Mechanical Behavior of Materials
Prof. S. Sankaran
Department of Metallurgical and Materials Engineering
Indian Institute of Technology - Madras

Module No # 11
Lecture No # 54
Fracture Mechanics – VI

Hello, I am processor Sankaran in the department of metallurgical and materials engineering.

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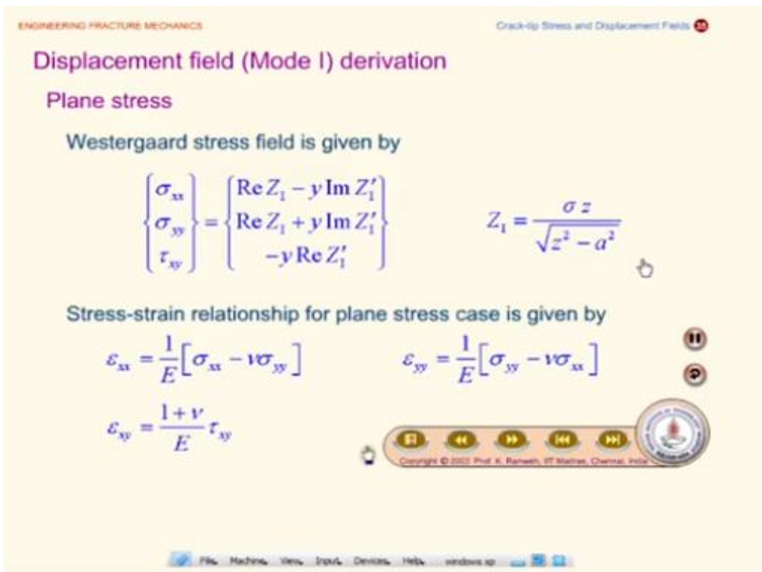

Hello everyone welcome to this lecture and in the last class we looked at the stress field near the crack tip and I would like to continue in this class to look at the displacement field in the Mode I derivation. So we will first look at the plane stress what is given by the Westergaard stress field

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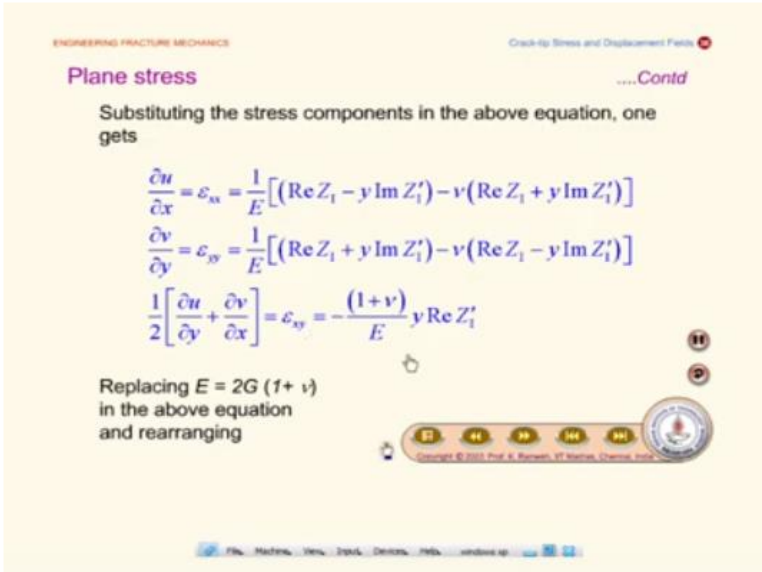

So this is already established, this is a Westergaard stress field for the biaxial loading in terms of Westergaard Z function. I mean stress function Z, it is called Westergaard function. So, this we have already know and then this is the stress field and this is the stress function which is already we have seen. So in order to find the displacement field we use the stress-strain relationship for the plain stress case and which also we know this from the theory of elasticity. We have already seen this so we use this relation and then try to get the displacement field for the Mode I crack problem.

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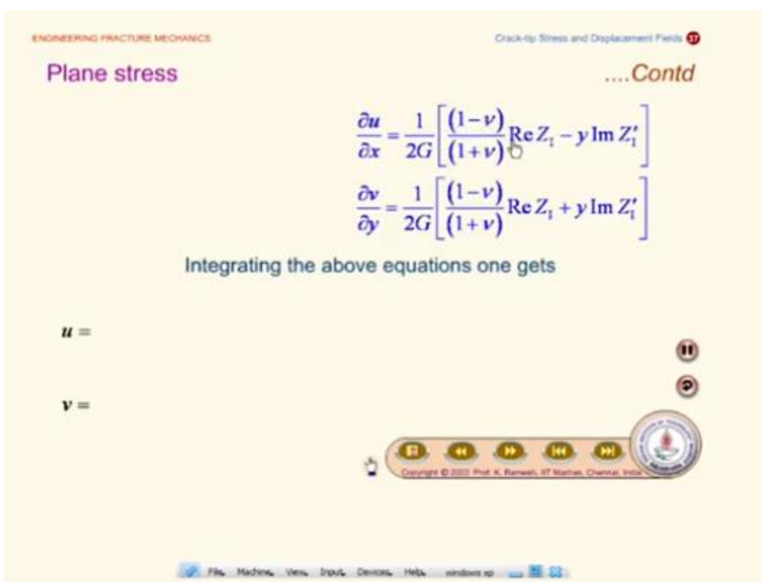

And for a plane stress substituting the stress components in the above equation, one gets this as

$$\frac{\partial u}{\partial x}=\varepsilon_{xx}=\frac{1}{E}\left[\left(\operatorname{Re}Z_{\mathrm{I}}-y\operatorname{Im}Z_{\mathrm{I}}'\right)-\nu\left(\operatorname{Re}Z_{\mathrm{I}}+y\operatorname{Im}Z_{\mathrm{I}}'\right)\right]$$
$$\frac{\partial v}{\partial y}=\varepsilon_{yy}=\frac{1}{E}\left[\left(\operatorname{Re}Z_{\mathrm{I}}+y\operatorname{Im}Z_{\mathrm{I}}'\right)-\nu\left(\operatorname{Re}Z_{\mathrm{I}}-y\operatorname{Im}Z_{\mathrm{I}}'\right)\right]$$
$$\frac{1}{2}\left[\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right]=\varepsilon_{xy}=-\frac{(1+\nu)}{E}y\operatorname{Re}Z_{\mathrm{I}}'$$

