Crystals, Symmetry and Tensors
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Lecture 86
Some Properties of Coordinate Transformation Matrix between Two Orthogonal Bases

The coordinate transformation matrix between two orthonormal bases have some interesting properties; and we will mention them in this video.

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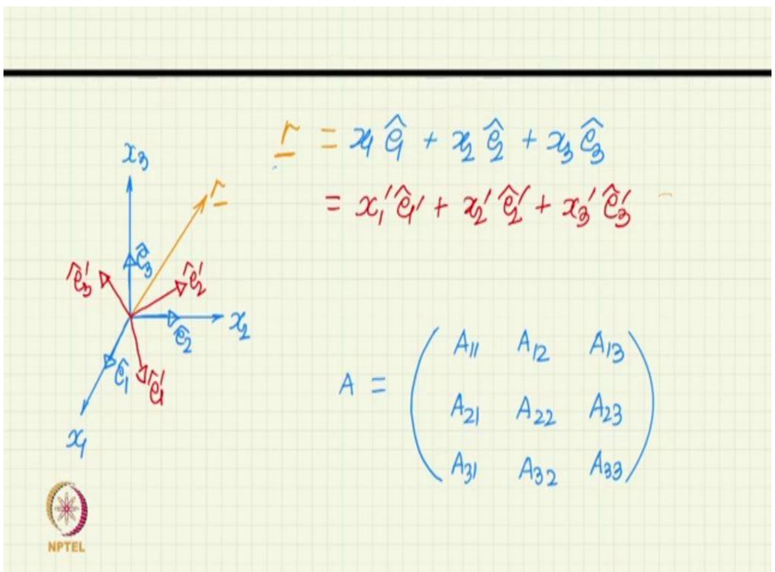

So, we have seen that a given vector can be expressed as in terms of its component with respect to a given basis. So, here vector r is expressed in terms of the orthonormal basis e1, e2 and e3. If we introduce another orthonormal basis, let us say represented in red with e1 prime, e2 prime, and e3 prime.

Then, the same vector will have different components x1 prime, x2 prime, and x3 prime; and we saw that how these two components are related. So, we have the coordinate transformation matrix A which is also; in the last video we saw that is also a direction cosine matrix. So, we just have this representation of the coordinate transformation matrix. Now, this matrix has a very interesting and unique property.

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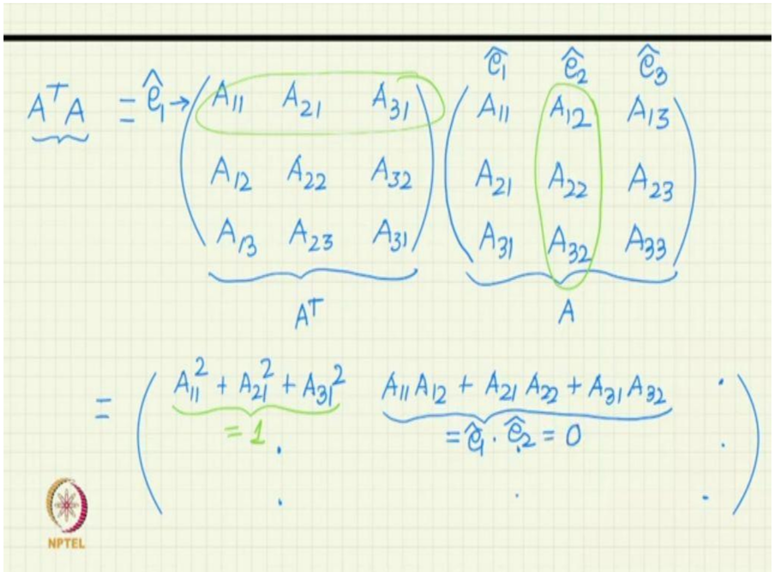

To look at that property, so let us look at the product A transpose A. So, this matrix is a transpose of A, and this one is the A. Now, if you know we had noticed that the components the first column, A11, A21 and A31 is nothing but components of the basis vector e1; the second column components the basis vector e2; and the third is the components of the basis vector e3 in the new coordinate system. So, let us look at this product. So, simply by using the multiplication rule for matrices, we get that the first term of the product will be A11 square, multiply this by this A11 square. Then, then we add this multiplied by this, so that is A21 square; and finally, 31 is square.

This is the first row first column of the product. First row, second column of the product will be A11 A12, plus A21 A22, plus A31 A32; and similarly, we can fill other products. Let us look at these products in a little bit more detail. What is this? What is this term A11 square, A21 square, A31 square? These are simply sum of the squares of these three components. But, these three components we said are the components of the bases vector e1 in an orthonormal basis, in the new orthonormal basis. So, sum of the squares of components of a vector is the square of the length of the vector; because our coordinate system is orthonormal.

