Elementary Stereology for Quantitative Microscopy
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Lecture - 09
Geometrical Probability – I

So, we will continue from the last lecture. So, what we have done so far is we have looked into events like intersection events in stereology, we have looked at the geometrical probability of those events and brought in the essentially the concept of geometrical probability. Now, going further with this thing, let us look at some basic stereological operations.

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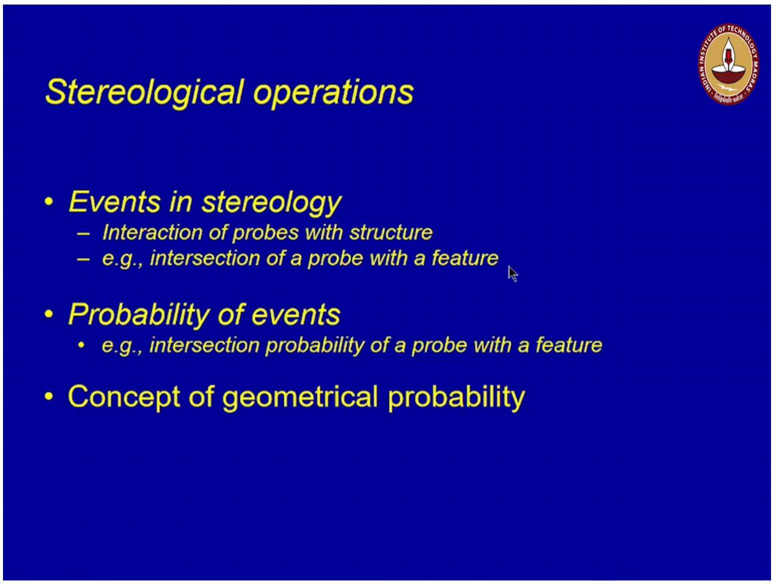

This is one operation that you have already seen it is probability of intersection with a point probe.

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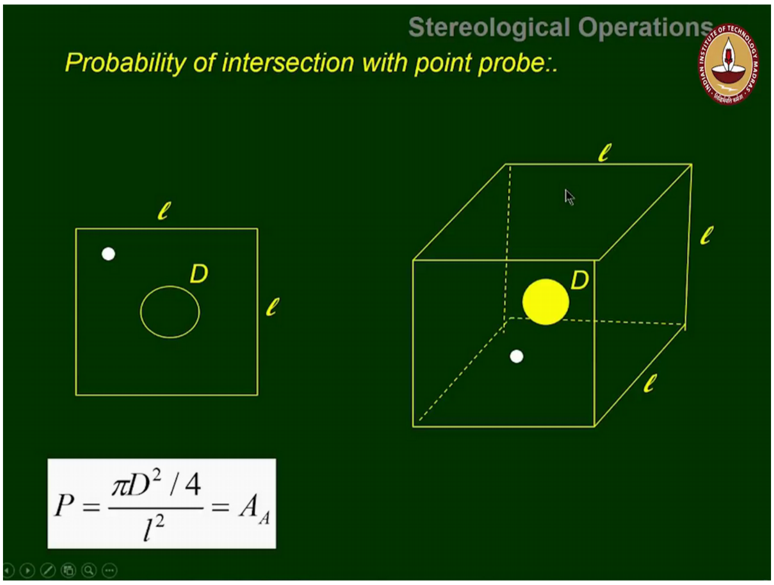

Here if you have an object of a circular object of diameter D contained in a 2 dimensional box of size l by l and if I put a random point then we have already seen that the probability that this point will fall in the circular object is given by the area of the object pi D square upon 4 divided by l square, which is also related to the area fraction in the structure.

Now, we can extend this idea from 2 dimensions to 3 dimensions, when now I have a spherical object in a cubic volume of l into l into l and if I drop a point randomly in this cubic volume, the probability that this point will fall inside the spherical object of diameter D is simply the volume of the object divided by the total volume of the cube. So, just a simple extension of the idea from 2 D to 3 D. So, that gives you pi D cube upon 6 divided by volume l cube and that is nothing, but the volume fraction ok.

