Elementary Stereology for Quantitative Microscopy

Prof. Sandeep Sangal
Prof. S. Sankaran
Department of Materials Science and Engineering
Department of Metallurgical and Materials Engineering
Indian Institute of Technology, Kanpur
Indian Institute of Technology, Madras

Lecture – 10
Geometrical Probability – II

So, let us first consider a 2 dimensional situation, ok.

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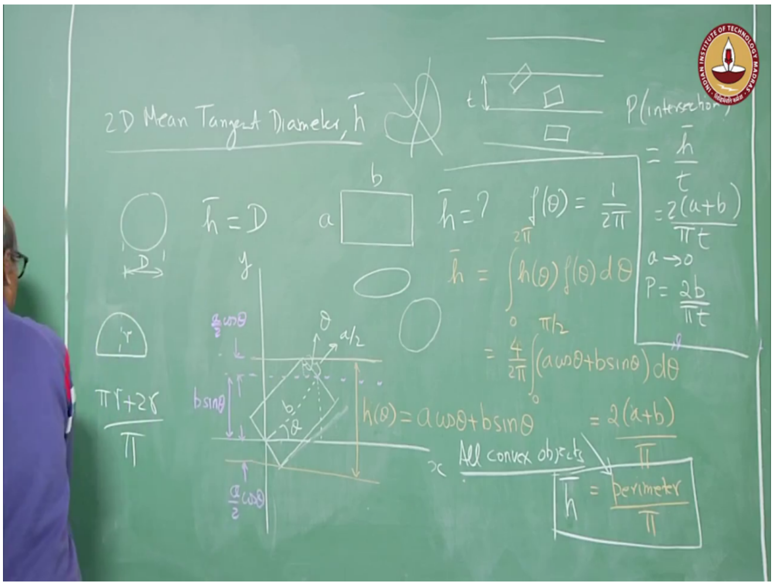

Let us say what would be 2 D mean tangent diameter; let me call that as small h bar. What would be the mean tangent diameter for a circular object of diameter D, what would be h bar for this? You take any orientation it remains as D.

Student: Yes.

So, average if you take an average remain D. So, it will be the diameter of the circle. Now, let me consider another object let us consider a rectangular object of dimensions a and b. What is the mean tangent diameter for this object? Here we will have to do that full integration, ok. Remember in the 2 D case, the density function for the orientation defined as f theta is the uniform density function given by 1 upon 2 pi. So, let me put this object in a coordinate system to understand this.

So, all I have done is, I have centered this object so that one end of the rectangle is centered about the origin and the orientation is defined by the central axis with respect to the x axis. Now, this theta can vary randomly over 0 to 2 pi. What I will do is, these are my two tangent lines parallel to the x axis and the distance between these two tangent lines would be H theta, no I am using small h so I will keep using small h here, then the mean tangent diameter h bar theta is simply 0 to 2 pi h theta f theta d theta.

So, what I need to do in order to do this integral? In order to integrate this I need to express h theta in terms of theta a and b. Once I am able to do that it will be a straight forward job to do this integration. So, let us try this. Let me draw some more lines here. Let us get different distances to get that h theta. First of all let us take this distance how much is this well this length is b.

So, clearly this length would be nothing, but if I consider this triangle this would be b sin theta, right. Now, what about this length here then I will add up all of these pieces to get at h theta; well if this angle is theta let me draw a perpendicular from here if this is theta this angle would be theta, this would become pi by 2 minus theta, then this would become theta ok, simple geometry.

Hence, this length is nothing but if you consider this small triangle well in this triangle what is this length? This is a by 2 hence this length would become a by 2 cos theta. Similarly, this length would also be a by 2 cos theta you can examine in this and therefore, I simply add up all of these h theta would become a cos theta plus b sin theta. So, I have expressed the tangent diameter in terms of a, b and theta, ok.

So, now if I continue with this integration this would become 0 to 2 pi instead of h theta now I can write a cos theta plus b sin theta and instead of f theta I put 1 upon 2 pi, 1 upon 2 pi being a constant I take it out of the integral times d theta I just need to integrate this.

Now, also look at the symmetry of the problem I really do not need to integrate all the way from 0 to 2 pi. I just need to integrate from 0 to pi by 2. If I do that then instead of 2 pi here I put here pi by 2 and then there are other there is an integral from pi by 2 to pi then pi to 3 they have four such integral. So, there will be a factor of 4 that will come out here, ok. This is what that integral would become, alright.

Now, this is a simple integral this will give you the result; I will directly write the result now this will give me the result 2 times a plus b divided by pi. Now, just take let us go back to the Buffon's needle problem going back to the Buffon's needle problem instead of throwing a needle I throw this objects rectangular objects of size a by b such that the diagonal which is the maximum length of that object is less than the spacing between these lines t. So, sometimes I will get an intersection sometimes I will not; what is the probability of intersection in that case? Probability of intersection with the lines this is nothing, but h bar divided by t which is nothing, but 2 times a plus b divided by t.

