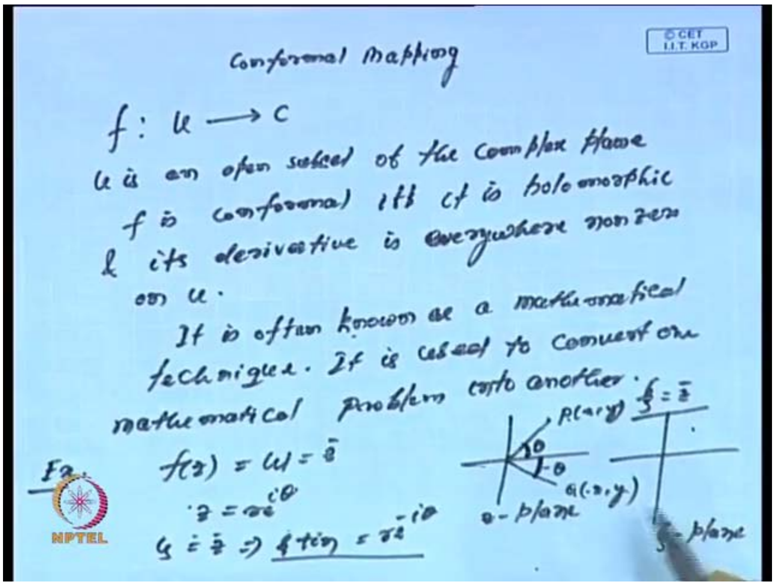

So, suppose I will take one example, because today we are talking about conformal mapping, what exactly it is, so this is function f from u to c and u is an open subset, it is an open subset of the complex plane complex plane. We say that it is confirm f is the mapping it is conformal, if and only if it is holomorphic and it is derivative is continuous everywhere is everywhere non zero is everywhere non zero on u, it is very basically it is mathematical technique it is often known as mathematical technique, and is it is used to convert one mathematical problem into another.

So, here suppose I will take one example, (( )) take one example suppose I say f z is equal to w is equal to z bar, so I have the plane, this is the z plane and if I have a point p, I will say I have a point p which is x y. Then this point it can be represent and the polar coordinate f z, z is equal to r e to the power i theta, if I say I have another plane, this is the z plane let me call this is the zeta plane the zeta plane, suppose I say zeta is equal to z bar which implies place zeta.

If I say zeta plane has i plus i eta that is called r e to the power minus i theta, so in the new plane, so what we do in the new plane zeta plane z bar is defined and this for each point if you look at this this is nothing but minus theta. So; that means, here the angle is it is mirror image of this angle theta and it is here so; that means, here we are writing this is our zeta plane, in the zeta plane every point is represented by, so zeta is nothing but z bar.

So, basically we see that the here the angle is just there is a mirror image of this angle theta, so that is what we see here and in other words we can also if you call it x y then this point can be q minus x y. So, in the z plane when you say f z is z bar; that means, if I say zeta is z bar then z plus i eta is after by minus i theta. So, basically what we are doing we are representing each point here in the x plane we have a point we have angle the corresponding angle is nothing but here it is if it is theta here, here the corresponding angle a point here is treated has as if it is minus theta. So, that is what, so now, I will come to another example orbit of clarity about the problem.

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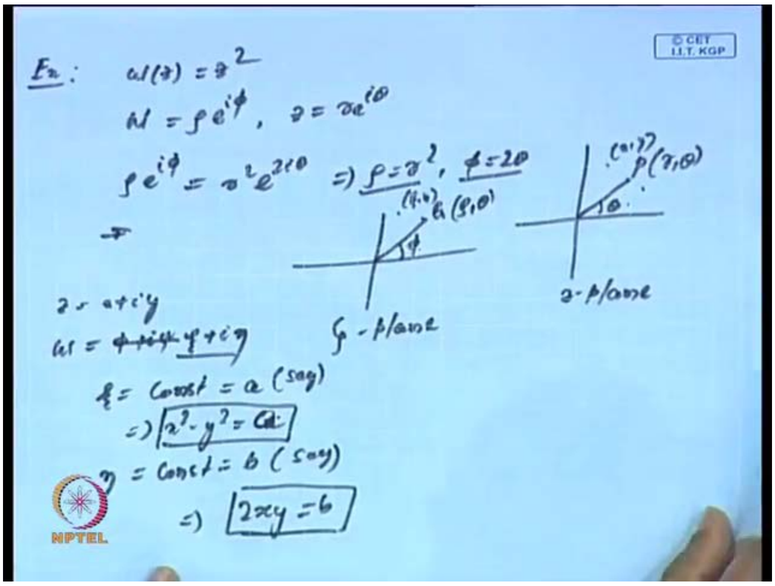

Now, suppose I will say another example, I will take because let us workout few examples to clarify what exactly it is if I say w z is equal to z z square, and if I right w is equal to rho e to the power i phi and I have z is equal two r e to the power i theta. So, then what will happen because will have rho e to the power i phi this is nothing but z square z is r square e to the power 2 i theta and which implies my rho is equal to r square my phi is equal to 2 theta.

