Mechanical Behavior of Materials
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Lecture No 53
Fracture Mechanics - V

(Video Time: 00:18)

Hello, I am Professor S. Sankaran in the Department of Metallurgical and Materials Engineering. Hello, welcome to this lecture again on fracture mechanics, we have been discussing the crack problems. In the last class we looked at the concept of energy release rate and then we also witnessed how nicely it explains the, you know the critical crack length that catastrophic failure in terms of stable crack growth and so on.

So, we will now turn our attention to you know the crack tip stress and displacement fields. So, in fracture mechanics, the crack problems are addressed by analytic functions. So, just give you a brief idea about what is analytic functions? So, what are analytic functions? Analytic functions have the following requirement. The derivatives of a function of a complex variable $w = f(z)$ is defined by

$$\frac{dw}{dz} = w' = f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

So, this is a derivative, as a function of complex variable, it is defined like this. And for the derivatives to exist no matter how Δz approaches zero, it is necessary that the limit of the quotient be the same. So, which is shown in this graph. So, no matter how Δz approaches. So, it could be of this function or where $\Delta x = 0$, $\Delta z = i\,\Delta y$ or it could be this function $\Delta y = 0$, $\Delta z = \Delta x$ or it could be a linear function which approaches zero or it could be a non-linear something like this.

So, that is what it means no matter how Δz approaches zero. It is necessary that the limit of the quotient be the same, this is what it is given. So, the function $w = f(z) = u(x,y)+iv(x,y)$, by definition

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

If Δz is real, $\Delta y = 0$. if Δz is imaginary, $\Delta x = 0$. So, this is a property of an analytic function. So, why we are looking at it because we are going to look at the stress function which is also called analytic function. So, that is how we should understand.

So, the analytic function which when we define as a stress function they should have these similar properties that is what it is we are discussing. So, Δz is real, in this case what happens we can just rewrite this differential form like this where

$$\frac{dw}{dz} = \lim_{\Delta x \to 0} \frac{\left[u(x+\Delta x, y) + iv(x+\Delta x, y)\right] - \left[u(x, y) + iv(x, y)\right]}{\Delta x}$$

then you can expand this, the equation like this.

So, if you can just you know regroup them into you know all this *i* side and *v* side then you can rewrite like

$$\lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x, y) - u(x, y)}{\Delta x} + i\frac{v(x+\Delta x, y) - v(x, y)}{\Delta x}\right]$$

then this can be rewritten like

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

That is all it is for Δz is real. For the Δz is imaginary then the expression can be rewritten like this and then what you will see is

$$\frac{dw}{dz} = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

which can be recasted. So, for the two cases we are seeing this, one is Δz is real and another case Δz is imaginary.

So, you can put it together like

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

So, this particular type of compatibility equations are called famously called Cauchy-Riemann conditions. So, for the analytic function, any analytic function they should obey this conditions, that is why it is quite popular. If the derivative *dw/dz* is to exist, the following conditions are to be satisfied. So, which is nothing but

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

So, this is called Cauchy-Riemann condition which you have to remember for defining any analytic function. These conditions have arisen from a consideration of only two of the infinitely many ways in which Δz can approach zero. This is again an important point to note. Cauchy-Riemann's equations are not only necessary but also sufficient conditions for the existence of the derivative of *w*.

So, in earlier case also we have looked at the necessary and sufficient condition for the stable crack growth. So, similarly we have this Cauchy-Riemann condition is not only a necessary condition but also a sufficient condition for the derivative to exist. So, we can keep this in mind. In another important point to note here from the analytic functions, if $w = f(z)$ possesses a derivative at $z = z_o$ and at every point in some neighbourhood of z_o, then $f(z)$ is said to be analytic at z_o and z_o is called a regular point.

So, these are some of the basic definitions for analytic functions just for given for the completion of the knowledge. If function $f(z)$ is not an analytic at z_o, but if every neighborhood of z_o contains points at which $f(z)$ is analytic then z_o is called a singular point of $f(z)$. A function analytic at every point of the region R, the function is analytic in R or the function is regular or holomorphic. These are all the definitions in the analytic functions just to keep it to our knowledge.