So, let us now look at a different probe instead of a point probe consider now a 2 dimensional probe a section.

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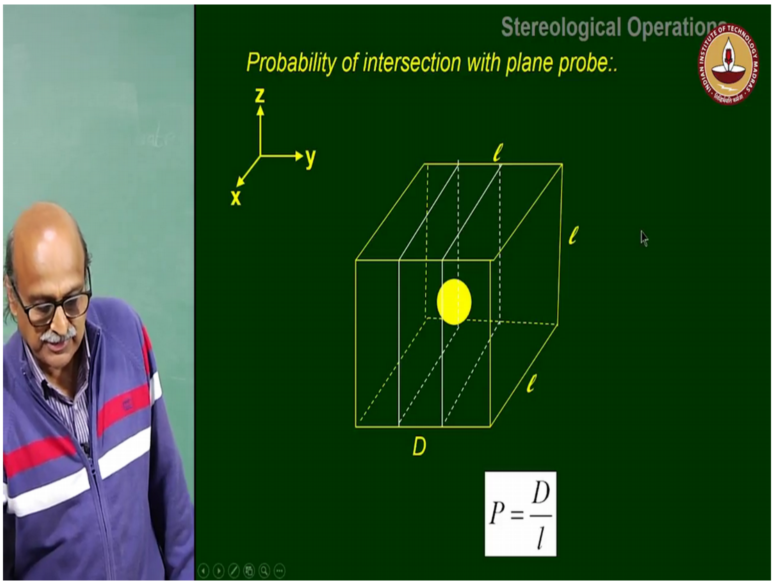

Now, here I have a again a cubic volume and in this cubic volume I have object a spherical object again of diameter D and if I want to find out what is the probability of cutting this particle by a section by a 2 dimensional probe.

Let us keep that probe parallel to the exit plane so; that means, we are talking about parallel to this face of the cube the side face, but that particular probe can randomly fall anywhere it is positioned along the y axis ok, parallel to the exit plane, but it is randomly positioned along the y axis. So, what would be the probability in order to figure that out, consider 2 tangent planes this plane makes a tangent to this object at this end and this particular plane makes it makes a tangent to this spherical object at this end.

The distance of separation between these 2 planes would be nothing, but the diameter of the sphere which is D. So, any plane now randomly placed anywhere in the cube parallel to any of these tangent planes the probability that plane would cut the object would simply be the ratio of the distance D the diameter D of the object divided by the length of the cube.

So, that would simply become so, it is a simple extension of the same idea, here we are talking about linear distance in the previous case when you are looking at the point probe we were looking at area and volume area in the 2 dimensional case and volume in the 3 dimension case. Now, consider a line probe now here I have again the same spherical object to diameter D inside the cubic volume.

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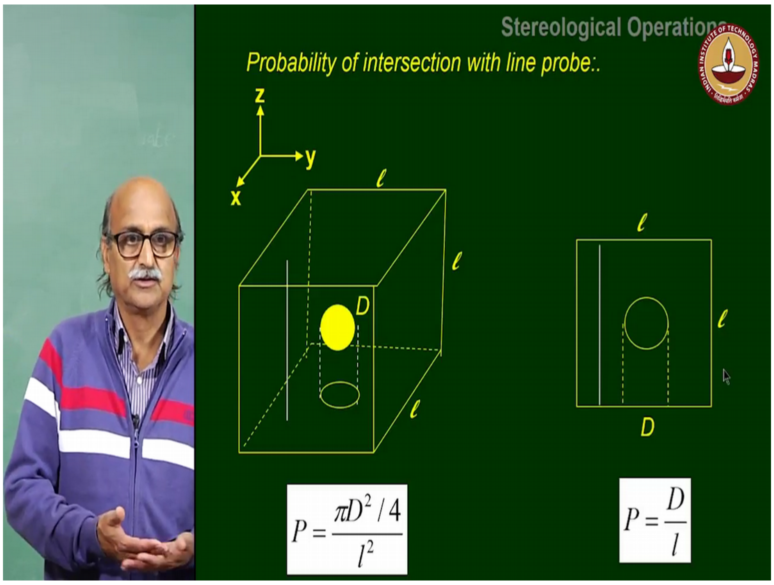

And I have now a line probe which is kept parallel to the Z axis , but it can be randomly located in the anywhere in the cube what would be the probability of the line probe cutting the object D any ideas here, how we could determine this? What we could do is, take a projection of this object as well as the line probe on to the X Y plane.