So replacing this *E* in terms of *G*, you can get the above equations rearranged in this form. You can write it like this

$$\frac{\partial u}{\partial x}=\frac{1}{2G}\left[\frac{(1-\nu)}{(1+\nu)}\operatorname{Re}Z_{\mathrm{I}}-y\operatorname{Im}Z_{\mathrm{I}}'\right]$$
$$\frac{\partial v}{\partial y}=\frac{1}{2G}\left[\frac{(1-\nu)}{(1+\nu)}\operatorname{Re}Z_{\mathrm{I}}+y\operatorname{Im}Z_{\mathrm{I}}'\right]$$

Now we are interested in getting the displacement field right, so what happens is since we have this expression for $\frac{\partial u}{\partial x}$.

So integrating this expression, you will get the *u* component, so that is what is given up here. So after integration this is I mean it is very simple because we have already defined this integral and differential form of this stress function here so it is very easy to rewrite this expression in terms of *u* now this is an elastic constraint which is remaining the same. But after integration becomes

$\operatorname{Re}\bar{Z}_{\mathrm{I}}-y\operatorname{Im}Z_{\mathrm{I}}$.

Because it was Z_1' earlier but now becomes Z_1 + *f*(y), this is an intergration constant. Similarly the *v* displacement is given by

$$v=\frac{1}{2G}\left[\frac{2}{(1+\nu)}\operatorname{Im}\bar{Z}_{\mathrm{I}}-y\operatorname{Re}Z_{\mathrm{I}}\right]+g(x)$$

g(x), this is which is the integral integration constant.

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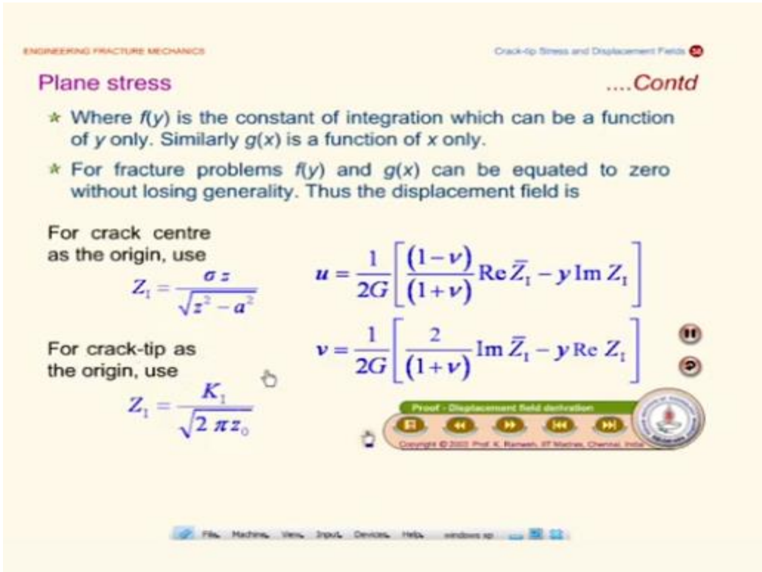

Where *f(y)* is the constant of integration which can be a function of *y* only. Similarly *g*(*x*) is a function of *x* only. For fracture problems *f*(*y*) and *g*(*x*) can be equated to zero without losing generality, So this is an advantage. Thus the displacement field can be written in this form for the crack center as the origin, we have to use this stress function

$$Z_I = \frac{\sigma z}{\sqrt{z^2 - a^2}}$$

for the tip as an origin we should use

$$Z_I = \frac{K_I}{\sqrt{2\pi z_0}}$$

So this is very important so this particular you know origin very near crack-tip this we have already seen in the stress field equation development. So that is not a problem. So, there has to be a clear cut distinction between where you look at the whether it is stress field or displacement filed whether you look at the origin as the center of the crack or the crack tip which is very important.

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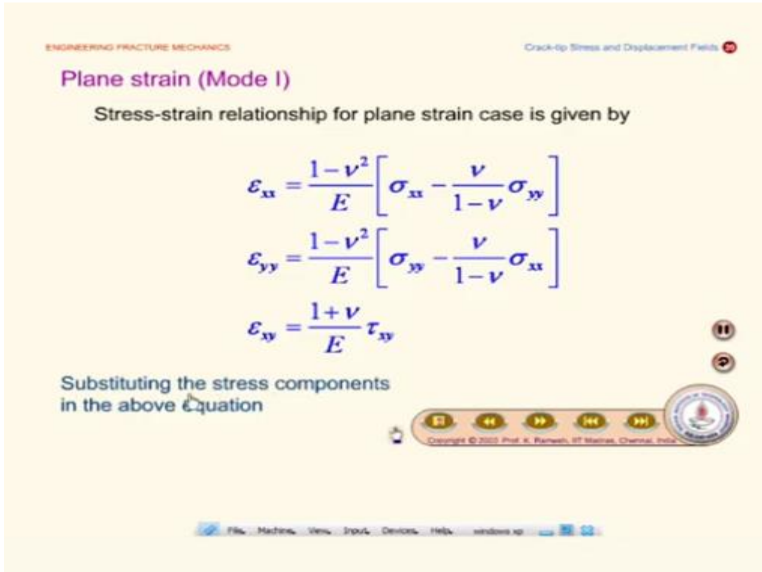

Similarly for the plane strain for a Mode I, stress-strain relationship is modified the count by the elastic constants like this then substituting this stress components in the above equation will read like this.

