Diffusion in Multicomponent Solids

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Lecture 18

Refresher on Laplace Transform

Welcome to the 18th lecture, in the open course on Diffusion in Multicomponent Solids. We will be solving diffusion equation for non-steady state diffusion, particularly for infinite boundary conditions. We will make use of Laplace transforms and hence, this lecture is a refresher on Laplace transforms.

One of the most important aspect of phenomenological expressions of diffusion is that we can use it to predict concentration profiles in different diffusion problems. With respect to different boundary conditions and given initial conditions, we can solve the diffusion equation to predict the concentration profile as a function of x and t. For that we need to solve the diffusion equation. Remember how we derived diffusion equation? We used continuity equation, substitute Fick's law expression into continuity equation, and we get the desired diffusion equation.

Now, in steady state condition the concentration at a given x does not change with time and we have seen the solutions for steady state condition as we solved the diffusion equation for steady state condition. Now, we will go over the solution for diffusion equation in non-steady state conditions, for different boundary conditions and initial conditions. For that, we will use Laplace transform as one of the very useful mathematical tool.

Now, you guys have already studied Laplace transforms in your mathematics class probably, but at that time the connection may not be very clear. The application of mathematics to the problems that we solve in our Material Science or Metallurgy may not be very clear so I would like to first give a refresher of Laplace transform for next couple of classes because we will be using a very important mathematical tool for solving diffusion equations.

Anybody remembers what is Laplace transform? What does it do? How do we define Laplace transform?

Student: It will convert one kind of function into another kind of, it will change the variable.

Professor: What kind of variable we typically use? Let us look at the definition of Laplace transform.

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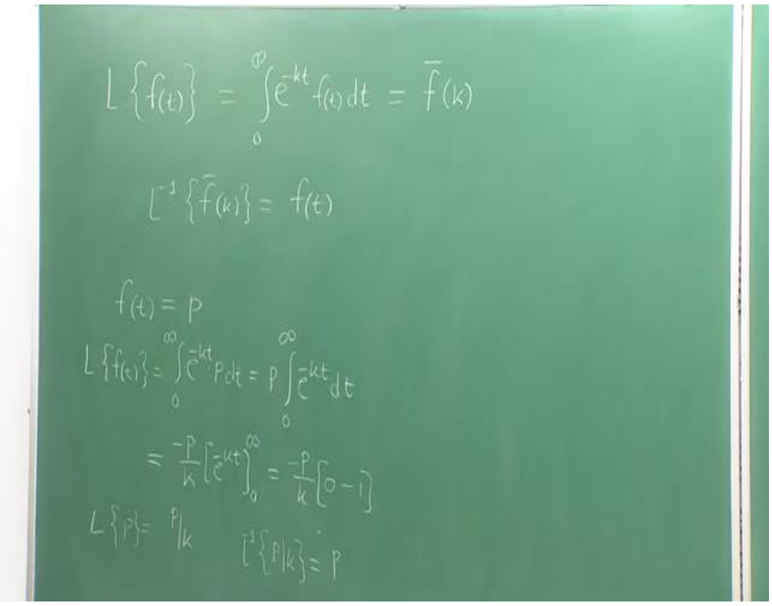

If we have a function $f(t)$, then we define the Laplace transform of $f(t)$ or we write:

$$L\{f(t)\} = \int_0^\infty e^{-kt} f(t) dt = \bar{f}(k)$$

This is a definite integral so obviously it is independent of time.

The Laplace transform is also denoted by the function $\bar{f}$:

$$L\{f(t)\} = \bar{f}(k)$$

and now, $\bar{f}$ is not a function of t, but function of k. If we take the inverse of $\bar{f}(k)$, inverse Laplace transform, we get back the original function $f(t)$. We write:

$$L^{-1}\{\bar{f}(k)\} = f(t)$$

Basically the function which was initially a function of time, by taking Laplace transform, we are making it independent of time, t. Let us see some simple examples.