Now, suppose I let a go to 0; what kind of an object this will become? This will become a line of length b. Say if I put a go to 0 my probability would become there is a pi missing here, 2 a plus b upon pi this will become 2 b upon pi t. You look at this is exactly the same result that we had the probability of needle intersecting lines. So, what we actually derived is a from both angles, we come to the same solution.

Another important thing to look at this result here; what is 2 times a plus b? Is the perimeter of the rectangle? So, this perimeter upon pi; it turns out and we can prove it in a very general case that for any convex object ok, this object need not be a rectangle it could have been for example, an ellipse or it could have been any other object as long as it is a convex object. So, for all convex objects first of all what are convex object well? For example, this is not a convex object, because it has some concavity, ok.

So, I am not talking about objects like this I am talking about only convex objects. This is a general result and it can be proved in a very general way, but we keep this we will not try to prove in this course that I am sorry this is not the probability this is h bar, the mean tangent diameter in the 2 dimensional convex object is nothing, but simply the perimeter of that object divided by pi, ok.

So, this makes things very simple or you can imagine what is the mean tangent diameter for a semicircular object, what would be the mean tangent diameter for objects there are sometimes you see this precipitate also. So, semicircular objects it would be take the perimeter of this so, if it is radius is r. So, perimeter would be pi r plus 2 r divided by pi that is the mean tangent diameter ok.

So, this was one example that I put it in 2 dimensions let us take one example in 3 dimensions situation and that also we will go through the derivation for that. So, for the 3 dimensional mean tangent diameter again a sphere of diameter D, what would be it is mean tangent diameter? It would be the diameter.

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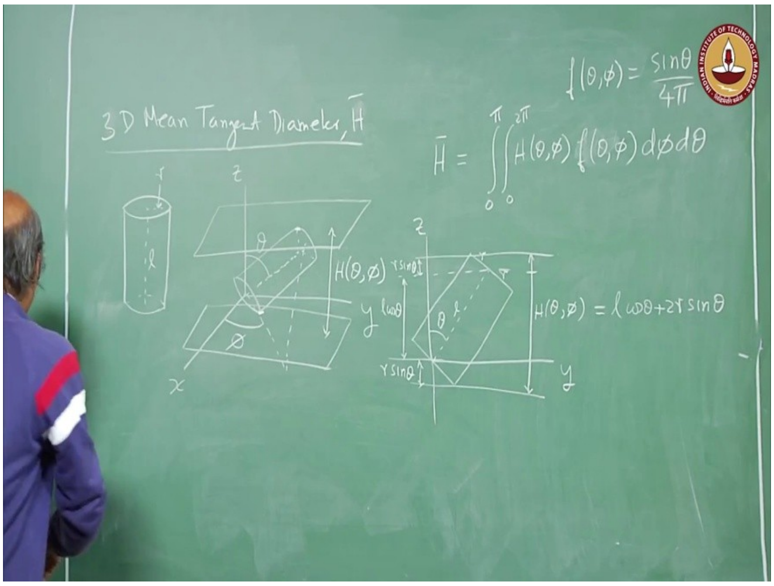

Now, consider a cylinder of length l and radius r; what would be its mean tangent diameter? So, now we let us first define an orientation for the cylinder. So, the orientation for the cylinder just like I put the rectangle I am going to put in a very similar way. So, take the axis of the cylinder, the axis of cylinder makes an angle of theta with the z axis and it makes an angle of phi with the x axis the projection of the axis. So, different value of theta and phi would give you different orientations of this cylinder.

Consider this is as one tangent plane and then there would be another tangent plane below the x y plane which will make tangent here, and make tangent here; that distance of separation between these two tangent planes is H theta, phi and in order to get the mean tangent diameter I have to do now a double integral over pi and theta H theta, phi times the density function in 3 D, d phi d theta and remember that the density function just derived as sin theta upon 4 pi.

So, again I will have to figure out what is H theta, phi in terms of the radius of the cylinder the length and theta and phi. Now, this problem is not as complicated as it looks. Let us try and visualize this as, how will first of all the tangent diameter will vary with theta. If theta is 0 degrees; what will be the tangent diameter then? The distance between the tangent planes if theta is 0, then the cylinder is upright.

Student: (Refer Time: 16:36).

It is simply length the length of the cylinder l. If theta is 90 degrees it is lying down, then the tangent diameter will be 2 r, in between it will vary. That is how it varies with theta. How does it vary with phi if I keep theta fixed? Let us say if I keep theta is 90 degrees and let phi vary it does not change. If I keep theta 0 it does not change with phi, if I keep theta is anywhere else it would not change because of symmetry of this object.