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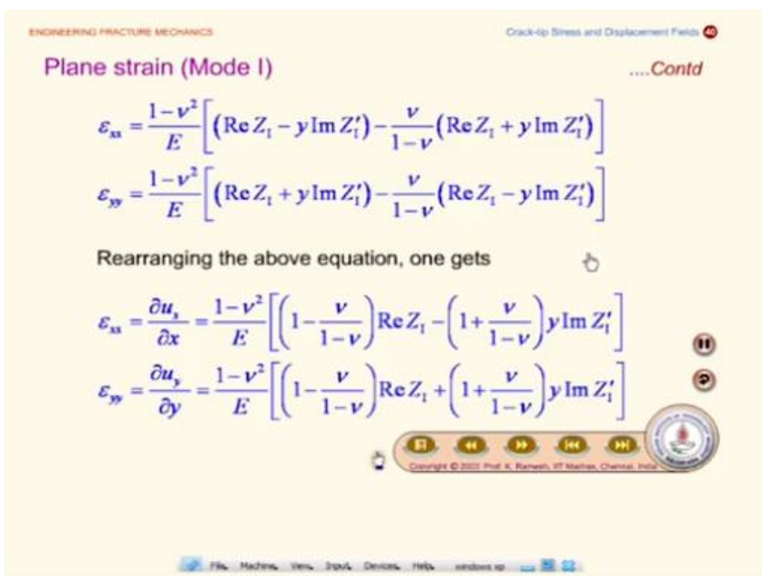

$$\varepsilon_{xx} = \frac{1-\nu^2}{E}\left[\left(\operatorname{Re} Z_I - y \operatorname{Im} Z_I'\right) - \frac{\nu}{1-\nu}\left(\operatorname{Re} Z_I + y \operatorname{Im} Z_I'\right)\right]$$

For ε_{yy} also it is a similar expression except this sign change here these two places rest all the expression is same and then you can rearrange this whole equation, group them into similar terms. Then we can get the ε_{xx} that is

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = \frac{1-\nu^2}{E}\left[\left(1-\frac{\nu}{1-\nu}\right)\text{Re}\, Z_{\text{I}} - \left(1+\frac{\nu}{1-\nu}\right) y\, \text{Im}\, Z_{\text{I}}'\right]$$

So similarly expression is for ε_{yy} except this the sign change right. So, these expressions are bit longer then it is very easy to understand and then substitute and then realize this displacement.

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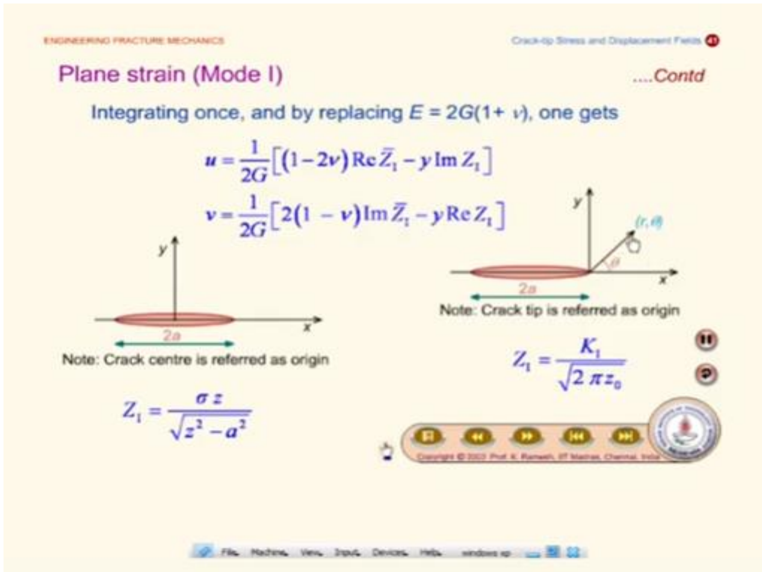

So integrating once again, replacing E = 2G(1+ *ν)*, we get this

$$u = \frac{1}{2G}\left[(1-2\nu)\,\text{Re}\,\bar{Z}_{\text{I}} - y\,\text{Im}\, Z_{\text{I}}\right]$$

Similarly *v* get to be

$$v = \frac{1}{2G}\left[2(1-\nu)\,\text{Im}\,\bar{Z}_{\text{I}} - y\,\text{Re}\, Z_{\text{I}}\right]$$

So what is important to note here is this is for the crack tip is referred as the origin here so this function is being used and we can also now I mean this is for the center as a origin. So we can also write this expression in terms of r and theta, like we have done in the stress field.

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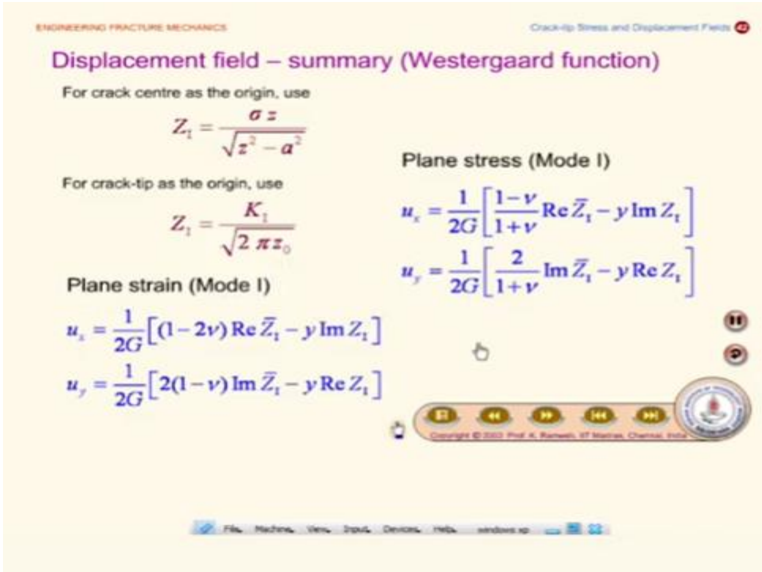

So for that what we need to do is this is just a summary to just put it as plain stress and plain strain displacement fieldd for Mode I and for the 2 conditions.