But, the vector itself is an unit vector. So, this number is nothing but 1, because this is the length of this is 1. What about this second term, the second element of the first row? So, again if you see, this is this product of the first row here and first row here, and the second column here; the first row in the transpose just like in the original matrix, the column was the basis vector e1 because of the transposition. Now, the row is the basis vector e1; and the column is basis vector e2, and we have sum of component wise product. So, this is nothing but the dot product of these two vectors; so, this is nothing but e1 dot e2. But, we are in an orthonormal basis; so, e1 dot e2 is nothing but 0.

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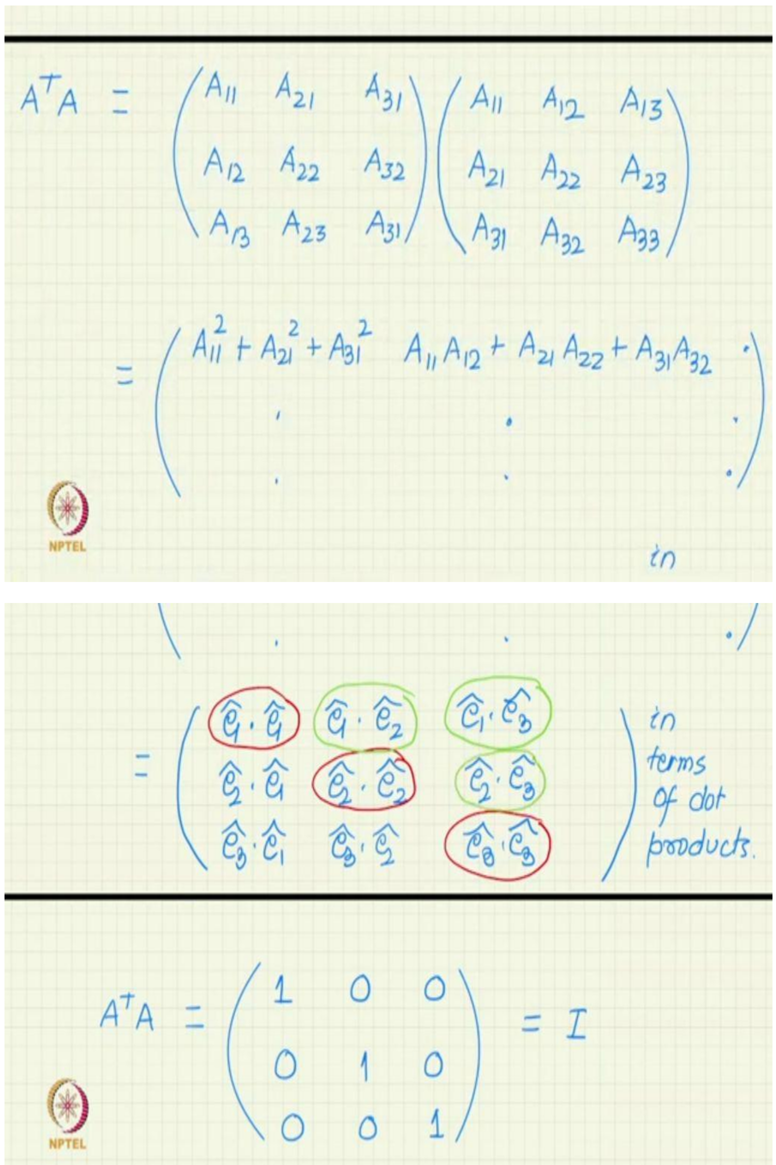

So, similarly, if you look at all these products all the products here; so, in terms of the dot product, if I write it in terms of the dot product is nothing but e1 dot e1, e1 dot e2, e1 dot e3. But, these dot products as we have already seen that the first one, first one here is 1; because it is the dot product of unit vector with itself. And the three diagonal terms all of them are the dot product of unit vector with itself. So, the diagonal terms are 1 and the off-diagonal terms are dot product of two different bases vectors of the orthonormal bases. And different basis vector of an orthonormal basis are always orthogonal; so, these dot products are all 0.

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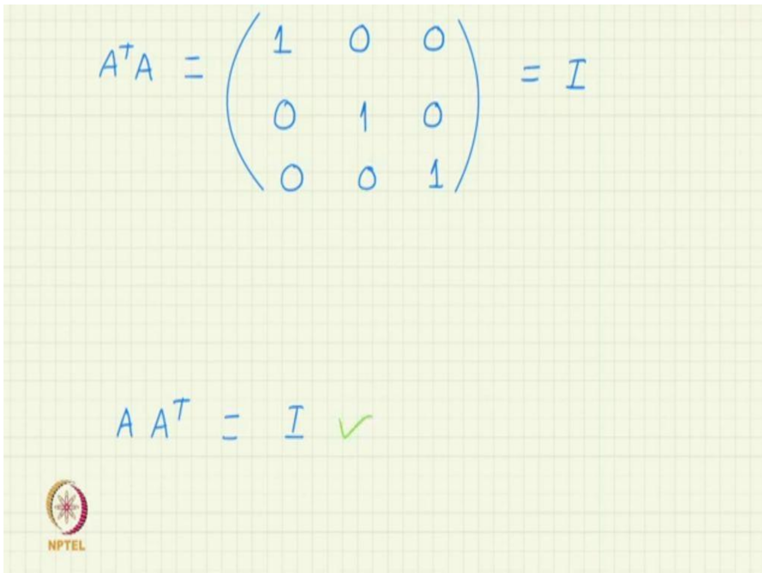

So, A transpose A is nothing but an identity matrix. Similarly, you can show the reverse is also true, if you were to take the product in opposite sense; AA transpose that is also an identity.

