Crystals, Symmetry and Tensors
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Lecture 13b
Frieze group-I

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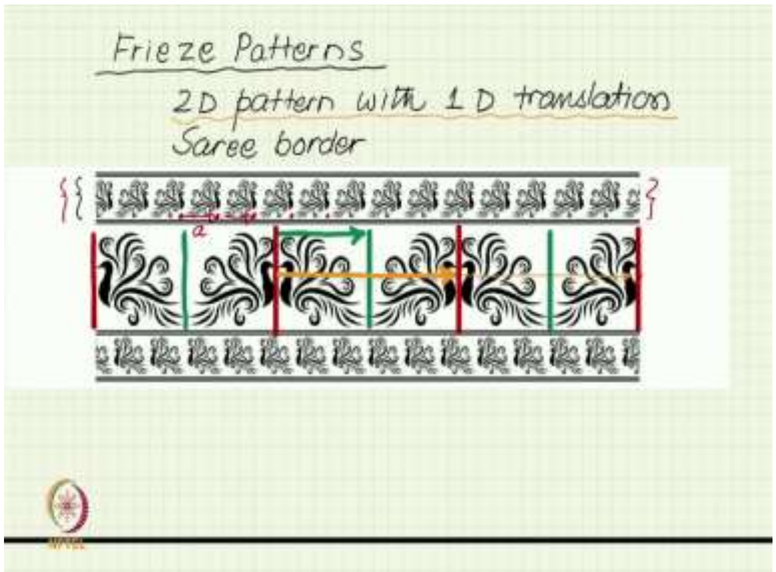

Professor Rajesh Prasad: So let us now come to the main topic of today's class and that is looking at, looking at the Frieze groups or Frieze symmetry. Frieze is just a technical name is actually an architectural phrase of in the buildings you, you will be seeing, that in the top of architectural building or beautiful temples. Also, there will be patterns repeating along 1 direction. So the Frieze pattern are 2-dimensional patterns. Pattern is 2-dimensional, but the translational symmetry is 1-dimensions. It is repeating in 1-dimensional. 2D pattern with 1D translation.

Architecture is one example. Say any kind of border, or you can call it a Saree. Saree border pattern is an example from textile. And these, this example I have collected from internet by searching saree border. So you will find many such beautiful example and you can see beautiful peacocks. And if you look at, the pattern is repeating. Of course there are different patterns, there are three patterns here. The top border, the bottom border, and in between there are this bigger pattern.

So let us focus only on the bigger pattern at the moment, not looking at that top and bottom, although they are also interesting in some sense. Or let us look at the top, let us look at this border only. Let us look at this border. So there are these little peacocks and if you select any point as your lattice point, then identical points close to each peacock will be your lattice point. So that is the translation direction. So the peacocks, these little peacocks are being repeated by these little translations.

Let us call them a. So it has a translational symmetry, but when you come to the central border, you find that it is having more than translational symmetry. Yes. One, so if I look at now again, let us say and that I take, let us take, I take this point as my lattice point. Then the next equivalent point is there and these are the only two equivalent points now. The third point will be exactly on the edge of my border. But in your mind's eye you can repeat this and see that the pattern is repeating.

So the whole peacock is a 2-dimensional object in this figure. Peacock is a, real peacock is a 3-dimensional object, but this is sketch, is a 2-dimensional. So, and these 2-dimensional peacock are being repeated in 1-dimension. So that is what we mean by 2D pattern with 1D translation. So it will be called a Frieze pattern. But this second pattern, the middle pattern has more symmetry it seems compared to the first one.

Why? Mirror. You have noticed correctly. So it has these nice mirror planes passing right in the middle of the two peacocks. This was not there in the top border. So you can see that all borders or all Frieze will have translational symmetry. But beyond translational symmetry they can have additional symmetries. Is this all the symmetry of this peacock pattern or you have something more?