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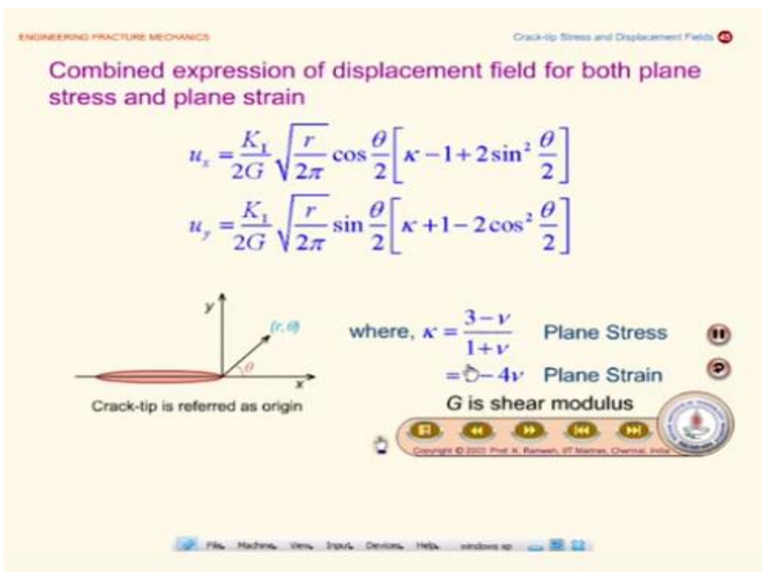

We will now just try to rewrite them into you know very near tip displacement field in the polar coordinates, that is we can write these functions in terms of r, theta. How do we do that? So this expression you have already known this is also we know how to write it. All that we required is Z_1 bar how do we get Z_1 bar of this expression is nothing but

$$\bar{Z}_1 = \left(\frac{2}{\pi}\right)^{1/2} K_1 \sqrt{z_0}$$

So z_o we can write in terms of r, theta like we wrote in the stress field equation, we can write it like this. So you get these two terms then now we can write the displacement field also in terms of r, theta like this for a plane stress and a plan strain. So, what you have to understand from this here is you see that for a plane stress this you have the 3 components u_x, u_y, u_z very important which is

$$u_z = -\left(\frac{2\nu}{E} K_1 \sqrt{\frac{r}{2\pi}} \cos\frac{\theta}{2}\right) B$$

B is the model thickness.

So for the plane stress, the displacement is three in fact we have looked at in the earlier basic equation also, when we write the stress tensor for the plane stress and plane strain we have realized that this is true. So, other point is in the stress field what we looked at the tip of crack, what we have seen that when r turns to 0. That means this stress goes infinity, that means to have the $\sqrt{r}$ singularity value.

Similarly but if you say that r = 0 here it becomes, the total *u* becomes zero. Because, the crack means it is free surface right that is y = 0. So that is also valid here, so we can re-verify that thought process by just putting at r = 0 then the whole displacement becomes 0. So that is one important point to keep in mind. So this is a combined form for the displacement field for both plane stress and plane strain which is given by this constant which is also given in most of the textbooks like this.

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(Video starts: 10:58) The other important points we have to now remember is crack opening displacement if you recall in the fracture parameter table, the one of the fracture parameter you know proposed by Bells it is especially practiced in UK. This is called a crack opening displacement so now we can just try to get an expression for crack opening displacement through this Westergaard's stress function.

So Z_1 is this and Z_1 bar is the integration of this integral. So, we need to look at the integral value of this and then try to take the expression for this stress field. So what is that we are just trying to get it here is this is the; so we are trying to get the displacement u here. So the total length is $2u$ but we are trying to get the u where on this line y = 0, but then we are interested in calculating the COD between $-a$ *to* $+a$.

So this opening displacement is still a valid parameter to calculate because on the x-axis we are; on the crack axis we are though we are considering y = 0 but as a COD, we are calculating between $-a$ to $+a$, that is how we want to understand. So this is what it is written here Z_1, this can be recast we will see how to recast this equation in the next slide.

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(Video starts: 12:41)So integration of this equation can be done by this some simple substitutions like this take m = $z^2 - a^2$. And differentiate this, $dm = 2zdz$ so now you substitute this zdz here that means then we can simply integrate this then it becomes very straight forward integral then the solution becomes

$$\sigma\sqrt{m} = \sigma\sqrt{(z^2 - a^2)}$$

and then we can put it like this and for y = 0 for any value of x can be this

$$\sigma\sqrt{(x^2 - a^2)}$$

And this can be rewritten like this and the minus sign outside then

$$i\sigma\sqrt{(a^2 - x^2)}$$

So the one advantage of writing it like this so we will simply take the imaginary part of Z_1 bar. So you can directly substitute this so that is the advantage of writing it like this.