So, on to take a projection on this cube face if I do this what will I get, I will get a circular projection, whose diameter will also be D and the projection of this line probe on the X Y plane would be nothing, but the, but a point on the X Y plane right. Now, what is the probability of this point because if this point falls inside the projection; that means, the line will cut the object what is the probability of a point falling inside the object?.

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Would be the area of the circular projection divided by area of this plane, which is l square. So, this is what would be the probability of the line probe cutting spherical object. Similarly, if you look at the 2 dimensional case of the line probe then it becomes similar to the tangent planes in the 3 dimensional case, where this line probe is kept parallel to this side of the square probability that this line will cut this circular object would be nothing, but this distance D which is nothing, but the distance between these 2 tangent lines divided by the length of the 2 dimensional box to which is the probability of the line probe cutting this object is D upon l.

So, these were some of the basic operations that we saw, but we had kept things simple we had kept either a spherical object or a circular object.

Now, my particles need not be spherical they can be of any shape, what happens if we take a now a non spherical object, a non spherical object like this particular object shown here.

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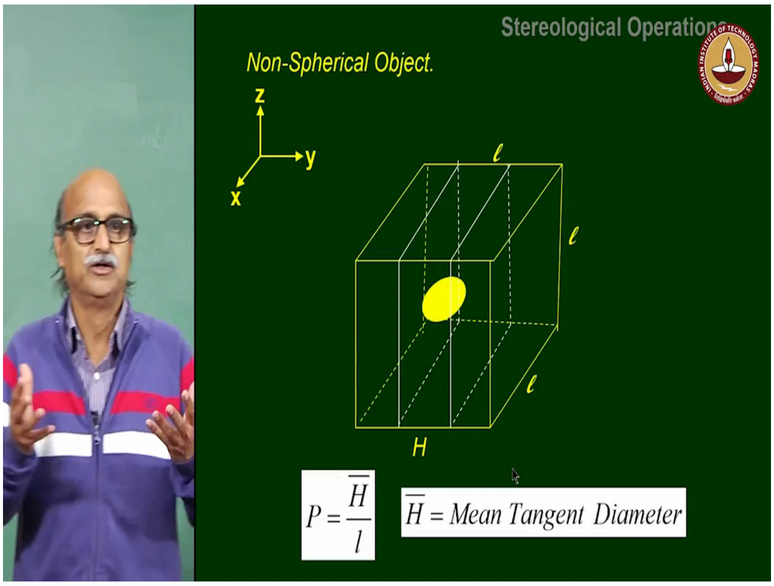

It is in some particular orientation and again I can draw 2 tangent planes on either side of the object and the distance of separation between these 2 tangent planes is kind of a diameter along of this object along a particular orientation. If this is the case then the probability of intersection of the tangent plane would simply be the distance of separation H divided by the total length of the cube that would be the probability of a 2 dimensional probe cutting this object.

But now the question arises is that this is a non spherical object and it can take up any orientation; that means, in a different orientation the probability would be different in another orientation the probability would be different. Hence, we need to come up with some kind of an average parameter for this object an average dimension of for this object and that average dimension divided by l would be the probability of cutting this object.

This H is what we call as the tangent diameter of this object or which is a distance of separation between the 2 tangent planes as, the object changes orientation the tangent diameter changes.

So, we can think of that there exists a dimension called the average or the mean tangent diameter and the probability of cutting by a plane probe this object then which takes up any random orientation would be the H bar the mean tangent diameter divided by l.