There is a head on mirror but there is a tail end mirror also. I am drawing it with different color simply to emphasize that in a sense they are different mirror in the pattern if you see. The red mirror where the peacocks are meeting face to face. The green mirror, the peacocks are meeting tail to tail. So mirror to mirror separation if you see, is half the lattice translation. Is this an accident or is this some sort of a symmetry formula?

It will turn out that this is a symmetry formula. You cannot, in this figure you found that there was this accidental mirror in between. You tried to put peacocks face to face to have a mirror, but you cannot avoid that peacocks will have mirror at tail to tail also. And you can show this by the use of Seitz operator, which we were just discussing. So we will come to that.

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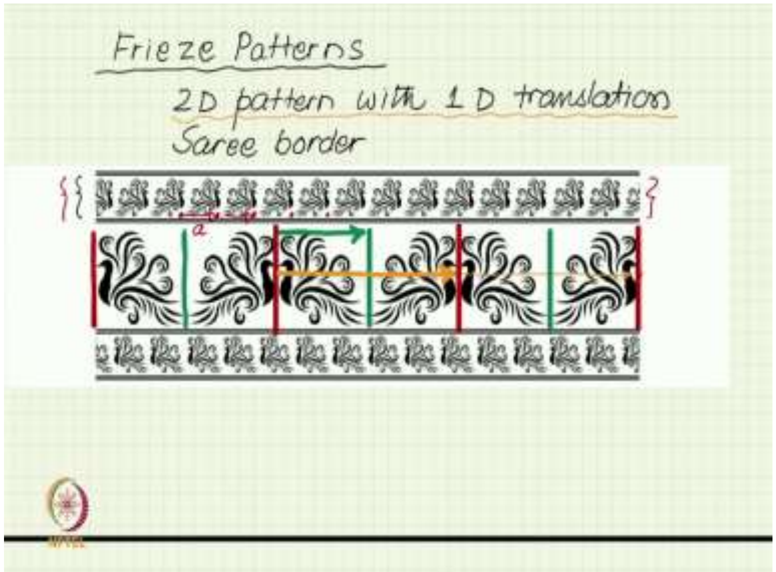

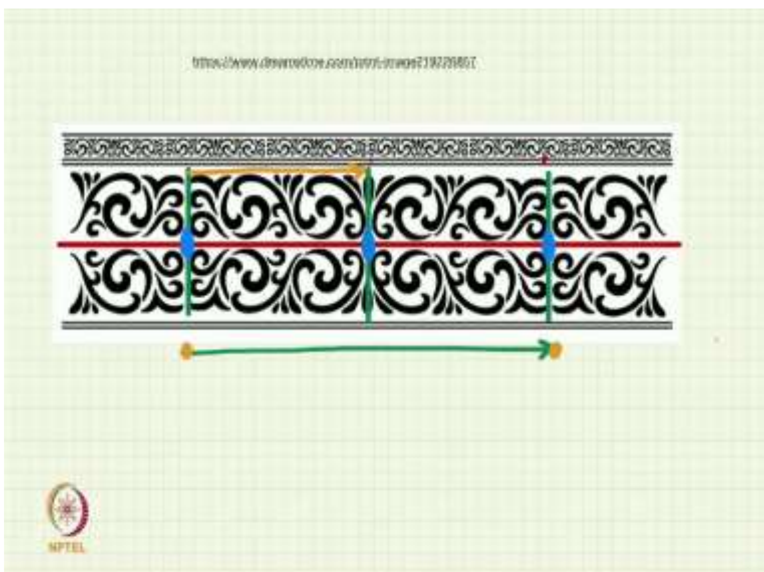

Professor Rajesh Prasad: Let us look at one more pattern and try to find out its symmetry. What are the symmetries here? There is a horizontal mirror, very good. Which was not there in the previous one, in the peacock figure there was only vertical mirrors. But now you have a horizontal mirror. Any other?

Students: Vertical mirror.