So, now we will look at the; what are the Modes of loading? This we have already seen. So, what you are seeing here is a Mode I loading that is Opening Mode and then we also know that Mode

II Inplane Mode or Sliding Mode and then Mode III is Out of Plane Shear Mode which is also called Tearing Mode. So, for all three of them K_{I}, K_{II}, K_{III} are the stress intensity factors and the corresponding fracture toughness parameter is K_{Ic}, K_{IIc} and K_{IIIc}.

So, we are now going to look at the stress field for the mode one case and we will see how it fracture mechanics proceeds to derive this K_{I} that is stress infinity factor. So, what is shown here is Westergaard's stress functions. Westergaard's has given simple stress function that is capital Z as a function of small z for studying the crack problems where

$$Z(z) = \text{Re } Z + i \text{Im } Z \qquad \text{where } z = x + iy$$

So, this is how it is defined. So, you should not get confused with the small z versus capital Z which is a stress function itself. The relevant Airy's stress function could be constructed based on the stress functions proposed by him. So, in the fracture problems Airy's stress functions are quite popular and for solving this kind of you know crack problems, this Airy's stress function is proposed. I did not get into the details of this but this Airy's stress function is one of the method by which this problem is being solved. So, the derivatives and the integrals of z are defined like this.

$$\frac{dZ}{dz} = Z' \qquad \bar{Z} = \int Z \, dz$$

$$\frac{dZ'}{dz} = Z'' \qquad \bar{\bar{Z}} = \int \bar{Z} \, dz$$

So, this is some of the basic definitions we are going to apply into the area stress function and then we will see. So, stress function is a very general function stress function which is being used in the crack problems I mean it is from theory of elasticity.

And the Westergaard's stress function is a very specific stress function for this particular crack problems. So, that we have to differentiate. If you use the Cauchy-Riemann conditions in terms of Z. So, we can write it like this

$$\text{Re } Z' = \frac{\partial(\text{Re } Z)}{\partial x} = \frac{\partial(\text{Im } Z)}{\partial y}$$

$$\text{Im } Z' = -\frac{\partial(\text{Re } Z)}{\partial y} = \frac{\partial(\text{Im } Z)}{\partial x}$$

So, you can see that there is a slight difference the flip here for the real part and then imaginary part. This you have to keep in mind. So, why are we seeing that particular Cauchy-Riemann condition because Airy's stress function has to obey bi-harmonic equation which are going to see and also has to obey this Cauchy-Riemann condition because that is a necessary and sufficient condition to be considered as an analytic function that is how you should understand that.

And now we will see that the mode one stress field equations. So, Airy's stress function for this problem that is the center of the crack is referred to the origin. So, what you are seeing here is an infinite plate subjected to biaxial loading you have to see this σ is here as well as this side. So, it is biaxial loading and the Airy's stress function considered the center of the crack is referred as the origin. So, this is an origin y and x and then this is r θ. So, this is referred in the polar coordinates r and theta.

So, this is an Airy's stress function

$$\phi = \mathrm{Re}\bar{\bar{Z}} + y\mathrm{Im}\bar{Z}$$

So, this function is proposed are taken from the theory of elasticity. It has been you know arrived at by different beings sometimes by even an intuition but this particular Airy's stress function is being applied in this crack problem and how this is going to give the solution that is the point we have to see. So, the stress function selected should satisfy the bi-harmonic equation.

So, this is something a condition for like you know for the; if you have any arbitrary stress function you have to check whether this like for to be called an analytic function you have some conditions. So similarly for this to be applied to the crack problem then again to be called an analytic function, these are the conditions it has to obey and then we will see. So, this is called the bi-harmonic equation that is

$$\nabla^4 \phi = 0$$

So, which is written like this

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

this is a equation.

So, $\frac{\partial\phi}{\partial x}$ can be written like this, since we know how to write the differential form and the integral form of the Westergaard's stress function which is given. So, we can write it like this

$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}\left(\operatorname{Re}\overline{\overline{Z}}\right) + y\frac{\partial}{\partial x}\left(\operatorname{Im}\overline{Z}\right)$$

So, which is yeah after differentiation it becomes

$$\operatorname{Re}\overline{\overline{Z}} + y\operatorname{Im}\overline{Z}$$

Then you can write it again for the

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial}{\partial x}\left(\operatorname{Re}\overline{Z}\right) + y\frac{\partial}{\partial x}\left(\operatorname{Im}Z\right)$$

you can differentiate this and then you will get

$$\operatorname{Re}\overline{\overline{Z}} + y\operatorname{Im}\overline{Z}$$

for this differential we have already seen.