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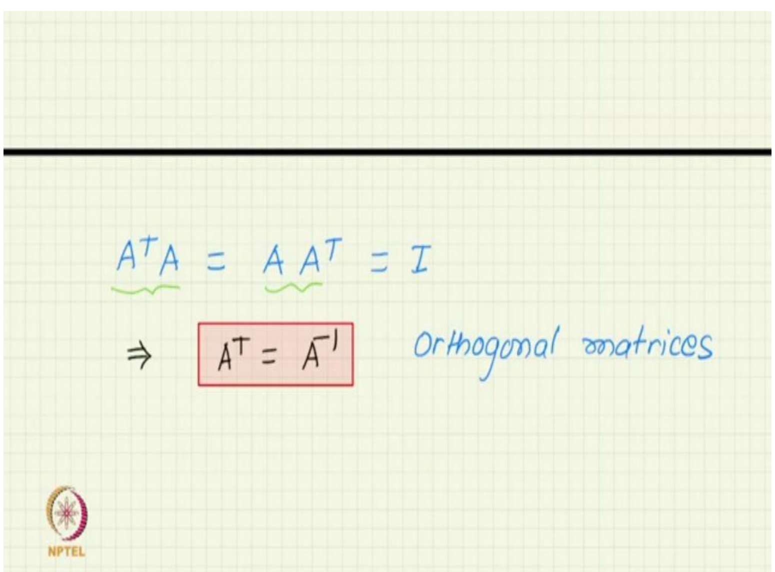

So, if both these products are identity, then you will know that this implies that A transpose is nothing but A inverse. So, this is an extremely interesting result for the orthonormal transformation matrices that if we want to find the inverse of the matrix, if we want to find the inverse of the matrix, it is nothing but the transpose of the matrix. This is sometimes called orthogonality condition; and such matrices are known as orthogonal matrices. This is a very interesting property. So, finding inverse of a transformation matrix in orthonormal coordinate system is extremely simple; all you have to do is to take the transpose.

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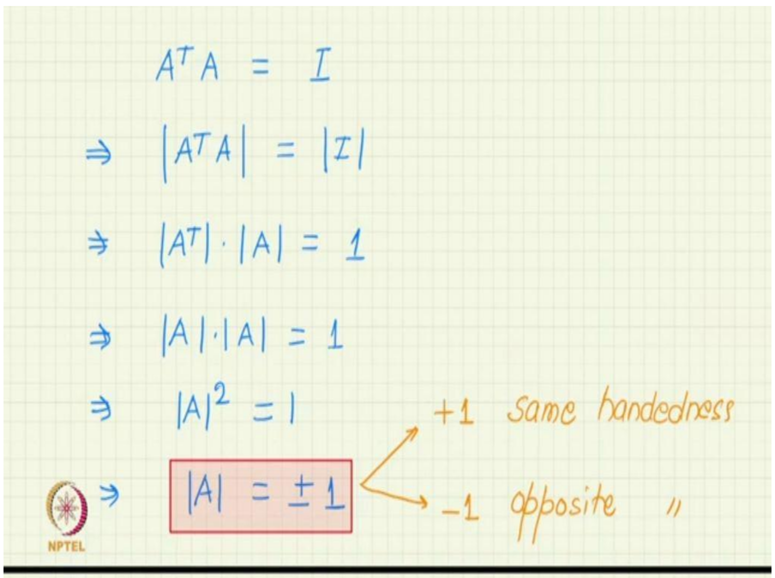

Now, this result also leads to an interesting result about the determinant of the transformation matrix. So, we have already seen that A transpose A is equal to I. So, taking the determinant of both sides, we can write determinant of A transpose A is determinant of I; determinant of product of two matrices is the product of determinants.

So, LHS I can write as product of two determinants and determinant of identity matrix is 1. And determinant of transpose is same as the determinant of the matrix. So, I can write this as A; determinant of A is equal to 1. So, the square of the determinant of A is 1. So, finally we can conclude that the determinant of A is either plus or minus 1.

So, the basis transformation matrix will always have determinant only either plus 1 or minus 1; and in fact, the sign has an important consequence here, that plus 1 means that the basis transformation is from between bases of same handedness. And minus 1 indicates that it is between bases of two opposite handedness.

So, these two interesting properties that the inverse of the transformation matrix is the transpose of the matrix; and the determinant of the transpose matrix is either plus 1, or minus 1. And since, we will always work; we will always try to keep our bases as right handed. So, we will never like to go from right handed to left handed basis. So, in our case, the determinant will always be plus 1. Thank you.