Elementary Stereology for Quantitative Microscopy

Prof. Sandeep Sangal
Prof. S. Sankaran
Department of Materials Science and Engineering
Department of Metallurgical and Materials Engineering
Indian Institute of Technology, Kanpur
Indian Institute of Technology, Madras

Lecture – 04
Geometric Probability

So, what you saw just now were some of the calculations done on some experiments that you did in the last lecture. In this you calculated basically some kind of statistical estimates for your area of fraction of those graphite nodules you estimated and if you consider them to be circles circular objects then the d bar give you some kind of an average size of these nodules. Now, as we were understanding that these relationships that have been used, how do we arrive at that relationship.

So, what we were discussing last class not really discussing, but I just mentioned that the bases of all of this relationship comes from stereology. And the bases of foundation of stereology is based on basically probability in statistics and the kind of probability that will be dealing with is what is called as geometrical probability. And these estimates that you saw, as I said was subjected to statistical errors that also we will see.

So, first I will deal with geometrical probability and then we will just briefly introduce a topic many of you would know probability distributions a little bit of an idea for probability distribution will kind of review it. From that we can then do some kind of an error analysis based on the based on this data on the cast iron microstructure.

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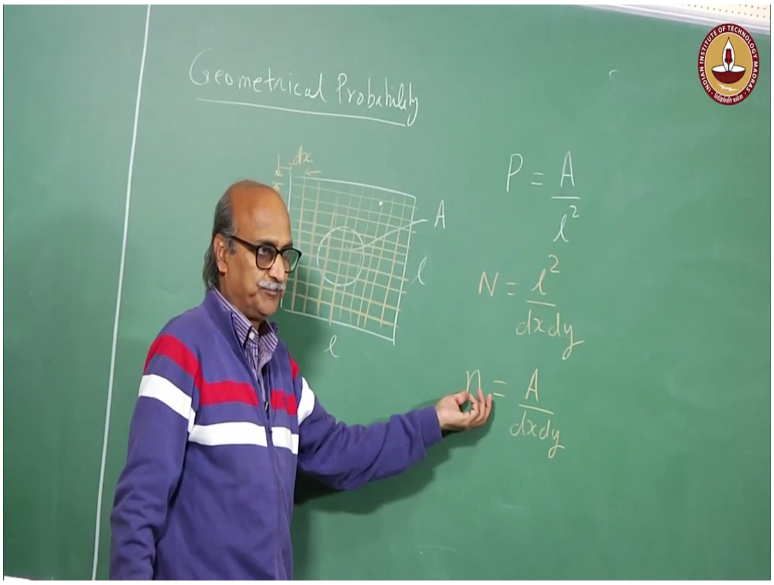

So, starting with geometrical probability let us try to understand what does this mean? Well just I am sure that you are pretty much aware as to what is probability, if I asked you to toss a coin and I ask you that what is the probability of getting a head; what would be the answer?

Student: (Refer Time: 02:34).

It is 0.5 or half or 50 percent that would be the answer you would give me if I threw a dice. What is the probability of getting a 3?

Student: (Refer Time: 02:46).

1 upon 6. What is the probability of getting a prime number? 2 3 and 5 are primes. So, it is half it is half again. How did we arrive at these results? Probability of getting a 3 is you look at what are the total number of outcomes you have right total number of outcomes are 6 in a dice experiment right 1, 2, 3, 4, 5, 6 getting a 3 means that is we have only one favourable outcome.

So, probability in this case is the ratio of favourable outcomes to the total number of outcomes right. So, 1 upon 6 getting a prime number 2 3 and 5 there are 3 favourable outcomes total 6 outcomes, so 3 upon 6 and that gives you half. In all of this of course, there is one assumption that all these outcomes are equally likely 1, 2, 3, 4, 5, 6, or in the toss of a coin head and tails are the 2 outcomes which are equally likely then only you can apply this simple relationship for probability the ratio of favourable outcomes upon the total number of outcomes.

We would be using this particular definition and then try to see understand geometrical probability one simple experiment with geometrical probability could be that I have this 2 dimensional space. In which let me say I have a object of area A ok.

Suppose I you know this is my this could be my dartboard for that matter and I threw a dart at it blindfolded and I am assuming that this dart can hit anywhere on this surface right. So, it hits within this square, but the dart hits anywhere it could hit inside the object, it could hit outside, so it could hit here.

So, essentially what we are doing is we are creating as I keep throwing darts I am essentially creating random points inside this square. So, now if I ask you the question what is the probability that the dart will hit this area A?

Student: (Refer Time: 05:40).

How do you get this?

Student: (Refer Time: 05:45).

So, you are saying this is area upon l square, but how do you come to this conclusion? That this is a you know this appears to be a reasonable result; can we reduce it down to our definition of probability saying that these are the favourable number of outcomes, and these are the total number of outcomes ok.

