Crystals, Symmetry and Tensors
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Lecture 87
Coordinate Transformation for a Second Rank Tensor

We have seen how a coordinate transformation works for a vector. Now, we want to extend this coordinate transformation for a second rank tensor.

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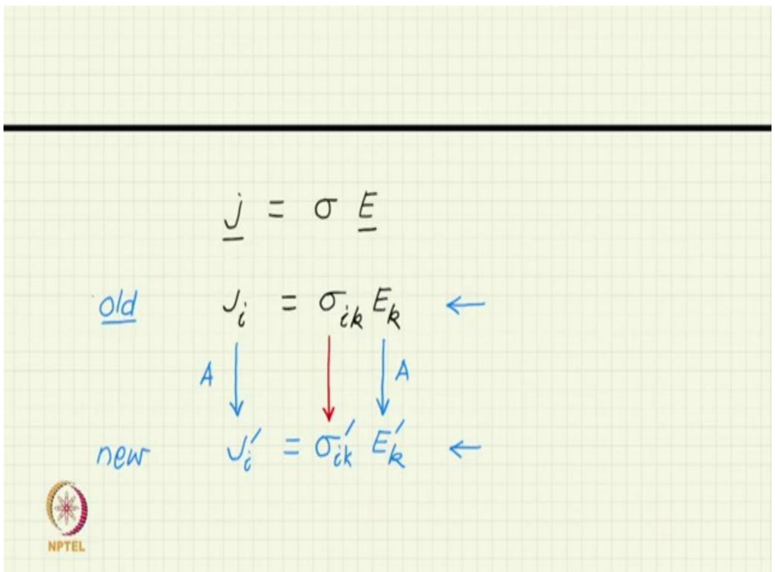

Because let us see, let us go back to one example of second rank tensor which we have introduced their subject is that, the electrical current density j is equal to conductivity tensor sigma times the electrical field E. Now, in terms of components if I write, so the ith component of j, which sigma ik Ek in Einstein's convention.

Now, notice that we have agreed that if we change the basis, so suppose this was in the old basis, some old basis; and we now introduced a new basis. In the new basis, the components of the same current density vector, same vector will now be different; and will be some other ji prime, some other ji prime.

Similarly, the electric field will also have different components Ek prime. Now, if sigma ik components, component sigma ik of the electric conductivity was working for old coordinate system; that same components cannot work for the new components; because, components of the current density and the components of the electric field have changed.

So, that means sigma ik will also have new components and will change to some other value, some different value represented here as sigma ik prime. So, as we coordinate transformation and we have established the coordinate transformation from ji to ji prime, and Ek to Ek prime through the coordinate transformation matrix A. The question is that, what is the transformation between sigma ik and sigma ik prime. How do we establish that? So, let us look at this problem.

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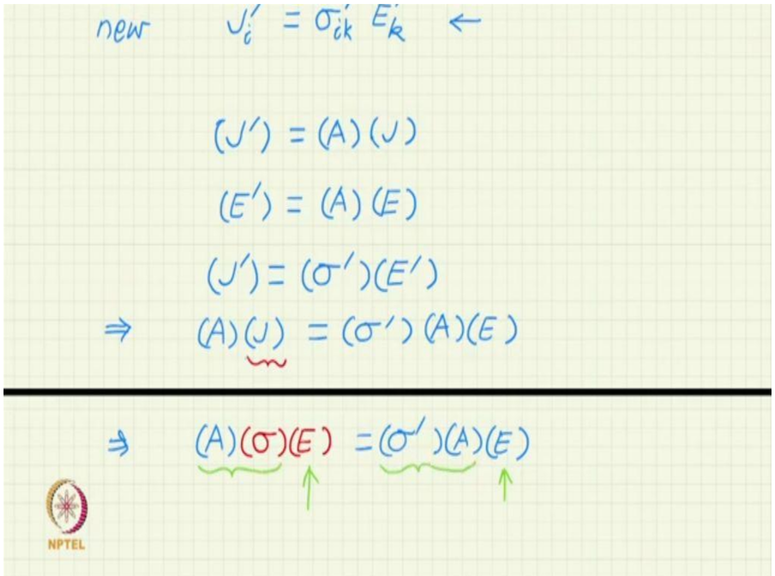

So, we have already seen that the new components ji prime can be written as. So, let us write it in the not in the component form, but in the matrix form. So, the new components, the column of new components is written as matrix A, the coordinate transformation matrix times the column of the j in old components.

So, this is our relationship from going from old j to new j; or new components of j with respect to the old components of j. You get it by multiplying with a matrix A; same thing works for E. So, the column of E prime is matrix A times column of E. Now, in the changed coordinate system, j prime will be the matrix of j prime, the column of j prime will be sigma prime times the electric field E prime.

If we use j is equal to sigma E relation, we get this in the old basis. Now, let us apply our coordinate transformation and write j prime as A times j; sigma prime is what we want to find out, and write E prime as A times E. But, in the old coordinate system we know that this j can also be written as sigma E.