Suppose:

$$f(t) = P$$

P is some constant, we can write:

$$L\{f(t)\} = \int_0^\infty e^{-kt} P dt$$

So, P is a constant you can take it out of integral:

$$L\{f(t)\} = P\int_0^\infty e^{-kt} dt = -\frac{P}{k}[e^{-kt}]_0^\infty = -\frac{P}{k}[0-1]$$

But:

$$L\{P\} = \frac{P}{k}$$

So, I can write:

$$L^{-1}\left\{\frac{P}{k}\right\} = P$$

We will write down some of the Laplace transforms here, on the right side.

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If $f(t)$ is P. Laplace transform of $f(t)$ is $\frac{P}{k}$.

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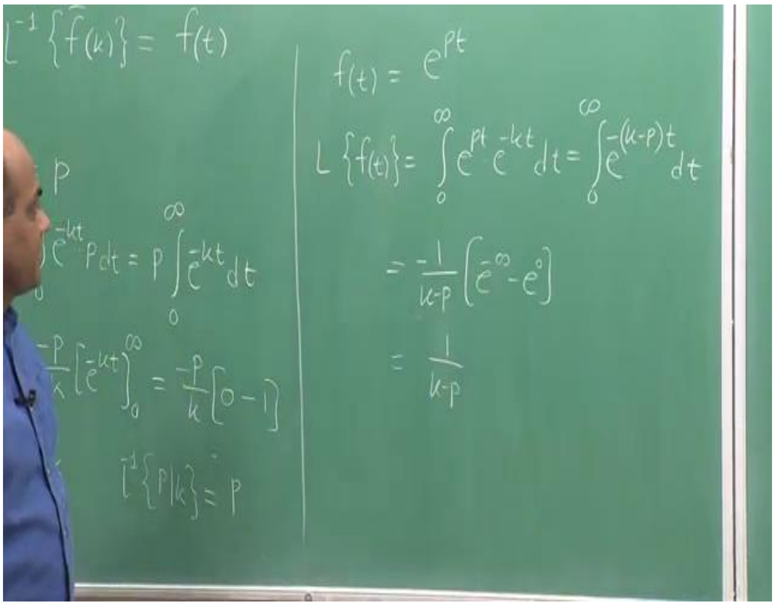

Consider, another example, if:

$$f(t) = e^{-Pt}$$

then Laplace transform of $f(t)$ would be:

$$L\{f(t)\} = \int_0^\infty e^{Pt}e^{-kt}dt = \int_0^\infty e^{-(k-p)t}dt = -\frac{1}{k-p}[e^{-\infty} - e^0] = \frac{1}{k-p}$$

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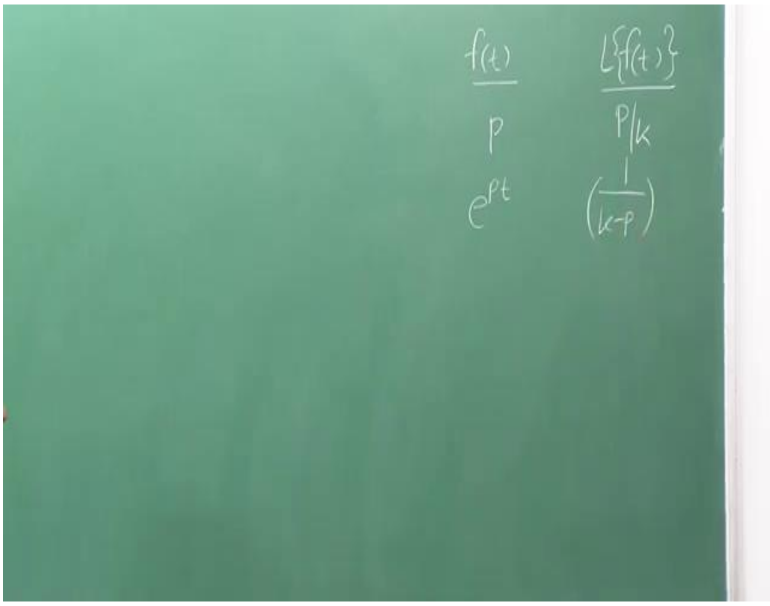

If our $f(t)$ is e^{Pt} its Laplace transform is $\frac{1}{k-p}$. Similarly, we can look at another example.