We sure that distance from there is in, if I have a if I have a this is my z plane and this is I say this is may zeta plane this is may zeta plane, then any point here this is represented by if I call it this is point p, and this is point q, this is rho rho theta, and this is p r theta. Then we have seen that if this angle is theta and this plus this angle is phi, then we have seen that rho is equal to r square phi is equal to 2 theta, so that means, if you look at this plane; that means, as if you are thinking of the distance square from the origin and the angle is twice then that of the angle here.

So, for each angle here we can get an angle here, and for each point here we will have a point here, so that is why now with this I will if I put the same thing in the Cartesian coordinate my z will be x plus i y and w is equal to phi plus i psi put it. Then my or again if I call it as a psi plus i eta then we have already we know that if psi is equal to constant is equal to a, then and what is my i plus i theta my psi is equal to constant a, and that is nothing but which implies x square minus y square is equal to constant is equal to a. Farther if I say my eta is equal to constant and that I say b, which is b say then I will get 2 x y is equal to b.

So, this is the… So, this is in the zeta plane, so for each point as I mean to say for each point here. So, if I have any point psi theta in this plane think of any point psi theta in this plane and it has corresponding x a point here, and here it is simple because x square minus y square is equal to a if it is constant; that means, line if I assume that this is a constant; that means, a point if psi is constant.

And again eta is constant; that means, if psi is constant and that will correspond it will give me a line in the plane and that is nothing but here x square minus y square. On the other hand if I say b is a constant 2 x y is b and then accordingly that is nothing but it will represent here p, so this is what exactly, so now, I in fact, we will go to a better example to have a look at what exactly is happening.

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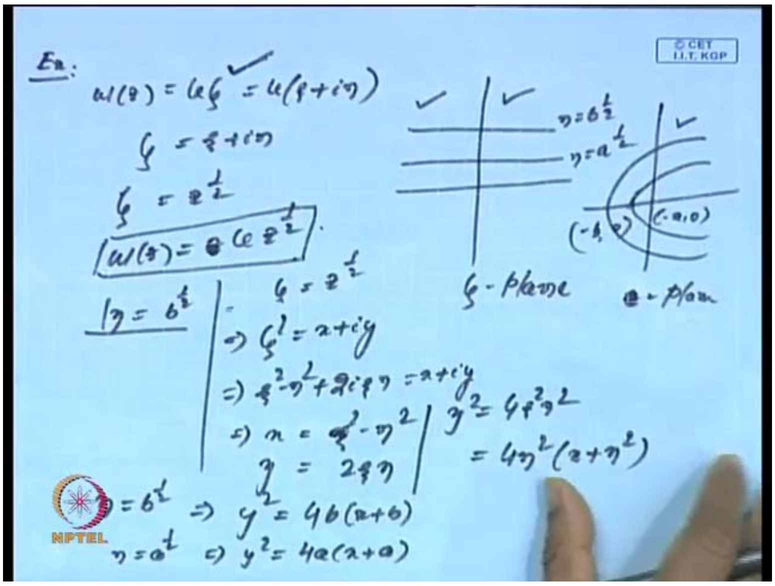

How will use this concept to transform various problems, suppose I look at my w z is equal to u zeta and that is nothing but u psi plus i eta, so my psi zeta is equal to psi plus i eta. So, let if I say if I put it at my zeta is equal to z to the power half so; that means, I am looking at w z is equal to z to the power u sorry u z to the power half. This complex this is a this is my w z and I have taken my zeta is z to the power half, so in the process w z can be written has u zeta, so it is a z this function and this is a most similar to handle then this function.

Now, if I look at this one let see what happen suppose, what will happen my eta is equal to b to power half, if eta is b to power half, so from here zeta is z to power the half. So, we have before that let see we have already zeta equal to the z to power the half which implies or zeta square is x plus i y because that is z and; that means, again zeta is psi plus i eta. If I put psi plus i eta psi square minus eta square plus two psi eta equal to x plus i y and once this is the; that means, x is equal to x square minus x is equal to psi square minus eta square and my y is equal to two psi eta.

So, which implies which implies my y square is equal to 4 psi square eta square, so I will put it as four eta square, and if I put psi square is equal to x plus eta square. And now if I say substitute eta is equal to b to the power half, which implies my y is equal to 4 eta is b to the power half. So, it will be b into x plus b so; that means, I have a eta is equal to b is half, it is a line along the, if this is the line eta is equal to b to the power half, in the zeta plane, and that will in the x y plane that will represent a parabola.

And this parabola is y square is sorry this is y square is 4 b x plus b, so here minus is b 0 minus b comma zero, similarly if I look another thing, suppose I say eta is equal to a to the power of half. Then my it is obese from here then y square will be four a into x plus a, so if I take another line here eta is equal to a to the power of half, then here also there will line here is parabola here.