Professor Rajesh Prasad: Vertical mirrors are also there. And again, if you see the periodicity wise…

Student: Translation.

Professor Rajesh Prasad: Yeah. So the periodicity wise the translation is from here to here. If you look at, if you try to put the lattice points, if you put a lattice point here, the next lattice point will be there. From the pattern feature you can see things are repeating at that distance. In between you do not have the identical feature. But in terms of mirror you can see that, again, the mirror is translating halfway. There is one more symmetry operation which we are missing in this way.

Student: Intersection of vertical on horizontal way.

Professor Rajesh Prasad: Intersection of vertical and horizontal. So that is also a symmetry principle. Whenever two mirrors intersect at 90 degree, you cannot avoid to have a two-fold rotation. So that will also be there. So you can see that in some sense this border is richer in symmetry than the previous two borders. So we can rank it in symmetry, the top most border was only having translation, was having least symmetric.

The peacock pattern was having vertical mirror, a little bit more symmetric. And this one is having vertical, horizontal as well as two-fold rotations. So the question is can we classify, or can we classify all possible borders? How many different kinds?

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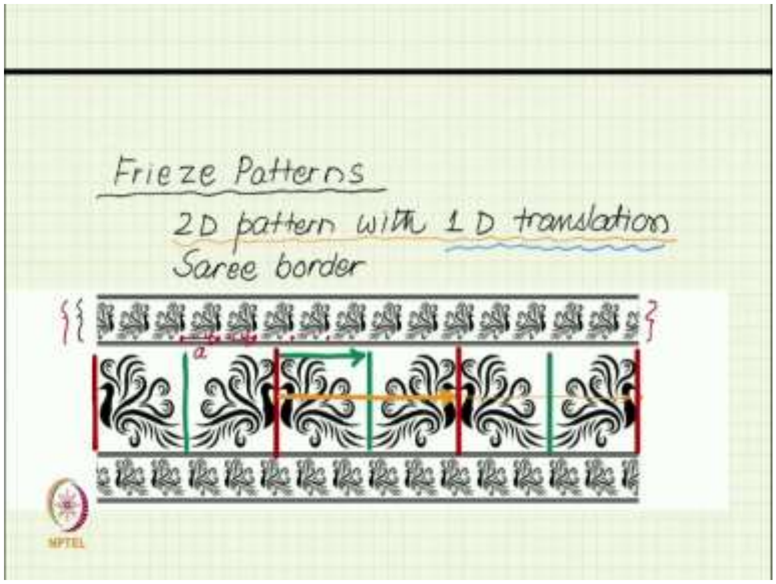

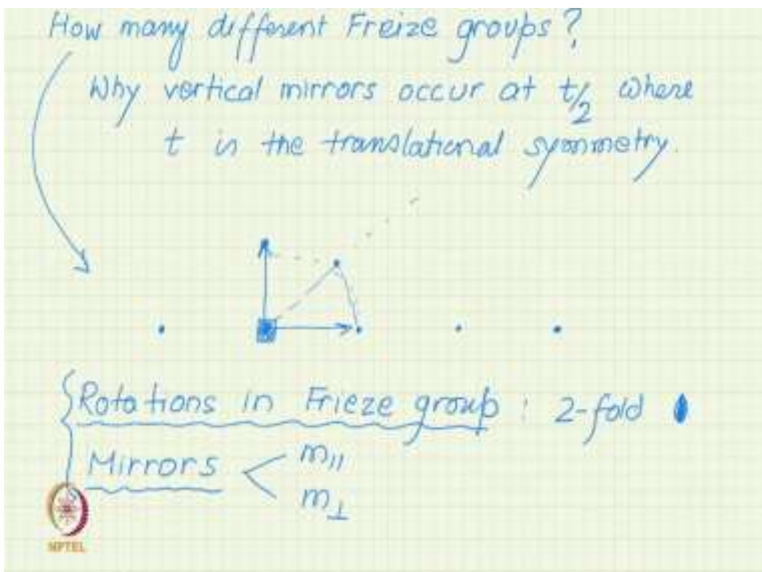

We have already seen three examples. Three, three kinds of border. How many different kinds of borders you can have? How many different Frieze groups? The other question is, how are the symmetry operation combining? Why vertical mirrors occur at t by 2, where t is the translational symmetry? So to answer the first question, first of all, let us see what are the symmetry operations we have with us. What are the ingredients which we can combine to get different patterns.