So, this probability will now depend on that what is the mean tangent diameter of this object; that means, we have to average it over all possible orientation. It is something similar to what we did, when we were deriving that relation for the Buford's needle problem, that we averaged the projection length of the needle over all possible orientations here we have to average the tangent diameter over all possible orientation.

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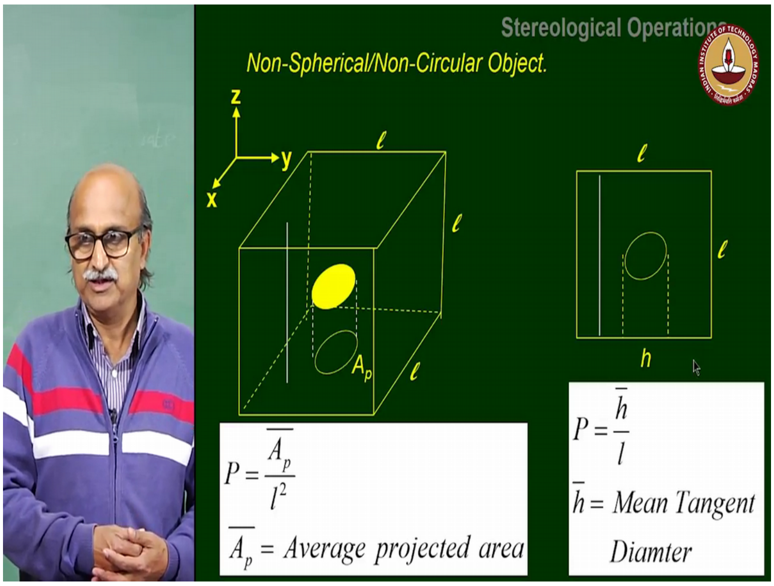

Similarly, you have a non spherical object and what will be the probability of a line probe cutting this object well now the projected area of this object which is A sub p here will is going to depend on what is the orientation of this object. So, again we have to come up with an average projected area and obtain the probability as the average projected area divided by l square.

A similar idea extended to 2 dimensions I have a tangent diameter again in 2 dimensions again for a non spherical object I will need some kind of an average value of this diameter to obtain the probability as H bar upon l. So, this what the basic stereo logical operations, but before you proceed further we need to understand how we can take these averages to obtain. For example, the mean tangent diameter and so on, which means we need to define orientation in space whether it is a 2 dimensional space or the 3 dimensional space, we need to have some kind of a definition for our orientation and then we would require some kind of a probability function for those orientation in order to do that averaging.

So, in order to do this I just go to the board and try to understand first how we can define orientation, we will take first the simple case of 2 dimensions. So, in 2 dimensions what I can do is.

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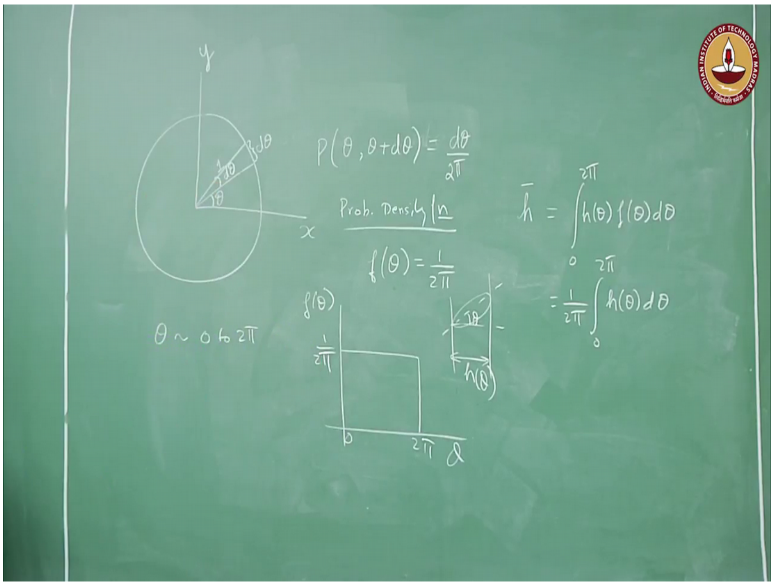

I can define a circle of unit radius in x y coordinates, then any orientation can be defined by the parameter theta and this theta can take up any value from 0 to 2 pi. So, in order to now define a probability function for theta let me consider an orientation which is lie which is lying in the range of theta and theta plus d theta, so, any orientation which is lying in this thin infinitesimal slice through my circle.