So, that means, a if I say that the flow along this will be the same along the flow along this between this 2 parabolas, and we call it parabolride, so that means this is very is in the new plan zeta plan this is a kind of uniform flow were as between the 2. If I say beta is equal to constant and beta is equals to both are constant, then can always say that flow between the 2 parallel lines and the other hand the (( )) present in the x y plan; that means, in the z plan. We should represent as if flow between a part 2 between a parabolite, those a straight lines in the zeta plan will correspond to a parabola in the z plan.

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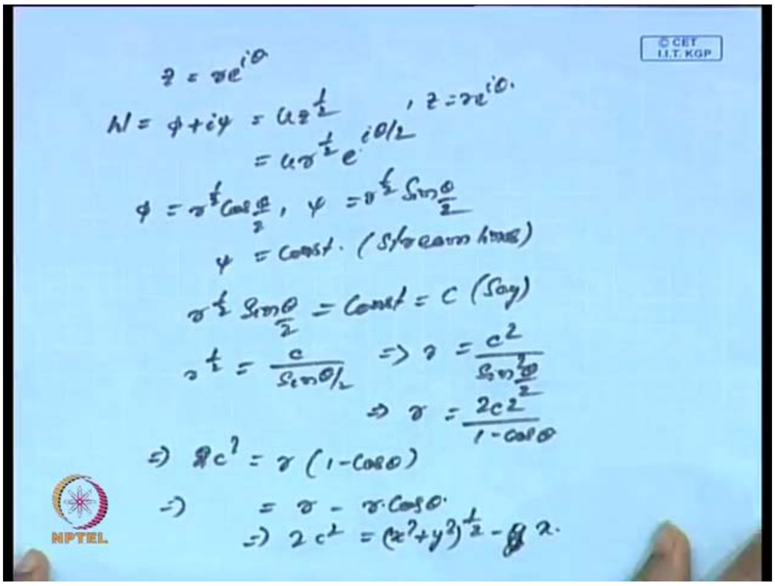

Now, if we look at this another plan another point of view, suppose I said z is equal to r into the power i theta, then we have w if i just considered this as the complex potential that is call phi plus i psi. And again we can call it we know that it is u z to the power of half and that is nothing but u r to the power half, e to the power i theta by 2. Where, z is equal to we have taken z is equal to r e to the power i theta, and once this is the then will get from this phi is equal to r to the power half cos theta by 2, and sign is equal to r to the power of half sign theta by 2.

And we changes sign is equal to constant this stream line the stream line then; that means, it is same as telling r to the power half sign theta by 2 is equal to constant, and this constant if I call this as c. Then by half r to the power half is c y sigh theta by 2 and that is nothing but which in place r is equal to c square by sign square by theta by 2 and this is which can be written as which is can be written as 2 c square by 1 minus cos cos theta. So, from this we can get 2 c square is equal to r into or 1 minus cos theta which in place 2 c square this is equal to r minus r cos theta which implies r is 2 c square is equal to r is x square plus y square 2 the power of half minus y or minus x r cos theta x will did algebra, but it is quite interesting.

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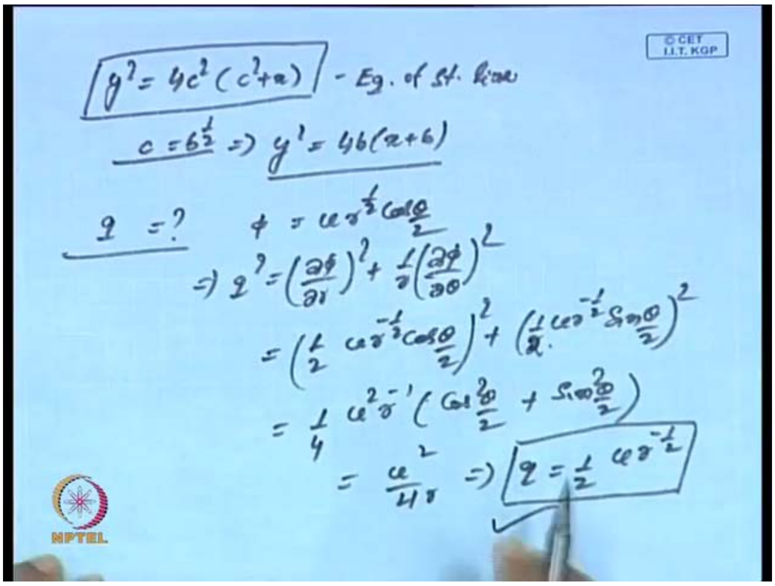

So, from which we can easily get y square is equal to four c square into c square plus x, because we have taken phi is equal to constant as given us the stream lines. So, c is equal to constant this is again gives us a stream line equation of stream line, so that why this can represent a flow. Now, if I take c is equal to b to the power half, but we have seen is called y square is equal to 4 b into x plus b.