Let us try to do that. So, what we could do? Is we can divide this 2 dimensional space into small infinitesimal elements ok. So, and each element let us say if I look at the dimensions let me these are infinitesimal element, so this is d x and d y. So, I have divided this entire 2 dimensional space into elements of size d x by d y. Now, this random point can fall in any one of these elements with equal likelihood ok.

So, now, for this random point to fall in any one of these elements if I look at what are the total number of outcomes that are possible? Are the total number of elements that are present in this space. What are the total number of elements present? Let us say that is capital N, can we find out what is the total number of elements is the area of this space divided by area of each of the elements ok.

If a point falls in an element which is inside this area A, then that point falls inside that area A. So, all the elements which are part of this part of this a object of area A. How many such a elements are there? Because all those elements would be the favourable outcomes if a point falls in them.

So, total number of such elements is the area same way, area of the object divided by area of each element. So, small n becomes the total number of favourable elements or favourable outcomes and capital N is the all possible outcomes.

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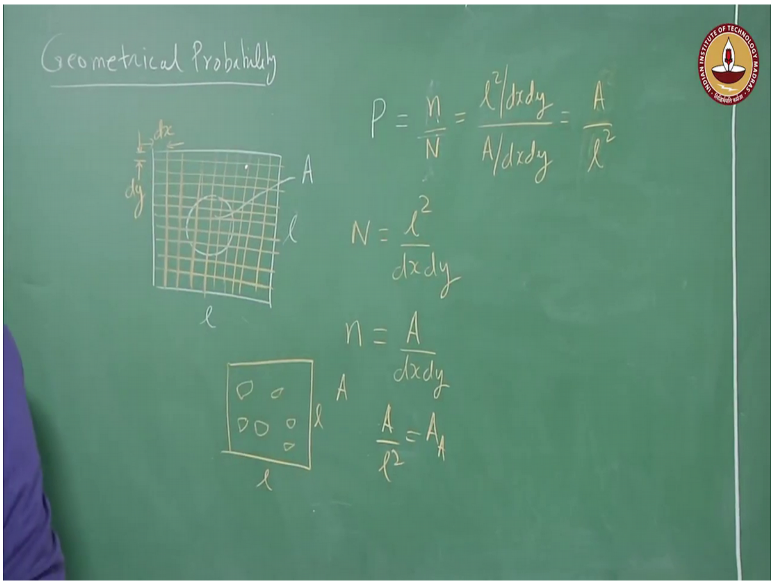

So, probability P instead of writing it here like this let me write it as small n upon capital N which is l square d x d y divided by A divided by d x d y. And this very clearly would give you the result that we had intuitively said is should be the result alright. Yes you have a question?

Student: (Refer Time: 09:26).

Yeah, this should be A by l square ok, now let me now pose a different problem, so this is one for problem. And in fact this particular solution that we have seen that the probability is the area of this object divided by the area of this space, in fact, would also be true if I had not just one element, but I had several elements.

Total area of those elements let us say was a and the space was still the same l by l then what is the probability of point hitting any one of these elements? Any one of these objects would be the total area of these objects A divided by l square the same. And what is this A upon l square if I look at this as particles let us say this is nothing, but area of fraction.

So, this point count technique that we did in the last lecture that was introduced to you this is the basis of that technique. After all what are you trying to find from point fraction? When I calculated the point fraction just now the point fraction was calculated from the data that we had collected. What are we trying to find out? We are trying to estimate this probability and that is why point fraction would directly give you an estimate of the area fraction ok. So, I have shown you one simple basis for the point count technique ok.

Now, let us take up some couple of more problems associated with geometrical probability let us take another example.

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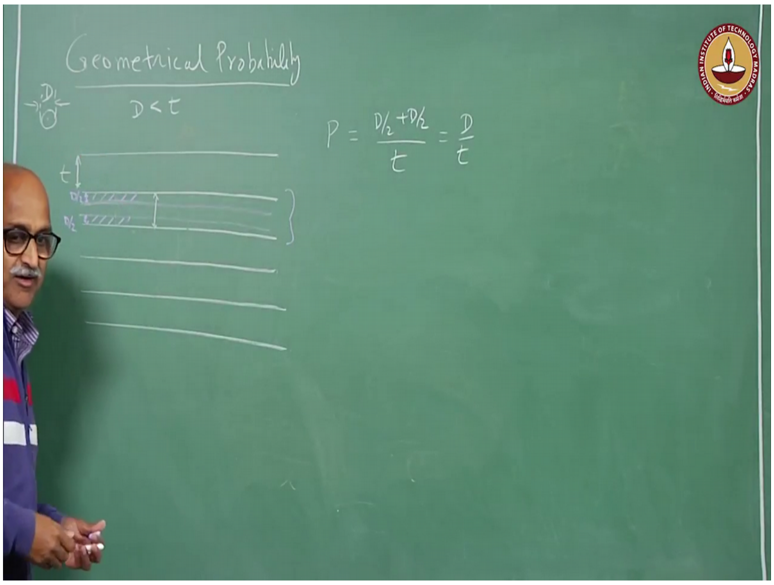

Let us say I have these set of parallel lines I have a coin whose diameter is let us say D ok. If I toss this coin on this set of lines what is the probability that this coin will intersect one of these lines given that D is less than t which means basically the coin will intersect only one line at a time cannot intersect more than one line.