So, it will be independent of phi it is only going to vary with theta. So, we are just now reduces the scale of the problem. Now, we can further simplify it that instead of looking at this diagram in 3 dimensions let us reduce it down to 2 dimensions. Since it does not vary with phi I position the cylinder first that the axis is in the y z plane and then take a section. So, what will I get? I will get a rectangle.

So, I will get this is r, this is l, this angle is theta, and the tangent planes would appear as lines in this 2 dimensional drawing and the distance between these 2 lines is H theta, phi. Now, this actually reduces down to my problem of the rectangle to find out my tangent diameter, but only catch is that here my theta is defined with respect to the z axis there it was with respect to the horizontal axis. So, my cos and sin will get reversed, is not it? And, I will get H theta, phi to b this again I can divide it just like before. So, I need to find this distance, I need to find this distance and this distance will be the same as this distance.

So, this one would be l cos theta, this would be r sin theta, this would be r sin theta, right and hence H theta, phi would become l cos theta plus 2 r sin theta. We can confirm this we have said that when theta is 0 the tangent diameter should be l. If I put theta is 0 this term goes to 0 this goes to one h theta phi is l if I put theta is pi by 2 this term goes to 0. This sin theta goes to 1 the tangent diameter is 2 r; the diameter of the cylinder.

Now, all I have to do is substitute this in this equation and get the mean tangent diameter by integrating it twice. So, let us do this partly.

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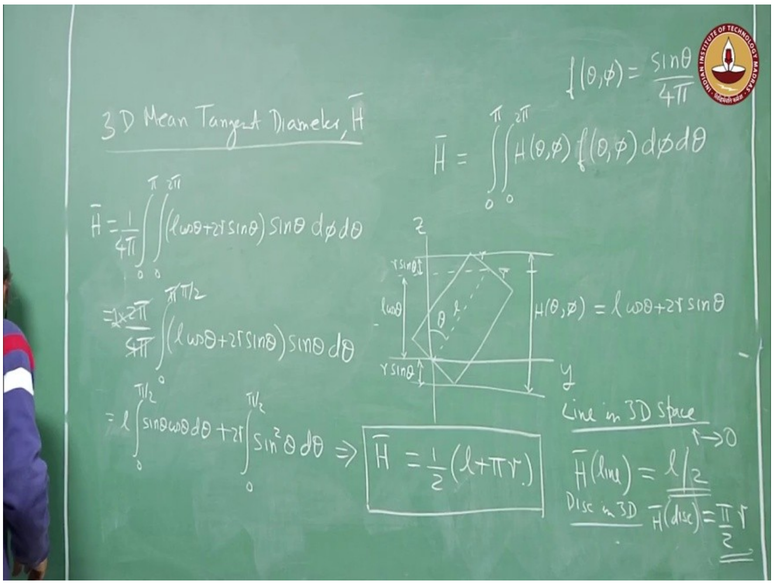

So, H bar; so substituted H theta, phi I substitute f theta phi with sin theta upon 4 pi sin theta will come here and 1 upon 4 pi I can take outside the two integrals being a constant d phi d theta. Let us first integrate with respect to phi if I integrate with respect to phi see these are all independent of phi. So, integral with respect to phi would simply be phi with the limits and then put the limit 0 to 2 pi. So, what will I get? Just 2 pi and 2 pi will come out of the integral with now pi gone you are left with theta.

Now, looking at the symmetry of this object I do not have to integrate all the way from 0 to pi. I can split this from 0 to pi by 2 pi by 2 to pi would be the same. So, I change this from pi to pi by 2 as a result I will have to add another 2 here. So, what happens, all of this cancel out so you are now left with 0 to pi by 2 and let me break up this integral l I take it out I have sin theta cos theta d theta plus another integral 0 to pi by 2, 2 r taken out I have sin square theta d theta.

Now, this is a simple kind of integration that I will leave it to you to perform, and the final results that you will get for the mean tangent diameter would be that H bar is equal to half of this term in brackets l plus pi r. From this I can get mean tangent diameter of couple of other objects.

Suppose, I let r go to 0, then what happens to the cylinder? It becomes a line. It gives me for a line in 3 D space the mean tangent diameter for a line of length l is simply l upon 2. Suppose, I let l go to 0 in this case I put r go to 0 if I let l go to 0 no yeah l go to 0 then what I am left with? I am left with a disc of radius r, ok. So, for disc in 3 D so it is a disc is a 2 D shape, but it can take orientations in 3 D the mean tangent diameter for a disc is simply pi by 2 times its radius.

In this way the mean tangent diameters of various other objects have also been derived. So, I hope I made this mean tangent diameter clear. We will be using some of these results now in order to derive some of the relations in stereology, which relate to surface area per unit volume, relating length per unit volume, number per unit area and so on. So, with this I close this lecture and we will continue in the next one.