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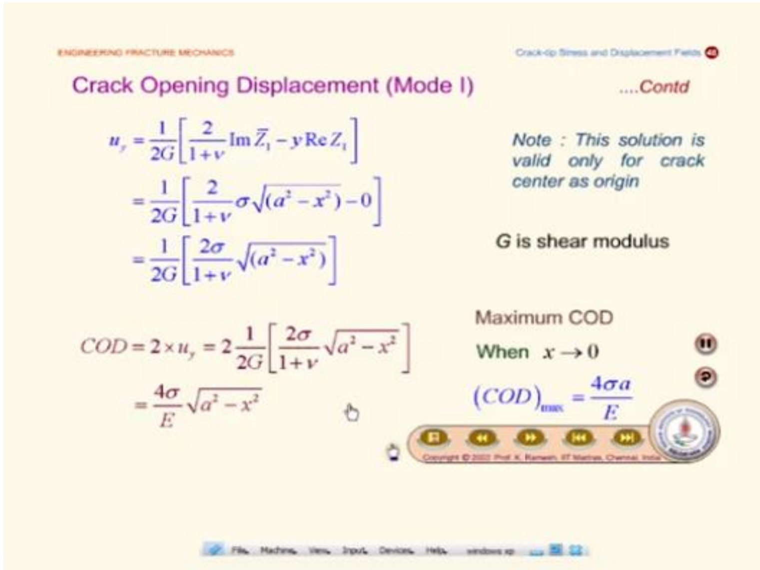

So if you take this and then put it into this equation what we will get is very interesting result. So this is the displacement u_y, you are getting this

$$\frac{1}{2G}\left[\frac{2\sigma}{1+\nu}\sqrt{(a^2-x^2)}\right]$$

And you get

$$COD = 2\times u_y = 2\frac{1}{2G}\left[\frac{2\sigma}{1+\nu}\sqrt{a^2-x^2}\right]$$

So, you get this in terms of this equation and then substitute *G*; replacing *G* and *u* by *E* then you get COD is equal to

$$\frac{4\sigma}{E}\sqrt{a^2-x^2}$$

And then also the maximum COD when x = 0

$$(COD)_{max} = \frac{4\sigma a}{E}$$

So, the another important point to realize is, this particular equation can be recasted into a equation called ellipse equation square by a square + y square by b squared. So we can recast this equation which is because this is a *y* displacement. So that also confirms that the every crack try to opens as an ellipse so it is an ellipse equation COD equation gives us an ellipse equation that is also an evidence that most of the crack they try to open it as ellipse as we shown in all the animation in this particular chapter.

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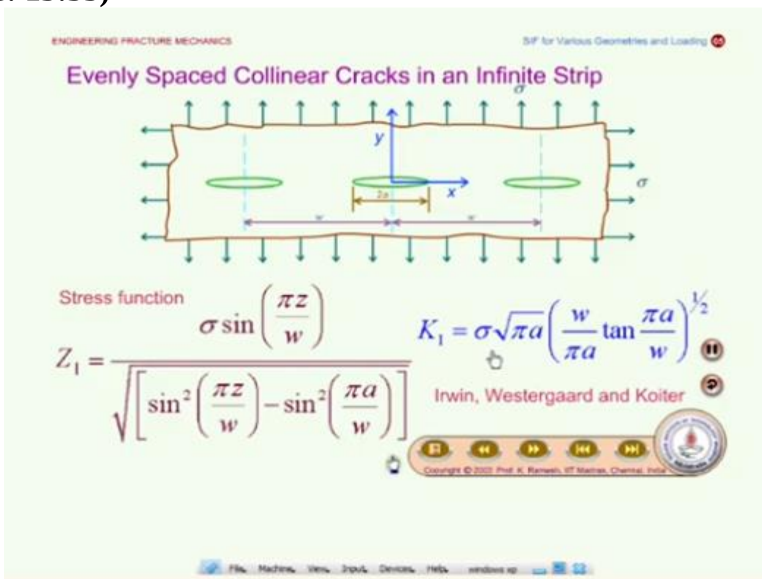

So now we will discuss a very important topic in this lecture that is evenly spaced to go linear cracks in infinite strip. See we have now looked at stress intensity factor for a central crack and then we also know how if we shift the origin to the crack tip how the stress field equations are derived and displacement fields are derived. Now I want to look at the other related topic which is evenly spaced collinear cracks in an infinite strip.

So Westergaard has given a very general solution for this kind of geometry where it is infinite plate which is subjected to again a biaxial loading. So and then you have the cracks you know the elliptical cracks they are placed in a equal distance by *w* and then westergaard as given stress function something like this

Stress function

$$Z_1 = \frac{\sigma \sin\left(\frac{\pi z}{w}\right)}{\sqrt{\left[\sin^2\left(\frac{\pi z}{w}\right) - \sin^2\left(\frac{\pi a}{w}\right)\right]}}$$

So the *a/w* is the one factor which you have already seen that you know when we introduce *a/w* then even in the energy release rate was normalized right by looking at the different different factor we just after introducing the factor *a/w* which was constant β then the *g* also converts into a single value point. So similarly when we bring the factor *a/w* there are so many problems can be solved.

So this particular type of you know evenly spaced collinear cracks problem is considered as a mother of all the solutions. So we will be able to derive so many stress intensity factor from this solution that is what I want to introduce. So let us see one by one quickly. So, the stress intensity factor you know given for this kind of problem is similar to $K_1 = \sigma\sqrt{\pi a}$. So this is already we know, this is already there so it has been multiplied by a factor at a time tangent.

$$\left(\frac{w}{\pi a}\tan\frac{\pi a}{w}\right)^{1/2}.$$

So this factor has to be added. So, now we will try to understand this why it is added which is basically provided by Irwin and Westergaard and Koiter. So we look at the details one by one.

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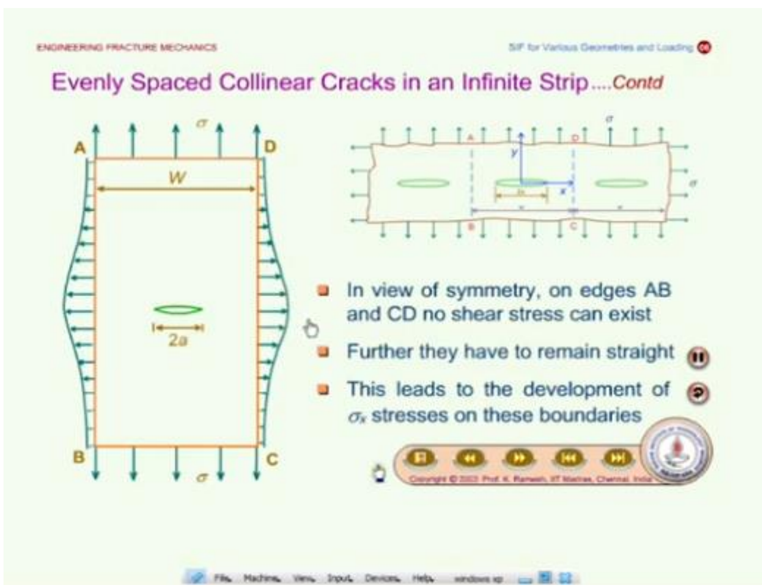

So in the view of symmetry, on edges AB and CD no shear stress cannot exist, suppose if you consider the strip like this which is subjected to uniaxial loading like this central crack is 2*a* and the width is *w*. So on edges AB here and CD no shear stress can exist further they have to remain straight. This leads to the development of σ stresses on these boundaries. So what is shown here is σ_x, is you know developing at along the crack axis.