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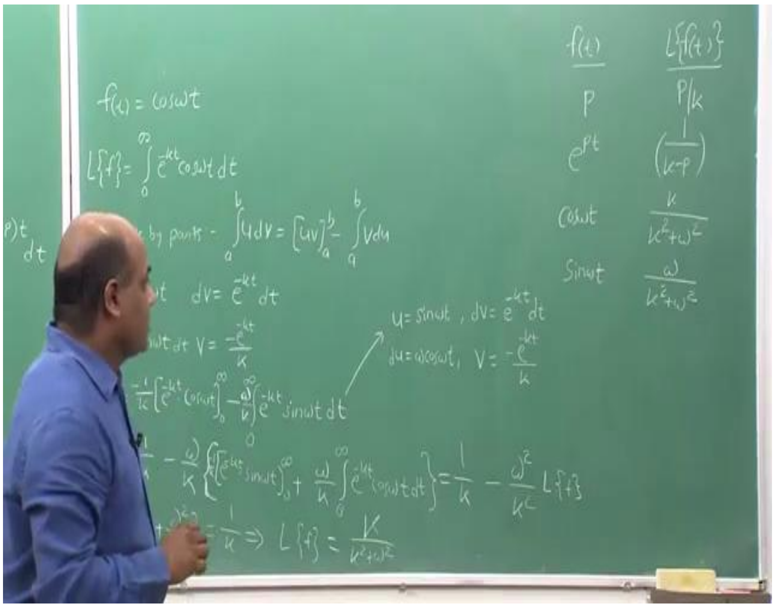

$$f(t) = cos\omega t$$

Let us try to get the Laplace transform of this:

$$L\{f(t)\} = \int_0^\infty e^{-kt} cos\omega t dt$$

Let us integrate by part, what is the formula for integration by part?

$$\int_a^b u dv = [uv]_a^b - \int_a^b v du$$

Here we can write:

$$u = cos\omega t$$

$$dv = e^{-kt} dt$$

$$v = -\frac{e^{-kt}}{k}$$

$$du = -\omega sin\omega t dt$$

$L\{f(t)\}$ becomes:

$$L\{f(t)\} = -\frac{1}{k}[e^{-kt}cos\omega t]_0^\infty - \frac{\omega}{k}\int_0^\infty e^{-kt}sin\omega t\, dt = \frac{1}{k} - \frac{\omega}{k}\int_0^\infty e^{-kt}sin\omega t\, dt$$

Second term on right hand side can be integrated by integration by parts considering:

$$u = sin\omega t \qquad dv = e^{-kt}dt$$

$$du = \omega cos\omega t \qquad v = -\frac{e^{-kt}}{k}$$

So we get:

$$L\{f(t)\} = \frac{1}{k} - \frac{\omega}{k}\left\{-\frac{1}{k}[e^{-kt}sin\omega t]_0^\infty + \frac{\omega}{k}\int_0^\infty e^{-kt}cos\omega t\, dt\right\} = \frac{1}{k} - \frac{\omega^2}{k^2}L\{f(t)\}$$

$$L\{f\} = \frac{k}{k^2 + \omega^2}$$

So, the Laplace transform of $cos\omega t$ we obtain as $\frac{k}{k^2+\omega^2}$. we can obtain Laplace transform of $sin\omega t$ similarly, and if you work on $sin\omega t$ we will get Laplace transform of $sin\omega t$ as $\frac{\omega}{k^2+\omega^2}$.

We can obtain Laplace transforms of some simple functions similarly, and you will see they are tabulated at various places. Now if we have to deal with complicated functions, then we need to look into more properties of Laplace transforms.