So, if I what it says that may consider (()) is a there to perhaps which have used two lines, that two lines in the gateau plan corresponds to unrivaled in the x y plain. Now, we see that if these two lines and we consider as two these constants can be considered as streamline, then in the z plain is can be again considered as streamlines and their parabolas. Now, what is the speed in this case, if we look at the speed this q we all know that pie is equal to your to the power half cos theta by 2 and that gives me q square alpha by theta square plus 1 by r alpha by theta square and that is nothing but 1 by 2 your power minus r cos theta by 2 square plus 1 by r your 2 power minus r sign theta by 2.

And this again gives us 1 by 4 this will give us u square r to minus 1 into this is u r square and this will give us del y by del theta, so this will give us cos square theta by 2 plus sign square theta by 2, and that will give me u square by 4 r implies q is equal to 1 by 2 u into r to the power minus half because we have del y by del theta. This will give us a half 1 by 2 times with the sign theta by 2 the alphabetic theta 1 by 2 into sigh theta it is fine, so we got q is equal to this once we know the speed then what will happen.

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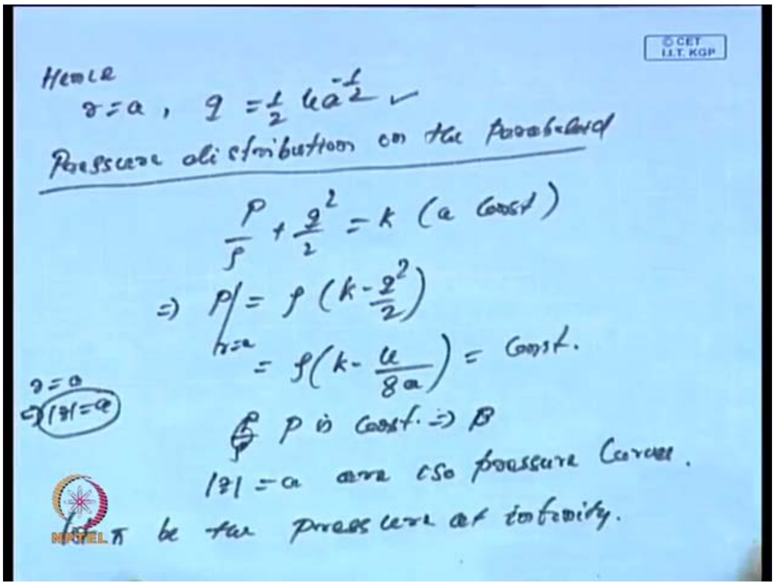

Hence, because we are looking at a parabola, when r is equal to a or q will be 1 by 2 q into a minus half, if I want to calculate what is the pressure distribution on the parabola. So, I have to use the one only sequence and that gives me p by rho plus square by two is q equal to constant, a constant and that constant I call it k, so which gives me p is equal to rho times k minus q square by 2. And if I just say q is equal to 2 on the line q is the only parabola q is equal to constant, (()) then I can always get rho k minus q square by 2 and this will give us u by that is 4 plus 2 is 8 a.

So, if I know that the pressure at the point a is known then that always I can get the, so if this is a constant that is the pressure at the point parabola, this is p at r is equal to a. And if this is then so; that means, circle mod z that is nothing but the circle r is equal to a implies mod z is equal to a; that means, on a circle mod z a p becomes this. Now, so this I can always call this p for pressure is same everywhere, on a chord on a circle along any streamline in the pressure; that means, it shows pressure along this streamline that this itself is showing me streamline, so along the streamline.

Since, p is constant this value is constant and once this is constant, so p is constant which implies p which is the circle implies the line mod z is equal to a, power are iso pressure curves. Then further suppose, I know the pressure at infinity let pie be the pressure at infinity, if pie the pressure at infinity then from this we have seen that the pressure at infinity is the velocity at infinity q at is 0.

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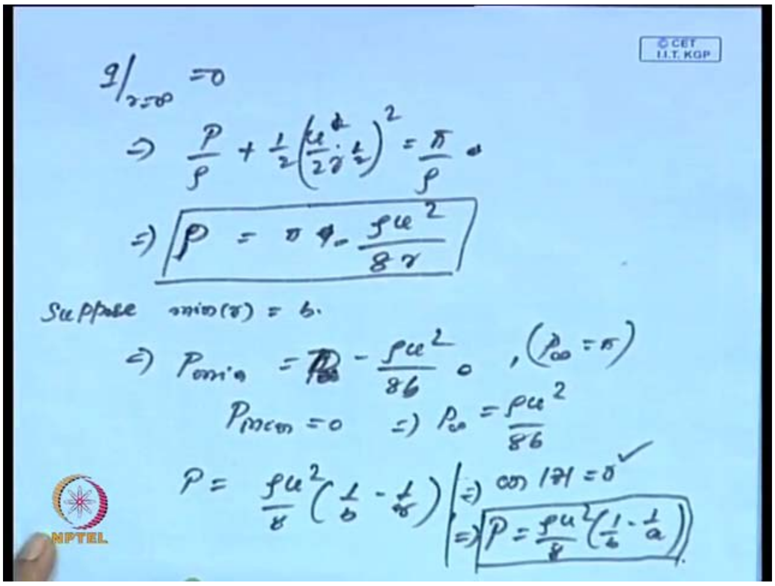