So that depends on the thickness the crack length and the width in fact that is why the factor *a/w* comes into the picture. So, as long as you know it all depends upon the *a* and the *w* length. The influence of this; the stresses which is σ_x developed on the side is going to be decided so that is why the *a/w* factor is brought so that is very important.

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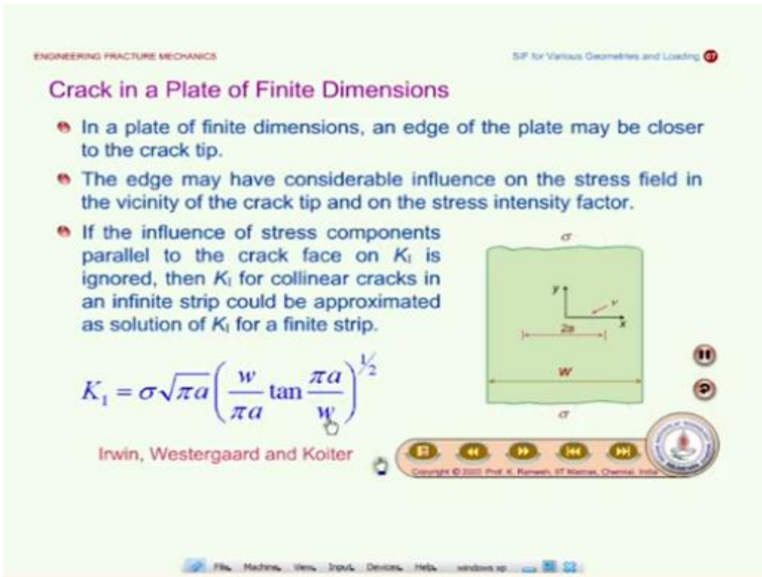

So now we look at the crack in the plate of finite dimensions so far we have looked at the crack problem in an infinite plate subjected to biaxial load. And then we looked at the origin of the central crack as well as the tip central as the object as well now deeper crack tip at the origin has seen the solutions. Now we look at the finite dimension, that means it is a fixed dimension here in a plate of finite dimensions and edge of plate may be closer to the crack tip.

So the edge what we are talking about whether it is close to the edge or not, that is under the question so that will influence the stresses. Therefore, the edge may have considerable influence on the stress field in the vicinity of the crack tip and on also on the stress intensity factor that is what it is shown here. So now it is a finite dimension, if the influence of the stress components parallel to the crack is on K_1 is ignored then K_1 of collinear cracks in an infinite strip could be approximated as solution of K_1 for the finite strip is in this form.

Suppose if you ignore this is the influence of the stress components parallel to the face on the K_1 then we can use this term stress intensity factor for the finite dimensions can be of this type that is

$$K_1 = \sigma\sqrt{\pi a}\left(\frac{w}{\pi a}\tan\frac{\pi a}{w}\right)^{1/2}$$

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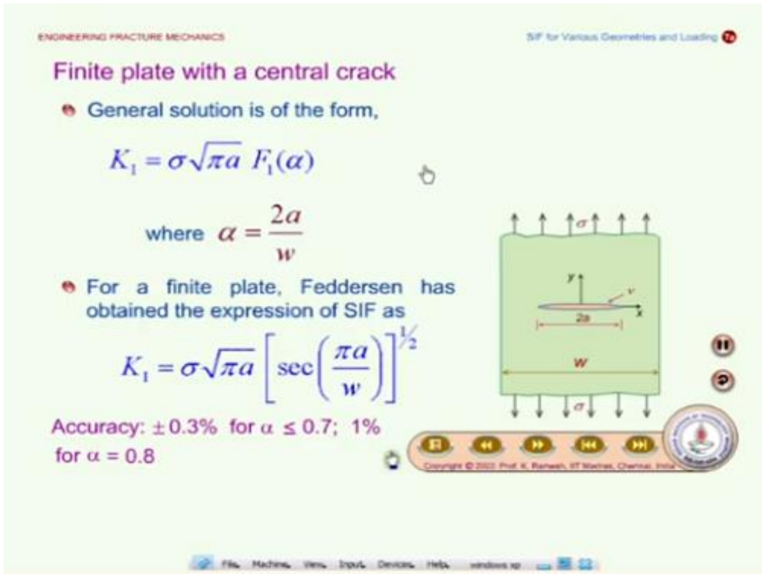

Finite plate with a central crack then the general solution is of the form, so

$$K_1 = \sigma\sqrt{\pi a}\ F_1(\alpha)$$

Well

$$\alpha = \frac{2a}{w}$$

So, this is a general solution can be written like this and several people have given some solutions for a finite plate. Feddersen has obtained an expression for stress intensity factor as

$$K_1 = \sigma\sqrt{\pi a}\left[\sec\left(\frac{\pi a}{w}\right)\right]^{1/2}$$

So, this is proposed by Fedderson and it is accuracy is ±3% for $\alpha \le 0.7$ and it is accuracy is 1% for $\alpha = 0.8$. So this is another important idea.