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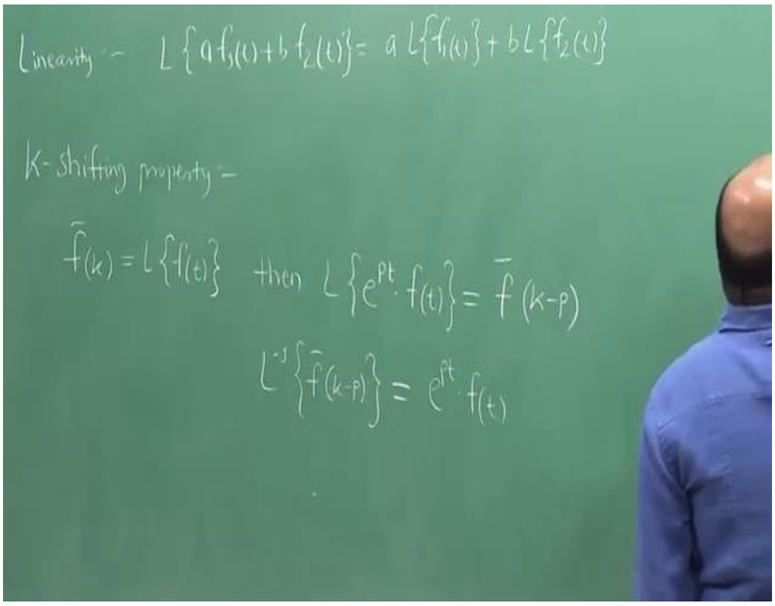

One of the property is the Linearity property which states that:

$$L\{af_1(t) + bf_2(t)\} = aL\{f_1(t)\} + bL\{f_2(t)\}$$

The second property which is called K- shifting property essentially says that if:

$$\bar{f}(k) = L\{f(t)\}$$

Then:

$$L\{e^{Pt}f(t)\} = \bar{f}(k - P)$$

It is shifting the k, that is why it is called k shifting. If I take inverse L inverse, I get:

$$L^{-1}\{\bar{f}(k - P)\} = e^{Pt}f(t)$$

We can prove this.

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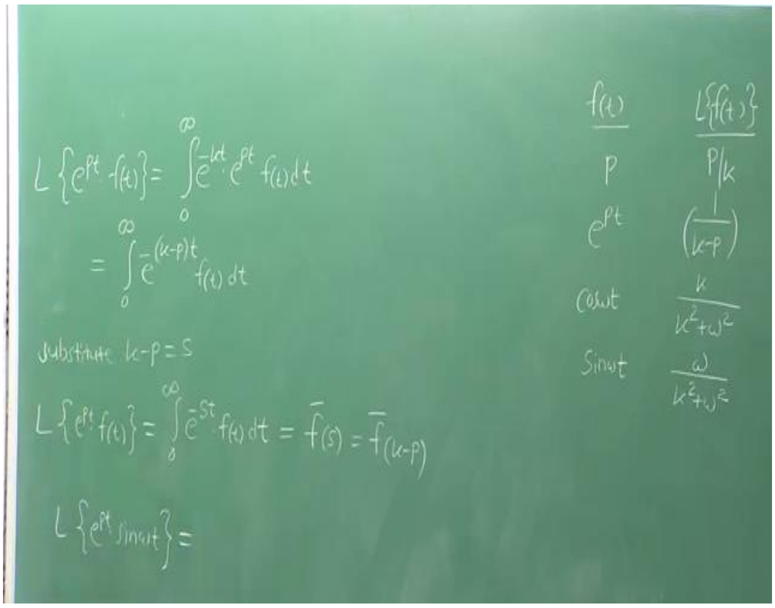

If you write the Laplace transform expression for $e^{Pt}f(t)$ this will be:

$$L\{e^{Pt}f(t)\} = \int_0^\infty e^{-kt}e^{Pt}f(t)dt = \int_0^\infty e^{-k+P}f(t)dt$$

and if we substitute:

$$k - P = s$$

we get:

$$L\{e^{Pt}f(t)\} = \int_0^\infty e^{-st}f(t)dt = \bar{f}(s) = \bar{f}(k-P)$$

$$L\{e^{Pt}sin\omega t\} = \frac{\omega}{(k-P)^2 + \omega^2}$$

For example, if you consider any function from here, let us say $sin\omega t$, then the Laplace transform of $e^{Pt}sin\omega t$ can be calculated by looking at the table. So you have to substitute k with $k - P$ which should be equal to $\frac{\omega}{(k-P)^2+\omega^2}$.

Similarly, what will be the Laplace transform of $e^{Pt}cos\omega t$? You have to substitute k with $k - P$. It will be $\frac{k-P}{(k-P)^2+\omega^2}$. Now, this property especially comes handy when we are trying to find inverse Laplace transforms of some complicated function. Let us see one example, let us try to find inverse Laplace transform of a function.