So, based on that we will just write it, it is quite easy because it is all very simple and straightforward we have already given the definite basic definition. So, it is like this. So, what you get as a final

$$\frac{\partial^4\phi}{\partial x^4} = \frac{\partial}{\partial x}\left(\operatorname{Re}Z'\right) + y\frac{\partial}{\partial x}\left(\operatorname{Im}Z''\right)$$

So, this is the final expression for this quantity. So, like that you have to take the other quantities like $\frac{\partial\phi}{\partial y}$ and then it will be like

$$\frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}\left(\operatorname{Re}\overline{\overline{Z}}\right) + \operatorname{Im}\overline{Z}\frac{\partial}{\partial y}(y) + y\frac{\partial}{\partial y}\left(\operatorname{Im}\overline{Z}\right)$$
$$= \operatorname{Im}\overline{Z} + \operatorname{Im}\overline{Z} + y\operatorname{Re}Z$$

then now you apply the Cauchy Riemann condition what we have seen. So, then it can be rewritten like this after applying this. Then it becomes y real part of Z. $\frac{\partial^2\phi}{\partial y^2}$ can be written like this and the final expression is

$$\operatorname{Re} Z - y \operatorname{Im} Z'.$$

So, we are trying to get all the quantities for the bi-harmonic equation. So, now you substitute this and then obtain this the cube term and which is

$$-2 \operatorname{Im} Z' - y \operatorname{Re} Z''$$

So, this is a simple substitution and rearrangement you will get and for this quantity you will get this for $\frac{\partial^4 \phi}{\partial y^4}$

$$-2 \operatorname{Re} Z'' + y \operatorname{Im} Z''' - \operatorname{Re} Z''$$

Now we can take this and because we know this quantity dou phi by dou y square we know already. So, we can substitute that and then we can get this quantity again. So, which is nothing but

$$\operatorname{Re} Z'' - y \operatorname{Im} Z'''$$

So, so now we will take this Airy's stress function and apply the bi-harmonic equation and we are substituting this and what we are finding is 0. So, the Airy stress function proposed for this crack problem is valid.

So, that is how we are validating this Airy stress function. So, it is satisfying the bi-harmonic equation and then we will also see that the summary of this just for the clarity. So, this is a function φ is equal

$$\phi = \operatorname{Re}\overline{\overline{Z}} + y \operatorname{Im}\overline{Z}$$

then it is a condition and this is a coordinate for this. So this is Airy's stress function summary whatever we have evaluated so far just to give you the recap.

So, then you can just substitute this into bi-harmonic equation which is equal to 0. So, that is just for the clarity. So, now this stress function also requires some boundary conditions. So, what is the boundary condition? First let us look at the crack geometry what is given here. So, if you look at carefully, if you look at the animation the crack is just opening this is y and this is x and the crack length is 2*a*.

So now we are looking at Undeformed condition and then while deform condition it opens up for deformed configuration or undefined configuration version. So, the coordinate is x and y and crack length is 2*a* and these are the boundary conditions, what are the boundary conditions? The one important point you have to understand before we get into the boundary condition is the moment you see that crack is opening then you have to realize that the crack surface is a free surface.

The crack surface is free surface you can see that. So, then if you think of that free surface on this top, another is bottom then you look at this conditions.

$$-a < x < a \text{ on } y = 0$$

$$\begin{cases} \sigma_y = 0 \\ \tau_{xy} = 0 \end{cases}$$

that is in plane shear stress is also becoming 0 because of the free surface that is one condition.

The second condition is when z → infinity, that means it is a far field it is an air field and this is a far field for field what happens $\sigma_x = \sigma_y$ which is nothing but σ. So, you know for field it becomes equal to σ and where τ_{xy} again is 0. So, this is the second boundary condition it has to satisfy and along y = 0 for any x, τ_{xy} again is equal to 0 due to symmetric.