So since we are insisting that we have a border. And border means 1-dimensional translational symmetry, so we have a line. And symmetry operations should leave that line? Fixed. Yeah. It should map the line to line. So what kind of rotations are possible? Possible rotations?

Student: Two-fold and four-fold.

Professor Rajesh Prasad: If you apply a four-fold, so where will it rotate? So it will make this line a horizontal line, a vertical. So then, and since this was a translation and a four-fold demands that this also is a translation. So now automatically you are having a 2-dimensional periodicity. So your restriction of the Frieze group, that it should have 1-dimensional translational periodicity, is getting violated if you allow for four-fold.

So when we, we will discuss this kind of patterns also, that is the 2-dimensional patterns with 2-dimensional periodicity. That is the pattern of tiles or any 2-dimensional say, let us say graphene crystal. So 2-dimensional patterns can have four-fold, but 1-dimensional pattern cannot have four-fold.

Six fold will rotate it by 60 degree, you will have the same issue. So since I have to confine my translations into 1-dimension only, yeah? The only allowed rotation is 180. Either no rotation, one-fold or 180 degree which will keep line to line, and you will still remain in 1-dimensions. So only allowed rotation is? Two-fold. And you, you can allow mirrors, we have already seen mirrors, but what kind of mirrors?

Student: Vertical and Horizontal.

Professor Rajesh Prasad: Mirror either has to coincide with the line or mirror has to be perpendicular to the line. If it is at any angle to the line, then again it will tilt the line into some other direction and it will start having periodicity in that direction which we do not want. We are restricting ourselves to 1-dimensions. So either a parallel mirror or a perpendicular mirror, only two kinds of this. So only these three ingredients are there, which we want to combine with translation.

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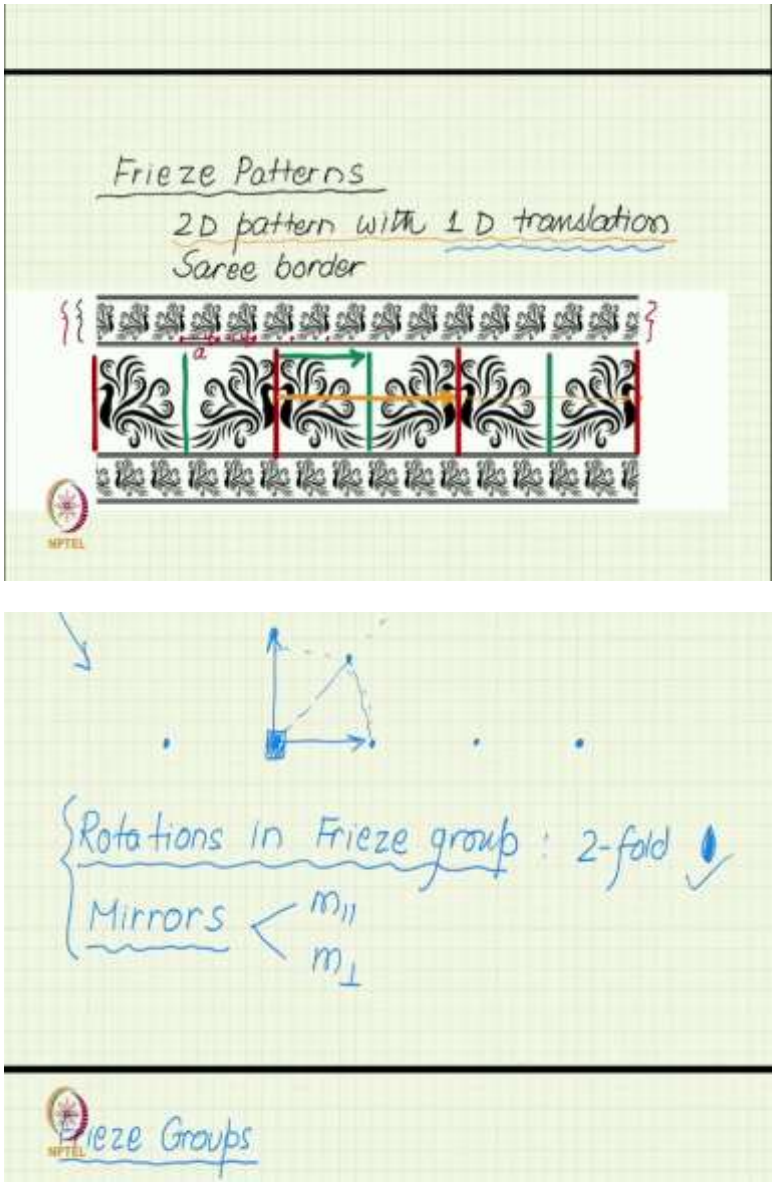