So, let us try to look at this let us say that let me draw this is D by 2 this is D by 2. Just consider a pair of lines within this pair of lines when would the coin intersect either the top line or the bottom line? Would be when the centre of the coin would fall either in this region or in this region right then only the coin will intersect, total length being the total distance between these 2 lines is being t.

And if the centre falls within D by 2 of this line or within D by 2 of this line then the coin will intersect one of the lines. Hence, what can you say about the probability of the coin intersecting either one of the lines the central point can fall anywhere from here to here. Horizontally where it falls does not matter it is only the vertical distance one has to look at. So, it can fall anywhere from this to this over the entire distance t out of which D by 2 here and D by 2 here. If the centre of the coin falls in that region then the coin will intersect the line hence what should be the probability then?

Student: (Refer Time: 14:53).

D by 2 plus D by 2 divided by t because this centre of the coin can fall with equal likelihood all along; So, this also you can break it up just the way I broke it up we can break it up into favourable outcomes and total number of outcomes by producing thin slices of thickness D t. And then counting how many slices are within D by 2 here?

How many slices are there within D by 2 here? Those becomes a favourable number of outcomes total number of slices within this pair of lines that becomes a total number of outcomes and hence you would get this D by t ok. So, probability that the coin will intersect one of these lines is simply the ratio D upon t. Now, let me take a similar problem, but slightly change it instead of a coin now I have a needle of some length.

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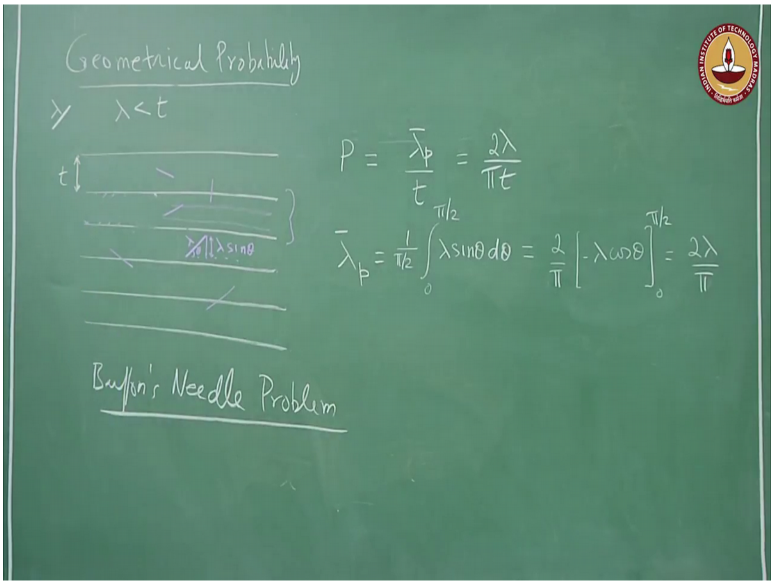

Let us say I now have a needle of length lambda and the spacing between these lines is again t and lambda is less than t and this is my needle or a stick or whatever you wish to call it. What is the probability that this needle will intersect one of these lines? But this is a slightly more complicated problem and the complication is coming from the fact that this needle now can fall in various orientations ok.

If let us say the needle was always falling, we have constrained it we are doing a experiment we have constrained the experiment in such a way that the needle is always falling perpendicular to these grid of lines. Then what would be the probability of this needle intersecting one of the lines?

Student: (Refer Time: 17:38).

Is lambda by t ok. If I constraint it to fall perpendicular, but I cannot constraint that it can fall in any random orientation. So, let us look at this problem in the following way let us take a particular orientation of the needle and I can define this orientation as an angle theta with the horizontal axis the projection of this needle perpendicular to this would be this length ok.

If I take a projection perpendicular to the, to these gridlines what would be this length? This length is simply lambda sine theta right. Suppose the needle is constrained in such a way that it always falls at a fixed angle theta then what is the probability of this needle intersecting one of these lines is lambda sine theta upon t right,, but it is not constraint theta can take any random orientation. So, the answer lies to this that what is the probability that a randomly thrown needle which falls in a random orientation would intersect one of the lines that probability then would be up average projected length upon t.