So, this is see what we are looking at is when we say along y = 0 still this condition is written keeping this mind as a straight line. So, since it is a very small deformation these are all assumptions. So, the inplane shear stress again is equal to 0 due to a symmetry that is again you have to remember. So, these are the boundary conditions. So, we have to see whether these boundary conditions are met by this stress function proposed by the Westergaard's.

So, we will see what is the Westergaard's stress function?

$$Z = \frac{\sigma z}{\sqrt{z^2 - a^2}}$$

So, this is a Westergaard's stress function proposed for the crack problems. In fact, this stress function is going to remain for all sort of you know Modes of failure Mode 1, Mode 2, Mode 3 we will see at least we will completely see the the derivation of at least for the Mode 1 then we can just mention how the second and third equations will look like.

So, now we try to verify this boundary conditions. So,

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = \operatorname{Re} Z + y \operatorname{Im} Z'$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -y \operatorname{Re} Z'$$

So, we will now take this the stress function given by Westergaard's and then try to see whether the boundary conditions are met here. So, what is the first boundary condition we know that where x between -*a* and +*a*, what happens to this term. So, this term becomes imaginary. So, that means the Re Z = 0. For y = 0 and *x* can be in the interval of - *a* and +*a*. So, this is the first condition, boundary condition then it becomes

$$\sigma_y = 0$$
$$\tau_{xy} = -y \operatorname{Re} Z' = 0$$

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So, this is the first boundary condition and what is the second boundary condition we have, that is a far field where z is much much higher than *a*. So, then it becomes *z*. So, it becomes *z*.

So, the z becomes a far field stress then what we have seen the stress function is equal to is simply sigma which is a constant a number it is a constant. So, the derivative of this number is a 0 that is Z' is equal to 0. So, now we will see we can apply this conditions to

$$\sigma_x = \operatorname{Re} Z - y \operatorname{Im} Z' = \sigma$$

$$\sigma_y = \operatorname{Re} Z + y \operatorname{Im} Z' = \sigma$$

$$\tau_{xy} = -y \operatorname{Re} Z' = 0$$

So, this secondary, all the second boundary condition also satisfied, that is what we are seeing. Now what is the next step? The next step is we are origin shifting. So, the originally we looked at the crack length as 2*a*. So, it was the center of the crack is kept as the origin of the description but, Irwin later proposed to shift this to crack tip to consider that as a, you know specific problem.

So, the origin is shifted from center to tip. So, then the z is defined as $z_o + a$. So, what is z_o, this is this distance small distance and this is really assumed to be very very localized and very close to crack tip that is an assumption. So, when you assume that then the Westergaad's stress function can be recast like this you just substitute z

$$z = z_0 + a$$

since we are talking about very near tip stress field the equation is obtained by making the approximation z_o is you know far smaller than the K, very very close to the crack tip.

Then with that assumptions, this equations becomes what, we will take only this form

$$Z = \frac{\sigma a}{\sqrt{2 a z_0}}$$

So, only this z_o is coming here because of the approximation taken here. So, this can be rewritten like this

$$\frac{\sigma a}{\sqrt{2 a}} z_0^{-1/2}$$

this is a definition given in by Irwin and Irwin's collaborator Kies. So, it is named the K is because of the name Kies but this is the proposed by Irwin. What is to be noted here is in the literature K_1 is also defined without π. So, since it is proposed by this Irwin. So, it is kept like that but even without π, the meaning of K_1 does not change. So, that is information you have to remember. So, if you write a σ using this equation then this equation can be recasted like this and if you replace z_o with r, theta and this can be written like this the Westergaard's function Z then becomes

$$\frac{K_1}{\sqrt{2\pi}}(re^{i\theta})^{-1/2}$$

and the derivative

$$Z' = -\frac{K_1}{2z_0\sqrt{2\pi z_0}}$$

So, this is straight forward from this equation. And now we can look at the mathematical definition of stress intensity factor. If $Z(z_o)$ is the stress function of the problem defined when defined with respect to the crack tip then

$$K = \lim_{z_0 \to 0} \sqrt{2\pi z_0}\, Z(z_0)$$

So, this is a stress function given by the Westergaard but what is now we are deciding is this function is very close to crack tip very close to that that is what it is. So, the stress intensity factors describe the strength very close to the crack tip. So, the interrelationship of the stress and the crack length in the fracture behaviour is nicely represented by K. Unlike the stress concentration factor *K* has units of MPa(m)$^{1/2}$.