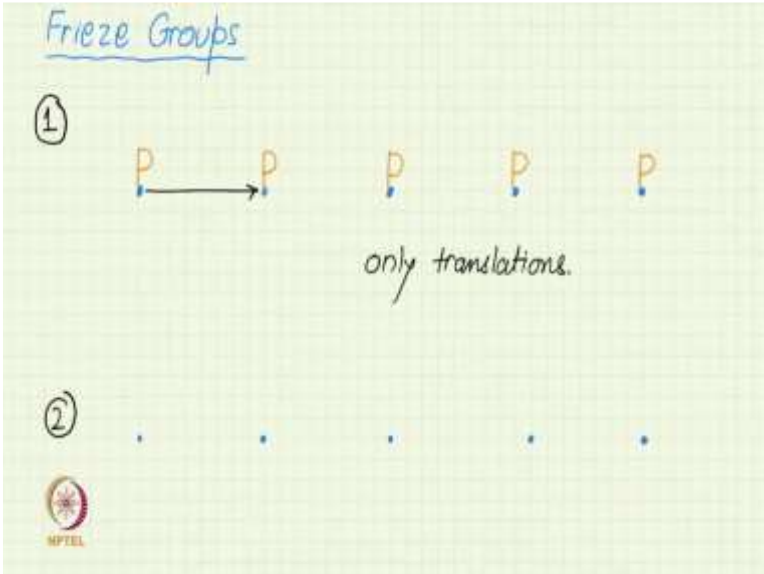

So let us see, let us do it one by one. So suppose this is a 1D lattice and of course you saw in the peacock pattern, in the most minimalist way, if you do not want any symmetry, you can go without symmetry. Without symmetry means only the translational symmetry. So you choose a motif which has no symmetry.

So let us recall our work with the, an English alphabet. So a letter without any symmetry, let us say the letter P. So if I place P at each lattice point. If I repeat P, I do not get any two-fold, any horizontal mirror, any vertical mirror, only translations. Let us call this our first group. Now suppose the other ingredient which we have is the rotation and the only rotation is a two-fold.