So what is this average projected length? Well then you have considered all possible orientations and average it over all the orientations. Now, what would be the what would be this average projected length lambda p bar well that all I have to do I have to integrate it over a certain angular interval. In this case looking at the symmetry of the problem I just need to integrate it from 0 to pi by 2 then the other orientations are identical.

So, 0 to pi by 2 lambda sine theta D theta, but I have to take an average value an average value of this would be 1 upon pi by 2 this will give me the average projected length. I have summed it over and divided by the difference between the integration limits this is a very simple integration.

So, this will become 2 over by pi and I will get lambda cos theta with the limits 0 to pi by 2. If I substitute this very easy to see that the average projected length is 2 lambda upon pi hence the probability of intersection then becomes 2 lambda upon pi t ok. So, this is the result that a needle thrown randomly on this set of parallel lines having a spacing t the probability of intersection is 2 lambda upon pi t.

In fact, this is a result we will see in a subsequent lecture that this result will be used to eventually show some of the other relationships that were written in the last lecture. And then again demonstrated today the relationship between p p p l etcetera we will see how that will come about.

Incidentally this particular problem is famously known as Buffon’s Needle problem. In fact, you can Google it you will you will be able to see this particular problem explained. This is a very old problem dating back to late 1700s Buffon was a mathematician who had constructed this problem from a point of view of estimating the value of pi experimentally by throwing needles on a set of lines and computing an experimental probability.

So, you know once the experimental probability then you can estimate pi there are any queries you can interrupt. Now, the third; third of the fourth problem another simple problem in geometrical probability let me take that up.

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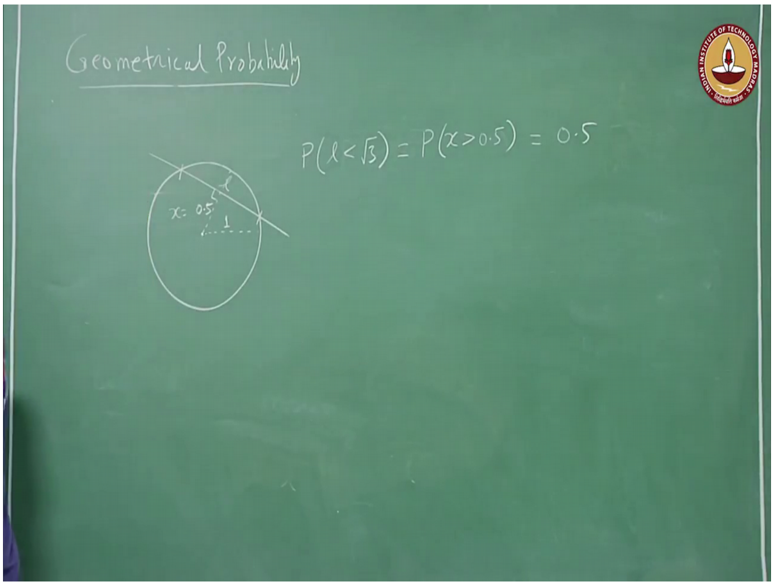

Suppose I have a circle of let us say unit radius, and this circle is intersected by a random line forming a random cord of length l. What is the probability that this random cord length would be less than square root of 3?

Now, in order to look at this particular geometrical probability question what I can do is? Let me drop a perpendicular from the centre very clearly the length of the cord is going to be a function of the perpendicular distance of the cord from the centre ok. As this distance increases the cord length reduces as the distance decreases the cord length increases and it is a maximum when the a line passes through the centre will be equal to the diameter of the cord.

So, the probability that the length will be less than root 3, let us find out what is the value or what is the perpendicular distance of this cord from the centre when this length is root 3. And the answer is quite obvious that when this perpendicular distance is 0.5 or half the radius in this case radius is 1. So, the when the perpendicular distance is 0.5 the cord length will be equal to root 3 you can work this out quite easily from the geometry of the problem.

So, probability of the cord length is would be less than root 3 I can rewrite this question as if this distance is x, probability that x is greater than 0.5 that is the midpoint of the cord with this the perpendicular will fall at the midpoint. So, the midpoint of the cord is at a distance of greater than 0.5 then the cord length will clearly be less than root 3.

Now, what is the chance of the cord falling at distances greater than half the radius is simply 0.5 because this midpoint of the cord can fall along this radius line anywhere from 0 to 1 in which half of those outcomes are going to be greater than 0.5 and hence the probability is half or 0.5 ok.

Similarly, you can you can further solve this problem what will be the distribution of cord lengths? But we will take that problem a little later right now I will just use some of these examples to show you what is the geometrical probability, and then we will later apply all of these concepts in subsequent lectures.