So, it implies p by rho plus and my q is half of u square by 2 r to the perhaps rather 2 r, this is square and this is equal to pie by rho because at the infinite pressure is the q is 0 which implies pie p can be written as pie plus rho u square by 8 r. And there will be minus sign by minus this is this and this may p, so at any point the pressure is this. Now, what will be p minimum, suppose minimum r is equal to b which implies p minimum p infinity minus rho u square by 8 b, and the p minimum is this.

And if this quantity is 0, if I see p minimum is 0,which implies p infinity is rho u square by 8 b, which implies in terms of the p minimum I will get my p infinity means pie will call it p infinity, but I will say p infinity is nothing but pie. And then p infinity is this it means, so I will get p is equal to rho u square by 8 into 1 by b minus 1 by r, so if at any point mod z r is equal to a, so if I have a circle which implies on mod z is equal to r my pressure will give me p is equal to rho u square by 8 into 1 by 2 minus 1 by a.

So, this is the pressure in terms of I know the minimum pressure then I can always get, if my I get minimum pressure unless, I can get that any point mod is z equal to r this, because pressure. Now, this example, so what here we have used you have transform these plane, because see it is porapoloide and, but we have not done the algebra we have run it here porapoloide, but we have mainly done it by considering that the stream lines are nothing but straight lines.

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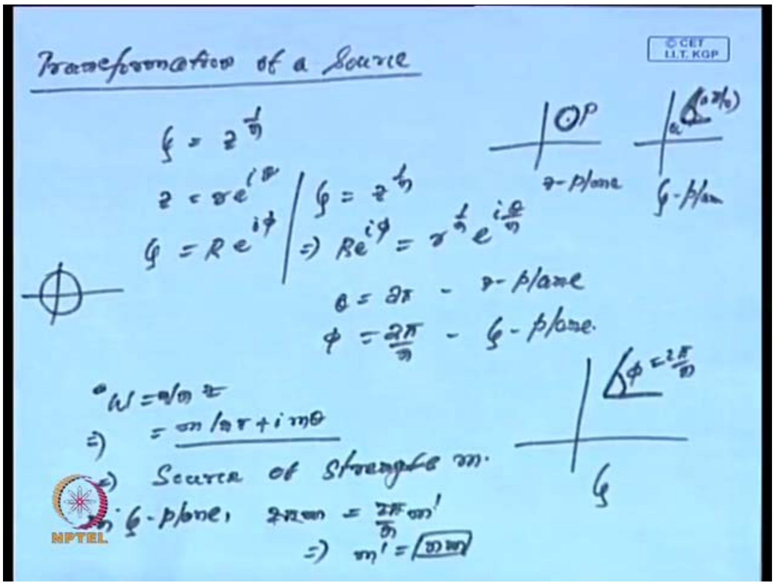

Now, I will go to another example suppose, how will get transformation of source, suppose I say my eta is z to the power 1 by n, if z eta is z to the power 1 by n zeta is z to the power 1 by n. Then what will happen suppose as I said z is r i theta where my zeta I call it r e to the power of i phi, then form this we can easily get that zeta is because in terms of z. This is z to the power 1 by n and which implies r e to the power i phi, is this becomes r to the power 1 by n into e to the power i into theta by n, which gives me.

So, if I now what is happening here, if my angle theta is in the then what will happen if theta is equal to 2 phi and that is in the z plane then phi will be two phi by n that is zeta plane. So, that means, if I take a point here this is measured plane and I consider another this is zeta plane, then if I say point here p and I consider another point here, if I consider a circle here. Then here I consider the same another point q, then the corresponding p point will be an angle on that angle is nothing but that is bit 2 phi by n, so this is just phi.

So, I just put it here in the zeta plane, this angle is phi and that phi is nothing but 2 phi by n. So, for an angle a circle here we have a angle here and that is 2 phi by n and circle means we are making. So, this emphasizes suppose I look at a source here, so in the z plane if I have a source, because if I said my w is equal to m n z, m log z; that means, I have a source in the z plane, which gives me my m log r plus i m theta.