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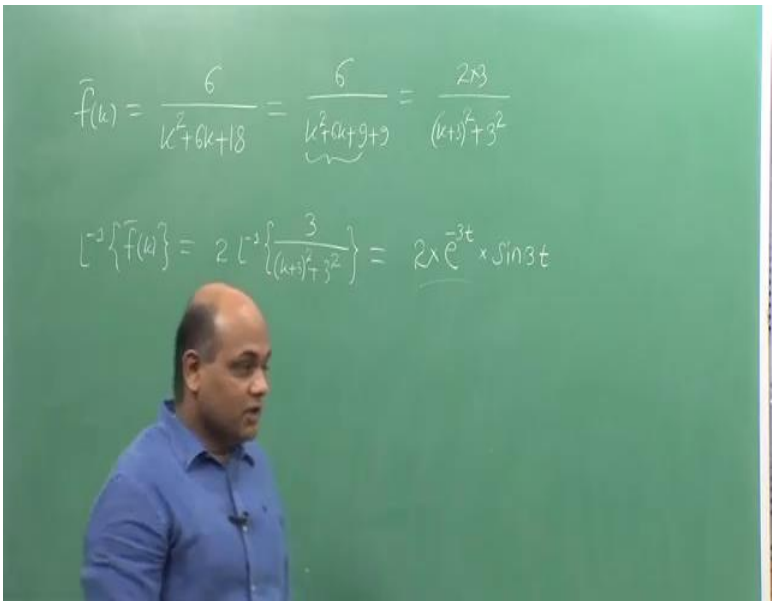

$$\bar{f}(k) = \frac{6}{k^2 + 6k + 18}$$

How do we go about this? This we can write as:

$$\bar{f}(k) = \frac{6}{k^2 + 6k + 18} = \frac{6}{k^2 + 6k + 9 + 9} = \frac{2 \times 3}{(k+3)^2 + 3^2}$$

This looks similar to the expression for Laplace transform of $sin\omega t$ with $\omega = 3$ and instead of k we have $k + 3$, then we use the k shifting theorem. This should be:

$$L^{-1}\{\bar{f}(k)\} = 2L^{-1}\left\{\frac{3}{(k+3)^2 + 3^2}\right\} = 2e^{-3t}sin3t$$

Now we will be solving differential equations with this. We are also interested in Laplace and inverse Laplace of derivatives. Let us look at the properties of Laplace transform of derivatives.

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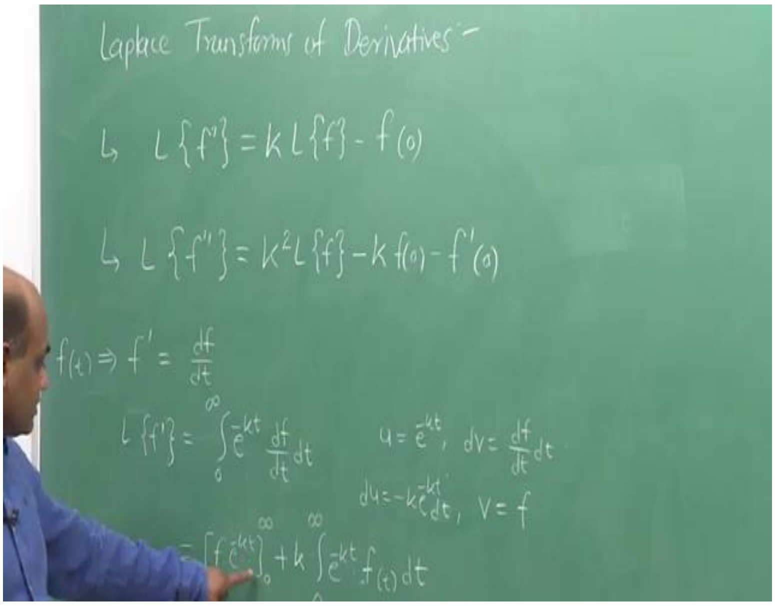