Now, you can see that we have in the right-hand side, the same column E; and we can vary this electric field, so this is a variable column. And the product the left-hand side is equal to the product on the right-hand side, which means the product of these two matrices on the left and product of these two matrices on the right has to be equal.

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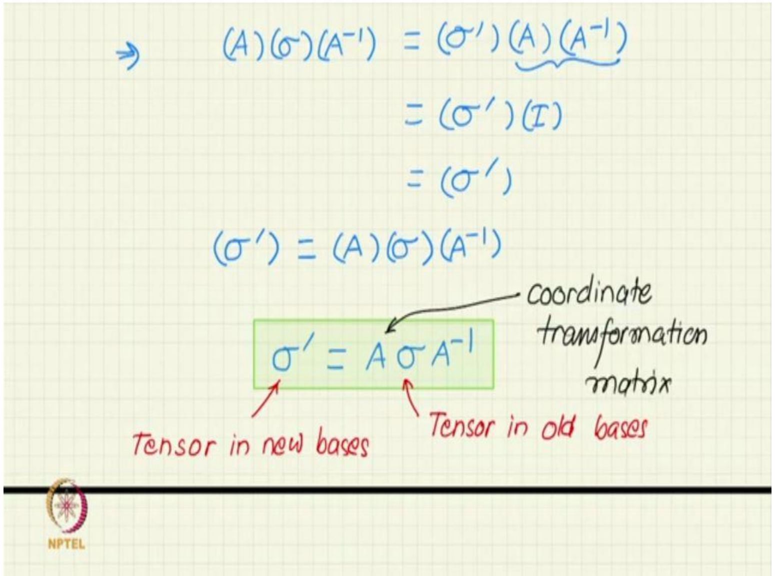

So, we equate that, so we can say A sigma is equal to sigma prime A; this gives us immediately the result because we want to find out the new sigma prime. So, I multiply on both sides by inverse of A. So, we are multiplying right multiplying the both sides by A inverse; here A and A inverse will give you an identity, which gives you sigma prime.

So, that means the representation of the tensor sigma in the new coordinate system, that is what is sigma prime is equal to A times representation of the tensor in the old coordinate system, times A inverse. For simplicity, we need not write the brackets; and we simply present this as sigma prime is equal to A sigma A inverse.

So, this is an interesting relation between a tensor in two different coordinate system; its components in two different coordinate system. So, sigma was the tensor in the old coordinate system; whereas, sigma prime new bases; and as you know A is the coordinate transformation matrix.

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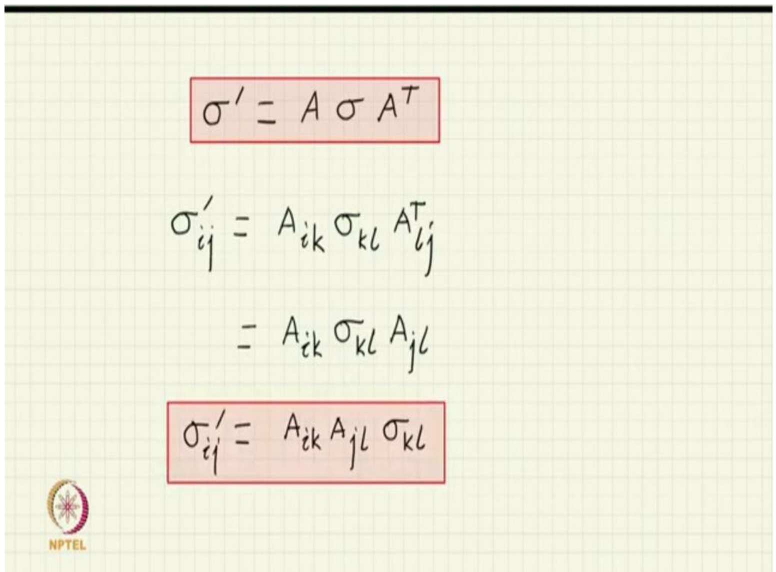