So, that means, if resource, so this represents if in the z plane I have a source of strength m, so because the here the stream lines will be circles. Whereas, see the circles will be just stream lines here, on the other hand in the zeta plane in the zeta plane what will happen to this, we can easily see that, so 2 phi m is the strength of the source. And then in the zeta plane the same thing that will be because it will give me 2 phi by m, 2 phi in to 2 phi by n into m m prime, m prime is the strength and that gives me because for each 2 prime the corresponding angle here 2 phi by n and if the source is here.

If the source here in the new zeta plane for this source is in prime, the I have 2 phi m is a must 2 phi by n into m prime which gives me m prime is because 2 pi 2 pi 2 pie two pie get cancelled m prime is n m. So, this is what, so corresponding as I have already mentioned, so a corresponding source of strength m here will have a source here whose strength is m prime the m prime will be n m times, n times n times m. Now, with this understanding on few examples on the conformal mapping; that means, how we relate it one point in a one plane to another point in the other plane one of the very important transformation, we will talk about that is called Joukowsk transformation.

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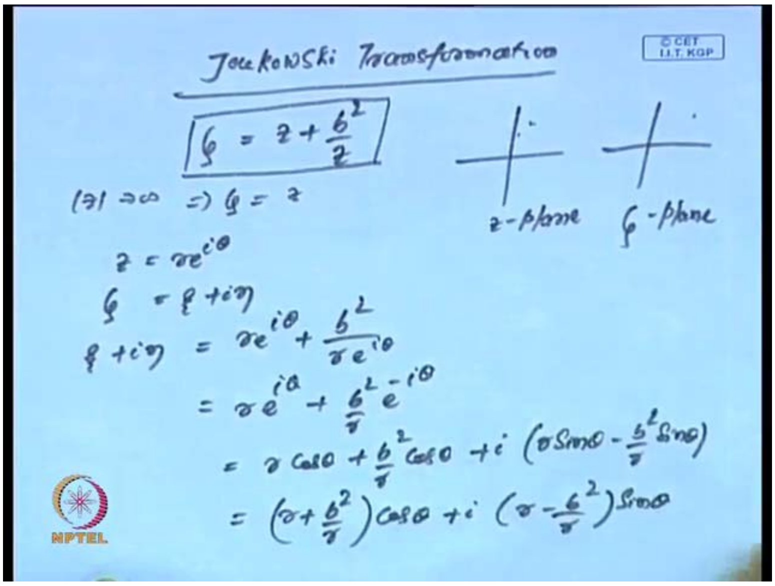

This Joukowsky transformation, it gives us very it is one of the very simple and important transformation and which was developed by the Russian mathematician Joukowsky. And then this transformation says that the transformation zeta is z plus b square by if this is the transformation, so I have a point this is the point in the z plane and this point in the zeta plane will be this is the zeta plane.

So, this point z here will have a corresponding point here, now we can by using this transformation we can have a point here and then have a respective point here and by vice versa. Now, it can be easily seen that what will happen when mod z is turning to infinity, then my zeta will be z and when zeta is z; that means, inside the fire field both zeta plane. Any point in the zeta plane is same as the another point in the z plane, but local points locally in the near field the transformation is the two function, two things are quite different a point in the z plane point in the zeta plane there by different.

Now, let us see what is the transformation means suppose I take z is a to the power i theta a point in the z plane, and let me say zeta is equal to (( )) plus i eta, when what will happen (( )) plus i eta if I put it using the transformation. Then I can easily get it r e to the power i theta plus b square by r e to the power i theta and which is nothing but r e to the power i theta plus b square by r into minus i theta and which gives me r cos theta plus b square by r cos theta plus i times r sin theta minus b square by r sin theta. So, that gives me r plus b square by r into cos theta, then plus i times r minus b square by r into sin theta.

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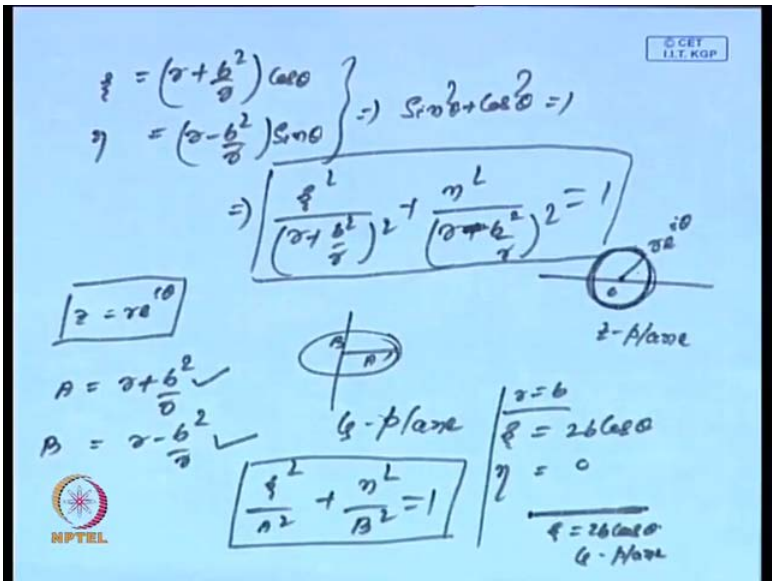

If I do this them my psi I will separate the real and imaginary parts my zie will be r plus b square by r into cos theta and my eta will be r minus b square by r into sin theta. If I from these two thing, I know cos square theta sin square theta plus cos square theta is equal to 1, which gives me psi square by r plus b square by r square plus eta square by r plus r minus b square by r square is equal to 1.