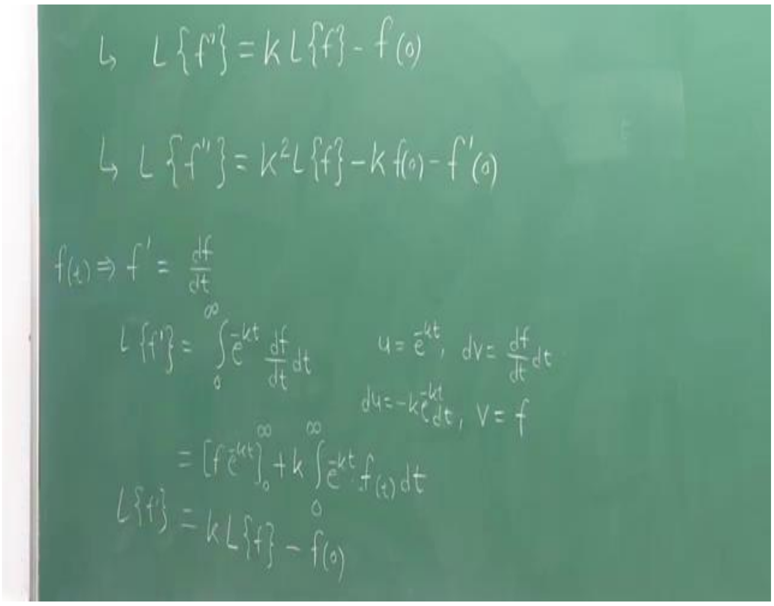

There are two formula, first:

$$L\{f'\} = kL\{f\} - f(0)$$

and the Laplace transform of second derivative of f that is:

$$L\{f''\} = k^2L\{f\} - kf(0) - f'(0)$$

Again let us try to prove this. Since f is a function of t, f' is nothing but:

$$f' = \frac{df}{dt}$$

Laplace transform of f' using integration by parts would be:

$$L\{f'\} = \int_0^\infty e^{-kt}\frac{df}{dt}dt = [fe^{-kt}]_0^\infty + k\int_0^\infty e^{-kt}f(t)dt$$

Here,

$$u = e^{-kt}$$

$$du = -ke^{-kt}$$

$$dv = \frac{df}{dt}dt$$

$$v = f$$

So, $L\{f'\}$ is nothing but:

$$L\{f'\} = kL\{f\} - f(0)$$

This proves our first formula.

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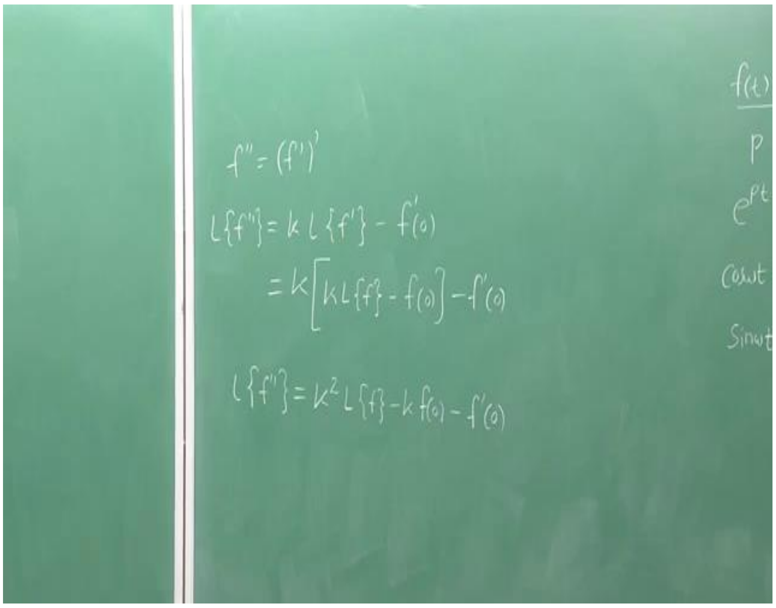

For the second one, if we have the first one it is simple because:

$$f'' = (f')'$$

We just substitute:

$$L\{f''\} = kL\{f'\} - f'(0) = [kL\{f\} - f(0)] - f'(0)$$

$$L\{f''\} = k^2L\{f\} - kf(0) - f'(0)$$

This proves our second formula. We can use this to find again Laplace transforms of more complicated functions, let us see an example.