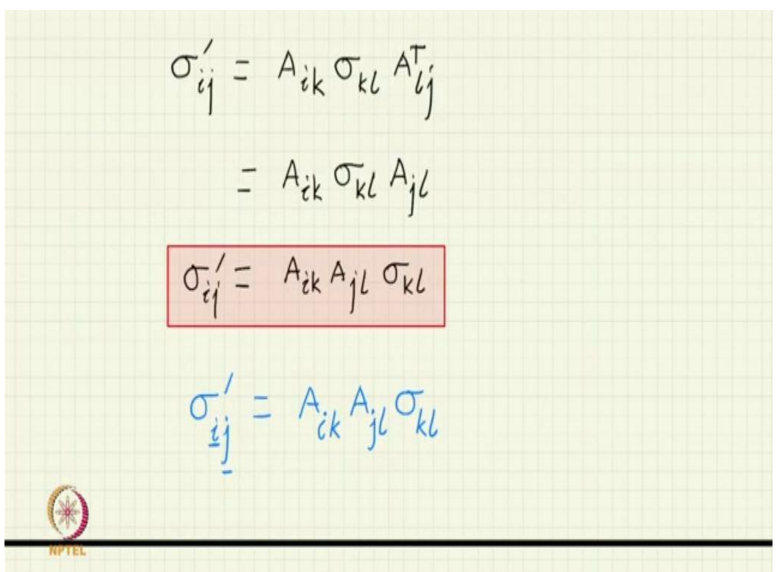

But now, use the property interesting property of. So, here the interesting property of A which we had established in one of the earlier videos that can be used that A inverse is nothing but A transpose. That makes calculations much simpler; you do not have to find the inverse of the matrix A; you simply have to find the transverse. And if we write this now in the coordinate terms, then we can say sigma ij is equal to Aik, sigma kL A transpose Lj. With the summation convention, I am not writing the sigmas. If we now use the property of transposition, so, A Lj transpose is nothing but Ajl.

And since, these are now not matrices, but components of matrices; so, they are scalars. So, they can be freely interchanged, they are commutative scalars are commutative. So, I write Aik Ajl sigma kL. So, the same relation which we wrote here in the matrix form, this was in the matrix form; this is now in the corresponding relation in the component form, using of course, the Einstein's summation convention. It is interesting to note here the order of the subscripts. So, let me write this again for you, emphasizing the order of the subscripts. So, let me first write them without any subscript; so, A, A and sigma. So, I want to transform the ij components.

So, the first subscript here, I write for the first A; and the second subscript here, I write as the first subscript of the second A. The second subscript, I introduced new subscript k and L for both A; and that K and L I use in the same sequence for my second sigma on the right-hand side. So, this is a way to remember and to avoid mistakes, while writing expressions like this. Now, let us see whether we can relate this relation, there is a graphical way of finding this relation.

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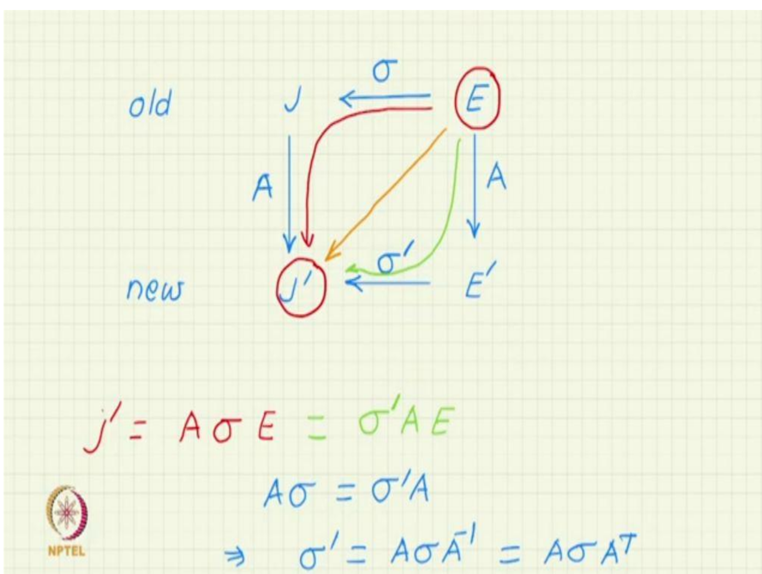

And that is let us say that we have an old basis and we have a new basis. In the old basis, we have the vector J; in the basis, we have the vector J prime. So, vector is not changing, the components are changing. So, in the old basis, the current density vector has components J; in the new basis the, it has the components J prime. J is also related to E, the electric field in the old basis; and J prime is related to electric field E prime in the old basis. The relationship between J and E, from E you go to J by multiplying by the matrix sigma, the tensor sigma; the electrical conductivity tensor converts E to J.

The transformed sigma in the new basis, sigma prime will do the same thing from E prime to J prime. J to J prime is the same vector, the coordinates are transforming. So, you are doing the coordinate transformation; and you know that for that you have to multiply J by the matrix A. Similarly, E to E prime is also just another vector, but the same coordinate transformation.

So, you also have the same matrix A. So, now let us start at this point E and come to J prime; I want to go from E to J prime. But, you can see that there is no direct route. So, either I take the red route, go from E to J and from j to j prime; or I take the green route. I go from E to E prime and then from E prime to J prime.

And since the starting and the end where do the start point and the destination is the same; the two routes should finally give me the same result. So, let us see how this is being done. So, if we take the red route, so, we first multiply E by the matrix sigma to get J; and then I multiply J by the matrix A to get J prime. So, this is what gives me the J prime.

But, if I take the green route, I first multiply E with A to get E prime; and then, I multiply sigma prime to get J prime. So, you can see that these two expressions are the same; we have got exactly the same result which we did previously. So, we have A sigma is equal to sigma prime A; or in other words, sigma prime is equal to A sigma A inverse, or A sigma A transpose. So, it is nice to keep this graphical picture also in mind to; and it avoids making mistake and gives you a good feel of the algebra which is happening. So, thank you very much.