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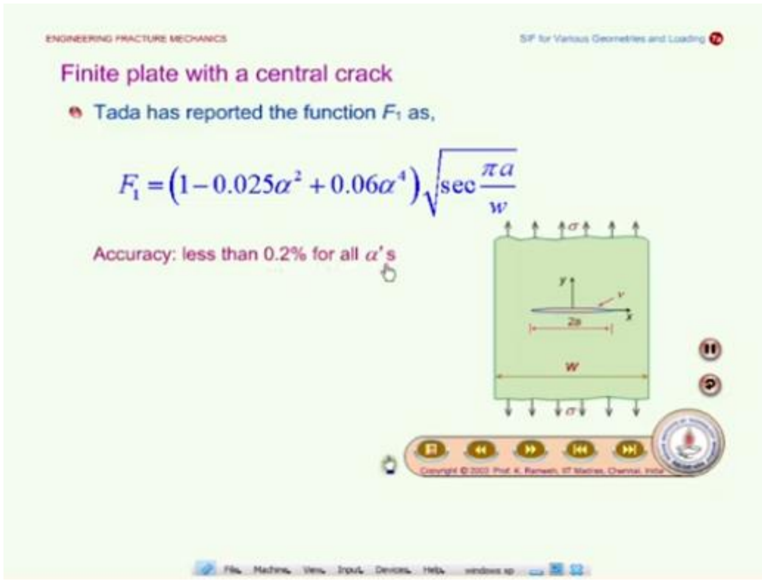

And Tada has reported the function

$$F_1 = \left(1 - 0.025\alpha^2 + 0.06\alpha^4\right)\sqrt{\sec\frac{\pi a}{w}}$$

This has got accuracy less than 0.2% for all α's.

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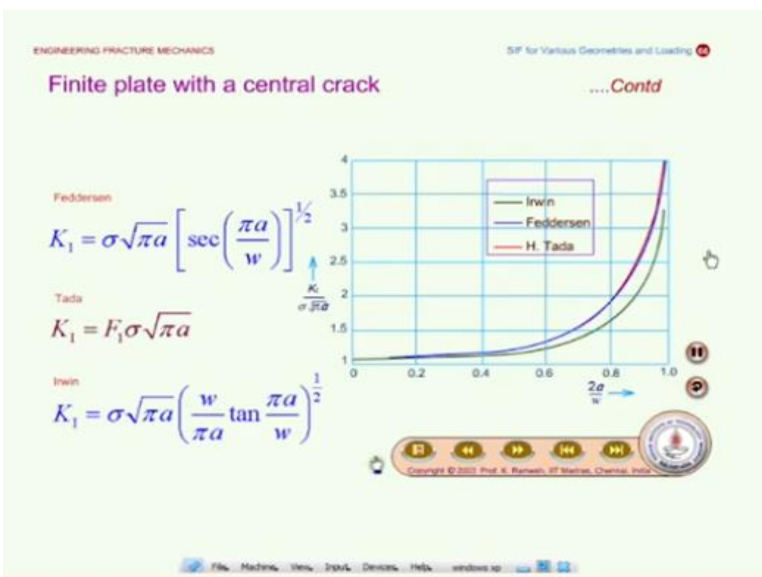

So now we have looked at 3 expressions for the similar problems so now we will compare this how this results appear. So this is a plot of you know

$\frac{K_I}{\sigma\sqrt{\pi a}}$ versus $\frac{2a}{w}$ that is influence of $\frac{2a}{w}$ on the stress intensity factor given by or proposed by these three people Fedderson, Tada and Irwin. So you can see that all are them are almost closely agree into each other.

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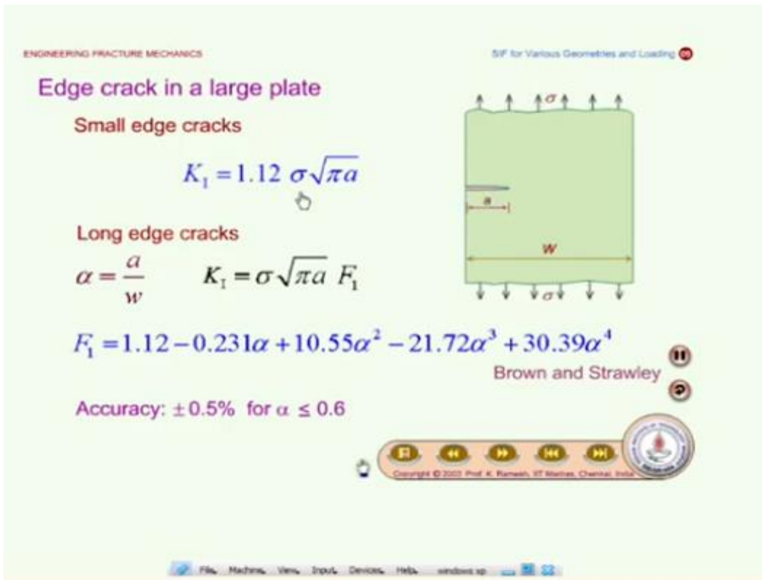

So the next one is the edge crack in a large plate, that means this is a finite dimension with *w* as width and then initially we looked at the central crack now we look at the edge crack the length *a* this half of the total length is *2a*. So half of it is *a*. So the stress intensity factor is given by

$$K_I = 1.12\ \sigma\sqrt{\pi a}\ .$$

So what is to be noted here is for the central crack, the stress intensity factor is

$$K_I = \sigma\sqrt{\pi a}$$

which is now 1.12 time more than that of central crack.

So that means the severity of the crack the edges more as compared to the central crack. Why it is so? This is because the phase which is there in this plane, it is a pack free surface. So this is a pack free surface and then since it is this end is already open and this edges will subjected to you know a free opening displacement again. So the stress which is you know required to open this also becomes higher keeping this track opened also is higher.