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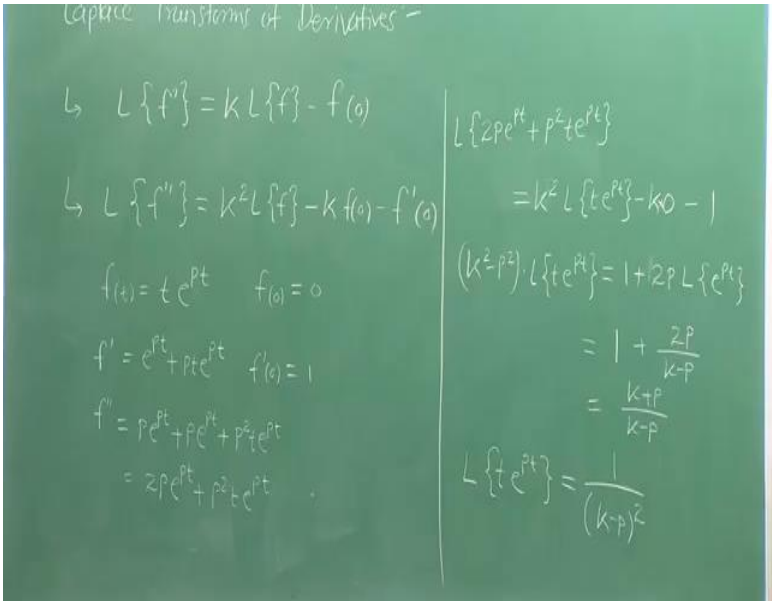

Suppose, we have to find the Laplace transform of te^{Pt}, so:

$$f(t) = te^{Pt}$$

Obviously:

$$f(0) = 0$$

And

$$f' = e^{Pt} + Pte^{Pt}$$

$$f'(0) = 1$$

and f'' will be:

$$f'' = Pe^{Pt} + Pe^{Pt} + P^2te^{Pt} = 2Pe^{Pt} + P^2te^{Pt}$$

And, if we use the second formula here, we will get:

Laplace transform of f'' should be equal to:

$$L\{2Pe^{Pt} + P^2te^{Pt}\} = k^2L\{te^{Pt}\} - k \times 0 - 1$$

On rearranging terms what we get is:

$$(k^2 - P^2)L\{te^{Pt}\} = 1 + 2PL\{e^{Pt}\} = 1 + \frac{2P}{kP} = \frac{k+P}{k-P}$$

We can get Laplace transform of te^{Pt} by this as:

$$L\{te^{Pt}\} = \frac{1}{(k-P)^2}$$

We can write another formula here.

$$L\{te^{Pt}\} = \frac{1}{(k-P)^2}$$

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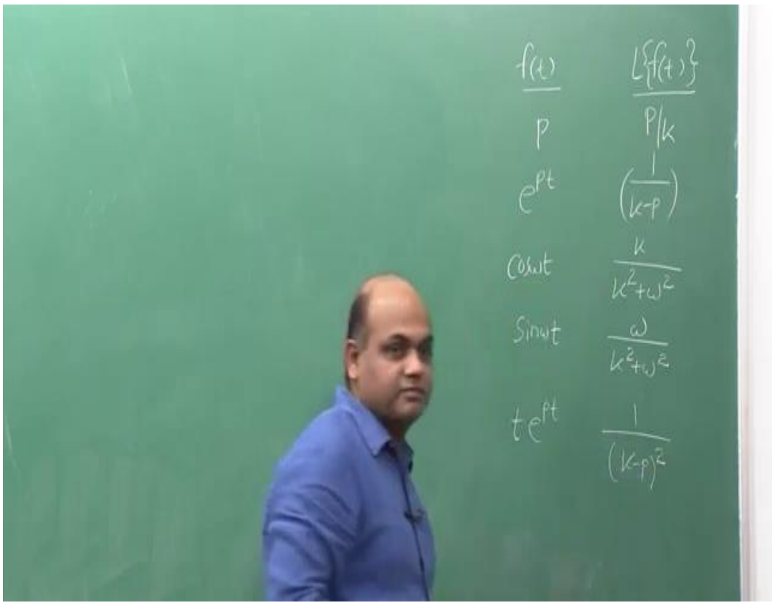

We will stop here today.