What would be so, now, the question that arises is that what would be the probability that a randomly chosen orientation will lie between theta and theta plus d theta.

So, I want to find this probability so, what do you think would be the probability for this to be random orientation with which uniformly spreads around the circumference of this circle, all positions must be equally likely which means that the probability that this orientation will lie between theta and theta plus d theta we can determine what is the length of this infinitesimal arc which is nothing, but I have a circle of unit radius. So, this is simply d theta and what is the perimeter of the circle, it is 2 pi right with unit radius.

Hence, probability that an orientation will lie between this in this particular section between theta and theta plus d theta would simply be length of that infinitesimal arc divided by the total perimeter of the circle. So, this d theta upon 2 pi which means now we can define the probability density function for these random orientation in 2 dimensional space as simply f theta is equal to one upon 2 pi where theta varies from 0 to 2 pi, what kind of a function is this, what kind of a probability distribution is this, where my density function is a constant it is a uniform probability distribution.

So, if I were to look at the density function it would be this is 0, this is 2 pi and this is 1 upon 2 pi ok. So, in 2 dimension the case is very simple. So, if I have to do any averaging of let us say tangent diameter then what do I have to do to get the mean tangent diameter for an object whose tangent diameter varies as h theta and where perhaps this is my theta for this kind of an object then this would simply be integral 0 to 2 pi h theta, f theta, d theta, this is a simple case this actually comes out to be simply 1 upon 2 pi integral 0 to 2 pi h theta d theta.

So, that is all you have to do if you are dealing with in 2 dimensions. Now, extend this idea to the 3 dimensional case, where it will be slightly more complicated, but not too complicated ok.

So, let us look at let me draw another figure for that, now instead of a circle consider a sphere of unit radius whose center is at the origin of my coordinate system.

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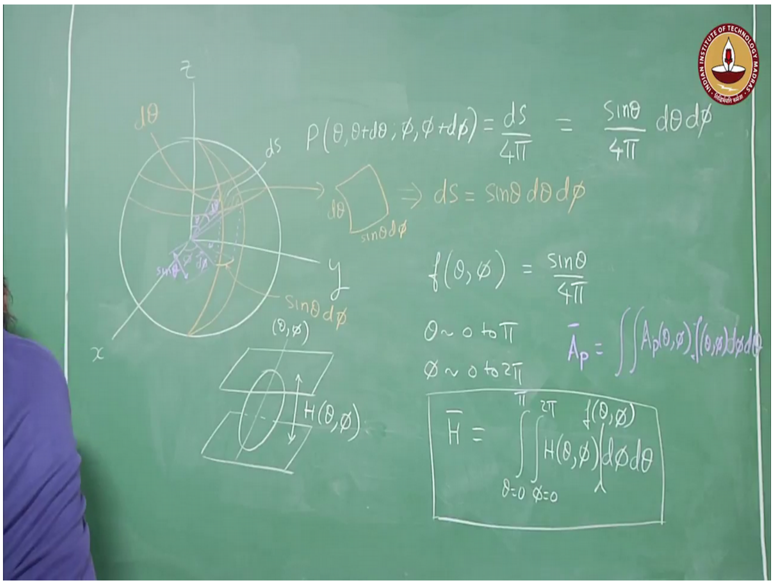

And in the previous case in the 2 dimensional case basically I divided the circle into small infinitesimal arcs r d theta. Here I am going to divide the entire surface of the sphere into small infinitesimal areas and then probability of any random orientation hitting one of those areas would be that infinitesimal area divided by the total surface area of the sphere.