So, what was now what is this I have taken a point z is r e to the power i theta; that means, I have a circle of radius r I think of a circle this is even this is the origin, I think of r to the i theta of any point the circle in the surface of the circle on the on the. Now, what is this represents this represents an ellipse so; that means, in the in the eta plane this is z plane, the eta plane this is an ellipse in a point and where as these distance. If I call this a; that means, my a is r plus b square by r and my b that my a and b will be r minus b square by r, so this is the semi major axis and this we called the semi minor axis.

So, we have seen that each point that we are able to by using these transformation, we are able to map every point on the psi plane to a point on the zeta plane. So, we are able to suffer that in process, we are able to map a circle to an ellipse and this is and here the origin of the circle is at the the center in the origin of the circle. Now, what I will do, I will just make it a very simple way I will look at it, so this I can say because of this substitution I will have the i square by a square plus zeta square by b square is equal to 1.

Now, what will happen if r is equal to b, if I say r is equal to b if r is equal to b, than this becomes my if r is equal to b than my will be here I will put r is equal to b, so this is b 2 b cos theta and write will beta r is equal to b r is equal to b means theta is 0. So, if that is the case than what will happen to the corresponding in the, so this will be line at this line because theta is 0, so this is a line is 0 present i is equal to 2 b cos theta. So, again we are able to see that from a circle we are able to if this the zeta plane the zeta plane from a circle we are again able to come to a line horizontal line. So, if we are look at a flow; that means, any point outside the circle can be related to any point outside the ellipse.

If I look at the invert transformation and again we can always see that any point outside the circle can also be related to any point outside the line of finite line, which is to be. So, if I look at the invert transformation what happens; that means, in the invert transformation any point outside the circle can be related to any point outside the ellipse here that any point outside the circle can be related to any point outside the this plane plane particularly.

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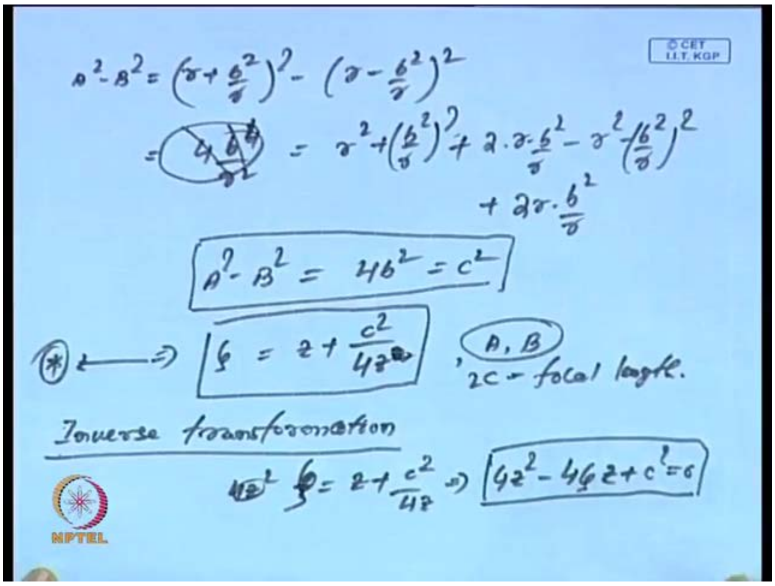

Now, even let us look at some of the characteristics, because I have taken now what will happen to my a square minus b square, because a square minus b square will have, so that will give me r plus b square by r square minus r minus b square by r. And that is give me that will give me square and that is which gives me r square minus r square b square by a r, so it will be 4 this is 4 into b square by b 4 by r square, for b fourth by r square. So, let be no sorry, sorry, sorry this will give me r square plus b square by r square plus 2 r into b square by r minus r square minus b square by r hole square than plus 2 r into b square by r, and that guess me 4 b square. So; that means, a square minus b square and if I call this as c square than that will give me which implies when zeta you can call it z plus mi this is b square y z which I can call it b square is c square by 4 z square.

If I consider this than if a b are the major axis than my c will be the 2 c will give me the focal length, because the point will be minus e 0 and z c 0, so this is the way if I consider this as the transformation, than it will represent a b will it present the major and minor as cease with is see two seeing the focal length. And again than this is one of the things, now it is this now what will happen let us look at from this zeta and look at the suppose I have an given zeta the transformation zeta in terms of z.