So that is what it is given as one of the reasons. And for long edge cracks, $\alpha = a/w$ and

$$K_{\mathrm{I}} = \sigma \sqrt{\pi a}\ F_1$$

where F_1 is given by this numerical relationship by Brown and Strawley. The accuracy of this expression is ±0.5% for $\alpha \leq 0.6$. So, the importance of this kind of expression is to alert that the edge cracks are much more dangerous as compared to the central cracks.

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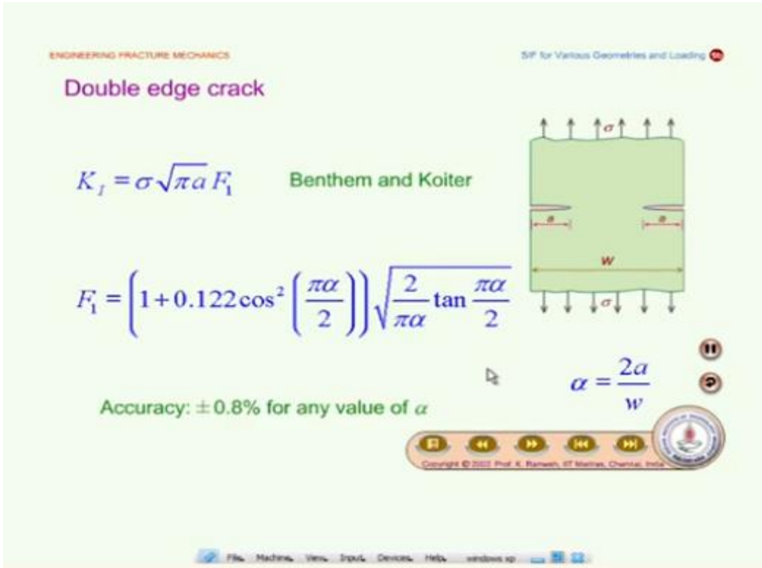

So that important message you should carry on the slides. Then again you compare this models with proposed by this different people. Plot of normalized stress intensity factors versus normalized crack length for single edge cracked plate. And you will see this variation was like that. Now, you can also look at the double edge crack earlier we have the single edge crack, now it is a double edge crack and Benthem and Koiter have given an expression like this.

You see that for all the expression this remains same, that is the beauty of this analysis the basic equation given by Irwin that is in fact and then if we keep modifying this function, which is basically you know the a/w, how the a/w is going to get influenced by this multiple cracks. So that is what it is given here so

$$F_1 = \left(1 + 0.122\cos^2\left(\frac{\pi\alpha}{2}\right)\right)\sqrt{\frac{2}{\pi\alpha}\tan\frac{\pi\alpha}{2}}$$

So this is given by Benthem and Koiter. It has got an accuracy of ±0.8% for any value of α, where α is 2*a/w*.

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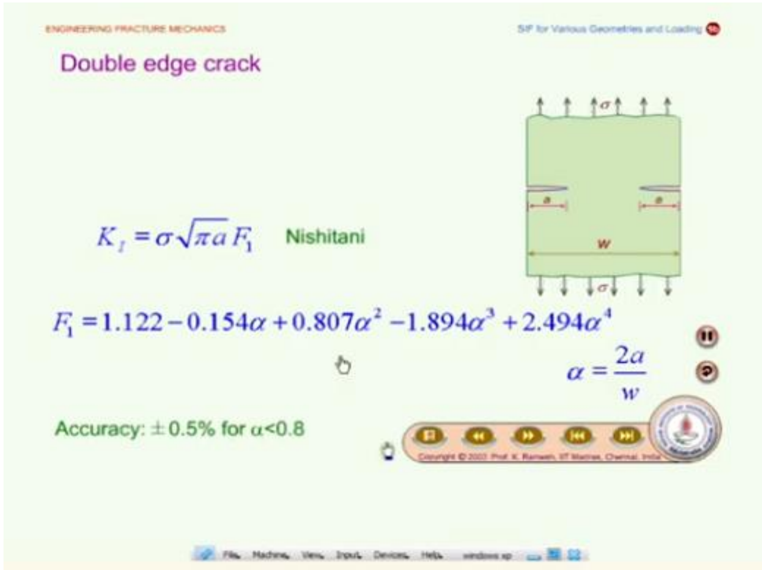

Now this is given by other person, Nishitani is given by other competitive relations which has got accuracy of pressure ±0.5% for α < 0.8.

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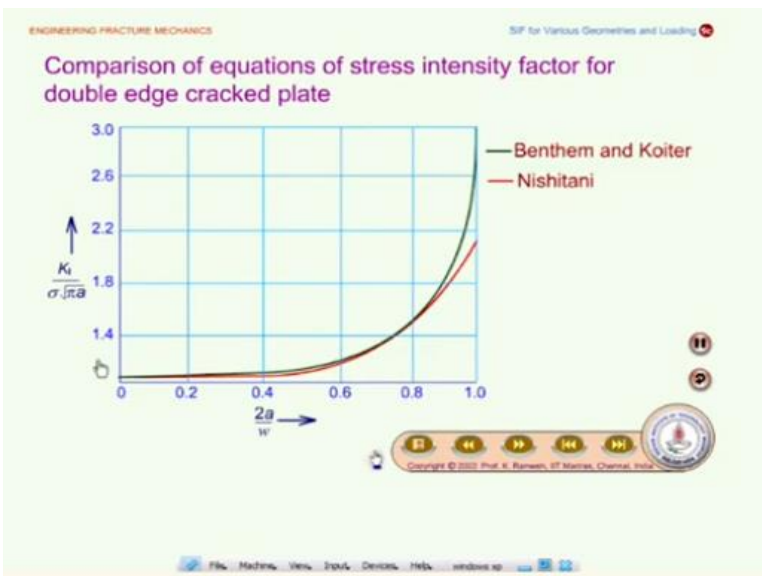

So we can compare the stress intensity factor again as a function of 2*a/w* given by the Benthem and Koiter and Nishitani. So, they agree very well with each other.