And the credit goes to Irwin for coining stress and stress intensity factor and since then the fracture mechanics took a giant leap forward. So, this is very important point because even though energy release rate gave quite a bit of confidence in dealing with this subject but the stress and after defining the stress intensity factor only people were able to quantify the near stress field solutions.

It is a near stress field solution, one can obtain by looking into stress intensity factor that is the idea. So, very near tip stress field equations for Mode I in terms of Westergaard's stress functions and if the stress function φ is known then the stress components could be determined from. So, what you have to remember is we have to identify φ which is given in the Airy's stress function that could be different for different crack problems or different fracture mechanics problems.

But in this case it is given by the Westergaard's stress function and then we are looking at it that is how you have to remember and also remember that this problem looked at the biaxial tension. So, the Airy's stress function is

$$\text{For } \phi = \mathrm{Re}\bar{\bar{Z}} + y\,\mathrm{Im}\bar{Z}$$

$$\text{and } Z = \frac{K_{\mathrm{I}}}{\sqrt{2\pi z_0}}$$

$$\text{where } z_0 << a$$

where z_o is very very small as compared to crack length that means it is very close to crack tip.

The solution is valid only to the very close to the crack tip. Now we can write this $\sigma_{x,}$ σ_y τ_{xy} expression in terms of real part and imaginary part what we have derived using this Airy's stress function. So, it takes this form. So, this is the stress field near the crack tip is given by this form. So, now what we can do look at it is we can now rewrite this near tip stress field using this Westergaard's stress functions

$$Z = \frac{K_1}{\sqrt{2\pi z_0}}$$

z_o can be replaced by r, theta then it becomes like this.

So what is written here is. So, r, i theta can be recasted like in terms of cosθ, sinθ like this. So, and

$$Z_1' = -\frac{K_1}{2z_0\sqrt{2\pi z_0}}$$

So, which can be written like this. So, what happens is this is nothing but $r^{3/2}$. So, then this can become like this then the cosθ, sinθ function takes this form. So, you have now Z and Z_1'. So, we can now substitute this value Z value and Z_1' value into the stress field equation and then you will get the complete form of stress intensity factor.

So, very near tip stress field equations in terms of r and theta is given by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{K_{\text{I}}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\begin{Bmatrix} 1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2} \\ 1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2} \\ \sin\frac{\theta}{2}\cos\frac{3\theta}{2} \end{Bmatrix}$$

So, I have not taken step by step in substitution which is very simple in the interest of time I have just skipped that step then if you substitute independently then also you will be able to derive this. So, what is the importance of this equation? The strength of you know all $\sigma_{x,}$ σ_y and τ_{xy} completely relies on all depend upon only this K_1 that is why it is called stress intensity factor.

So, what happens if r $\rightarrow$0? If r tends to 0, then this becomes infinity and this part it takes care of the orientation and this is the one that is the magnitude of the K_1 only dictate the strength of this parameters. So, in the above equation when r tends to 0 the stress field has the singularity of $\sqrt{r}$. So, $\sqrt{r}$ singularity is quite important in structure mechanics that means at this point at one single point r tends to 0 this values becomes infinity that is why it is called $\sqrt{r}$ singularity.

So, the practical utility of this equation for the evaluation of stress intensity factor in experimental mechanics is limited, as the region of its validity is very small. Though this expression gives a complete idea of the stress field around the crack tip, but the utility in a practical aspects like experimental determination of the stress field is limited and people have to do so many other multiplications which is actually beyond the scope of my discussion here.

But at least my intention was to show how this stress intensity factors are derived at least for the Mode I. Even for Mode II and Mode III the basic forms will not change they will all be the same, all the basic forms whatever we have seen is not going to change. But I urge all of you to look at the Professor K. Ramesh's course in NPTEL and you can continue or verify this other quantities like Mode I, Mode II derivation Mode III derivation, everything you can learn it from there if you are really interested in pursuing that.

So, I will stop here for the by discussion on stress intensity factor and then we will continue to discuss the other displacement fields and then I will go to the crack tip plasticity plastic zone in the next lecture, thank you. **(Video Ends: 37:24)**