I want to get the inverse transformation, what will happen to the inverse transformation, in the inverse transformation what I have to do I have to just we do the algebra. It is very interesting actually sometimes it my took a little complex to the algebra, for this is c square by 4 z, sorry this is 4 z not for z square, because I have taken b square by z. So, than I will have 4 z square, if I call this represents an a star, so from zeta rather I will say we have sorry, zeta is equal to z plus zeta is equal to z plus c square by 4 z which implies I will get this is 4 z square four z square and minus 4 zeta minus four zeta z plus c square is equal to 0, this is the quadratic equation in z.

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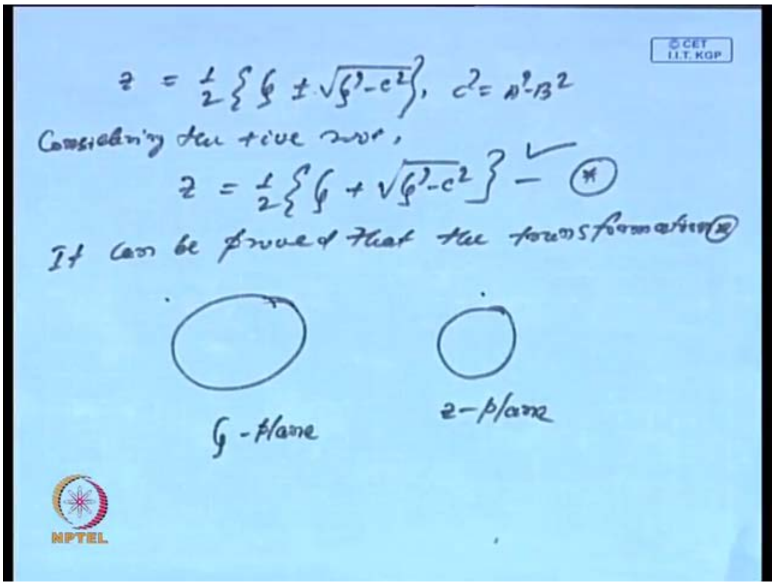

This is the quardratically z what will be z then because that will give mean the inverse transformation than my z will be 1 by 2 in terms of zeta plus minus zeta square minus c square. And here c square is I have seen my c square is an in what an a square minus b square, and whole the bracket. So, now, if z is this if I look at only the considering only the positive root positive root, in z than our z will be half of zeta plus zeta square minus c square. And it can be seen this transformation, if I call the star star, we have seen the earlier transformation that it maps every point on a circle to every point lives, than if I look at the inversion it can be said can be proved, that the transformation. Star represents any point on outside the lips to every point outside the circle, because there earlier we have seen from z to zeta.

Now, we are seeing zeta to z, so we can see that every point out of the side the circle out side the each will map to a point outsides, so this is the zeta plane this is the z plane. This can be easily proof and I am not going to the detail of the proof again this can be found in any of the reference proof what of suggested. Now, it this, so this is the very another interesting result that form Joukowsk transformation, we are seen that in very point in on a ellipse can be obtain from a point on the circle and here we are seeing the table that any from in see can also get a point in a circle.

So, to of better understanding on this will a come to the left a coordination system and there from the left coordination system we can see that how this a point a related what exactly the left the coordination system. And again we can get it from this transformation the in words transformation of the is equal to transformation and after that that will help us, because of every point outside in a ellipse can relate to a point can map to a point outside a circle.

That means, if have a flow have a flow particularly if I think of flow of a ellipse, particularly ellipse, it cylinder for plastic cylinder a can always relate it for plastic circular cylinder. So, that means, from a if I know the result for a elliptic cylinder are a circular cylinder, I can always get the result from a elliptic cylinder, because of every point outside the a ellipse can be relate can be map every point of side the circle.

That means, in a two dimensional flow pattern, if a put a cylinder, circular cylinder the flow, then I can apply circle theorem to to get the uniform (()) a circle cylinder in the same manner why using the same result, because the using this transformation. I can always in obtain the result an a elliptic cylinder, this is a what will you do in our next class, and with this today will stop this these transformation Joukowsky the transformation or the conformal mapping has a will.

In fact, not only for ellipse later in the future classes in the next few classes will see how more complex transformation, how more complex problems can be sold be using this joukowsky the transformation particularly, when it comes a hydro foil or a aerofoil. There are two types of aerofoil will consider, one is the symmetric aerofoil, one is the combed aerofoil. Again will see if the will see because we have already seen from a point to my point on the circle we can related to a point on the plane, so if I have uniform flow faster line particularly finite length may be applied of finite length.

Then or finite width and we can always think of your flow passed a plate obtain it from the user result of the flow past a cylinder, so this will do in the coming few classes. So, this this Joukowsky transformation or conformal mapping as is will be very help for in solving large number complex flow problems and will see in the next few classes how this is help full in analyzing some of the flow problems.

Thank you will stop here.