So, how do I divide so, first of all how do I divide the surface of the sphere into small infinitesimal area how can I do that you know in a simple manner, if you look at the globe how are areas divided on the globe in terms of latitudes and longitudes. So, we can do the same thing on this surface I can define I can divide the entire surface into latitudes and longitudes with angular distances between latitudes and longitudes as infinitesimal angles so, I will require 2 angles one for latitude other for longitude.

So, in order to define an orientation in space in 3 D I require 2 angles instead of one angle in the 2 dimensional case I require one angle. So, let us take a pair of longitudes and a pair of latitudes. So, I this will define one small element on the spherical surface and let us define this point.

Now, this the orientation of this point is defined as angle theta that the line makes with the z axis and take a projection of this line on the x y plane and it will make the projection will make an angle of phi with the x axis. So, this defines theta and phi defines the orientation of this point.

Let us consider now this point now if I take a projection of this it will come somewhere here. So, let us consider the angle between the 2 projections on the x y plane that angle to be d phi ok, then consider this point here draw this line consider this angle to be d theta. So, this entire element now is defined by the angular range theta to theta plus d theta that gives me essentially the angular spacing between the 2 latitudes and the angular spacing between the 2 longitudes by d phi.

So, now it is a simple matter by saying that the probability that a random orientation lies between theta and theta plus d theta and phi and phi plus d phi, this is the probability that I want to find. If the area surface area of this element is ds then this would simply be ds divided by the total surface area of the sphere which would be 4 pi for a sphere of unit radius any doubts here ok.

Now, let us try to find out ds in terms of my angles theta and phi. So, if I look at this length basically this length into this length would give me the area of this infinitesimal element. Now, what would be this length, this length would be the same as this projected length on the x y plane and the projected length on the x y plane is nothing, but this particular length the projected length multiplied by d phi.

Now, what is this projected length, this is nothing, but radius of the sphere times sin theta. So, I can write this length as simply sin theta or sin phi so, this will become sin theta. As a result this length would become sin theta d phi what about this length here, this length would nothing, but r d theta r is one so, this length is d theta.

So, I have essentially if I draw this element separately this is sin theta d phi and this is simply d theta. Hence, ds is equal to sin theta d theta d phi which means that the probability that a random orientation will lie between theta and theta plus d theta and phi and phi plus d phi would be sin theta upon 4 pi d theta d phi.

Now, this can be written in terms of the probability density function f theta phi which is now a function of 2 random variables theta and phi in the 2 dimensional case the density function was a function of one random variable only theta and this would simply be sin theta upon 4 phi. Now, we should also know what is the range over which this function is defined, the range of theta and the range of phi the range over which this is defined is should be such that every point must be visited on the sphere and no point should be visited more than once and that is simple to see that if I really theta from 0 to pi and phi from 0 to 2 pi then every point will be visited exactly once ok.

So, this probability density function is defined over the range of theta going from 0 to pi and phi going from 0 to 2 pi ok. So, now, if I have a non spherical object at some arbitrary orientation so, this is some arbitrary orientation given by theta and phi then the tangent diameter for this orientation is nothing, but the distance between these 2 tangent planes I can call that as H theta phi and this is going to vary as a function of theta and phi and if I want to get the mean tangent diameter H bar then I will have to integrate over phi as well as theta so, phi going from 0 to 2 pi theta going from 0 to pi h theta phi d phi d theta.

So, we have written down the functions by which we can get the mean tangent diameter, similarly for projected area I will simply be replacing s theta phi by projected area as a function of theta and phi that would give me the average projected area. So, if I want to get A P bar then it will be again double integral A P as a function of theta phi this there is a missing this thing I did not I have to introduce the density function also to get the average f theta phi here also it would be f theta phi d phi d theta ok.

So, this is the way we can get averages now best way to understand this is to do actually an exercise and get up mean tangent diameters for couple of objects we will do that in one case it would be a 2 dimensional case and for a 3 dimensional case ok. So, let us now take up subsequent exercise on this to get this idea of mean tangent diameter in more concrete terms.