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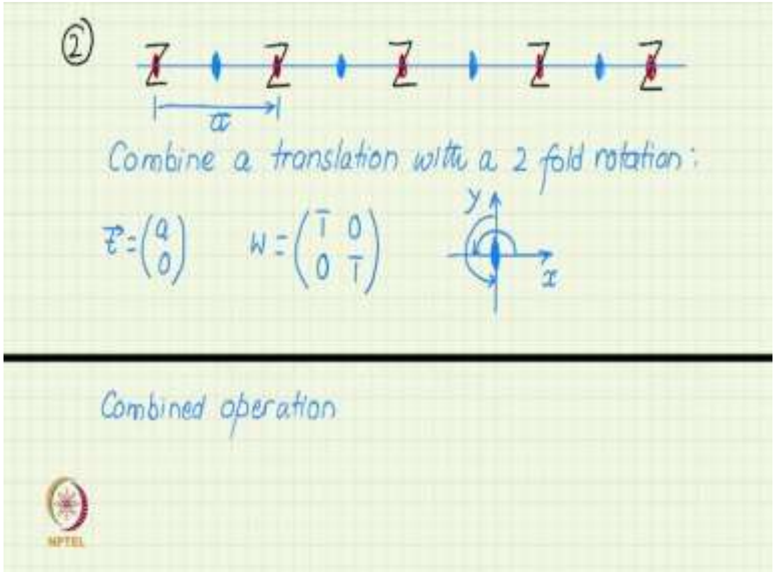

Professor Rajesh Prasad: So let us try to create a pattern with a two-fold symmetry. To create a pattern with two-fold symmetry, I will need a motif with two-fold symmetry. And which letter has the two-fold symmetry?

Student: Z.

Professor Rajesh Prasad: Z. So let us take this Z as our motif because that will automatically give me the two-fold symmetry and place at each lattice point. So translational symmetry was anyway there, but now interestingly, since the motif itself is a two-fold symmetry. Now when I am saying two-fold symmetry and I am placing, although I selected a motif of two-fold symmetry, but when I am putting two-fold now in the pattern, these two-folds are of the pattern not of the motif.

If only locally the motif is two-fold symmetry, but not the whole pattern, I will not count that. But you can see actually because the motif is having two-fold symmetry, they are inducing the two-fold symmetry of the entire line, entire pattern. Because if you draw this pattern on a tracing paper and flip the pattern 180 degree, so the whole line will go from one place to other place. This Z will rotate by 180 degree to this Z about this two-fold.

So when I am drawing this two-fold, I am meaning that not only of the local Z there but of the entire line. Can you see other symmetry emerging from this operation between two letters? Not easy to see, so you will have to experiment with drawing it on tracing paper and then you will find that if you rotate by 180 degree about the central points, they are also there at two-folds. You cannot avoid it. You cannot avoid it. And why cannot you avoid it?

So Seitz is telling you that you cannot avoid it. So let us look at, let us do Seitz analysis of this pattern. What we are saying that how to combine, let us combine a, because this pattern see, when we say these are Frieze groups, so all operations are actually forming group. So for any two different operation, the combination of the operation should also be present. So we are saying the translation t are present and the rotation two-fold is present.

Then the combination of translation and rotation should also be present in the group, for the symmetry group property to be satisfied. So what is the combination of a translation and a two-fold rotation, this is what we are asking. So the translation is t0 or let us call it a0. Because in lattice parameter wise we use the translation a, a and b. So a0 is the translation in the x-axis. And the rotation is a two-fold. Two-fold about the origin. Now how will you describe two-fold about the origin? So 180 degree will rotate x to minus x, minus 1, 0 and y to?

Student: Minus y.

Professor Rajesh Prasad: Minus y. 0 minus 1. So that is the rotation part. So the net part, the combined operation, combined operation in Seitz notation.

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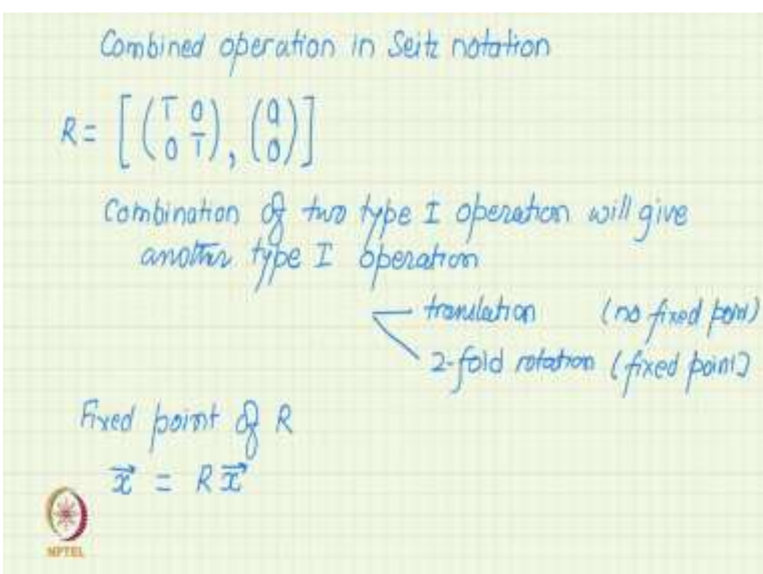

Professor Rajesh Prasad: Seitz notation is, the matrix part is minus 1, 0, 0 minus 1. The translation part is a0. So this is your, this is your symmetry operation, the combined operation. Let us call this some R. Now what is it? It is a combination of a rotation and a translation. Okay?

So both are of type one, both of them do not change handedness. So combining rotation and translation, you cannot get reflection. You can either get rotation or you get translation. So translation will not leave any point fixed, we have seen. Rotation will leave some point fixed. So our question that is this, so it cannot be reflection because it is of type I. We are combining two type I operation. And we have seen that we have only rotation vertical mirror, horizontal mirror and translation. Out of which horizontal and vertical mirrors are type II.

So the only possibility of for this is to, either be a translation or be a rotation or be a two-fold. Rotation also is only one possibility, so it is a two-fold rotation. But we know the property of the translation will not leave any fixed point. Two-fold rotation will leave a fixed point. So let us look at, so we can fix R, that what is R? A combination of two-fold rotation with a translation by distance a, by just trying to find its fixed point. So what are the fixed point of R? Fixed point of R means that x is equal to Rx.

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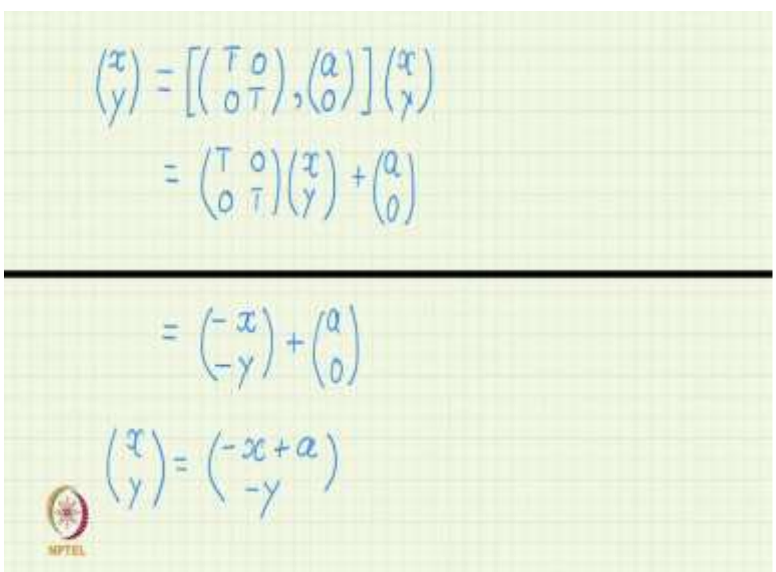

After applying R whatever point is not moved by R is the fixed point. So let us say that the components are x, y, R is minus 1, 0, 0, minus 1, comma a0 and we are applying it to x, y. We know how to multiply the Seitz operator, we multiply by the matrix, we multiply the column vector x, y with the matrix and add a0. This gives you, minus x, minus y plus a0. This then gives you minus x plus a, and minus y. So x, y, if x, y is a fixed point of R, they should be satisfied.

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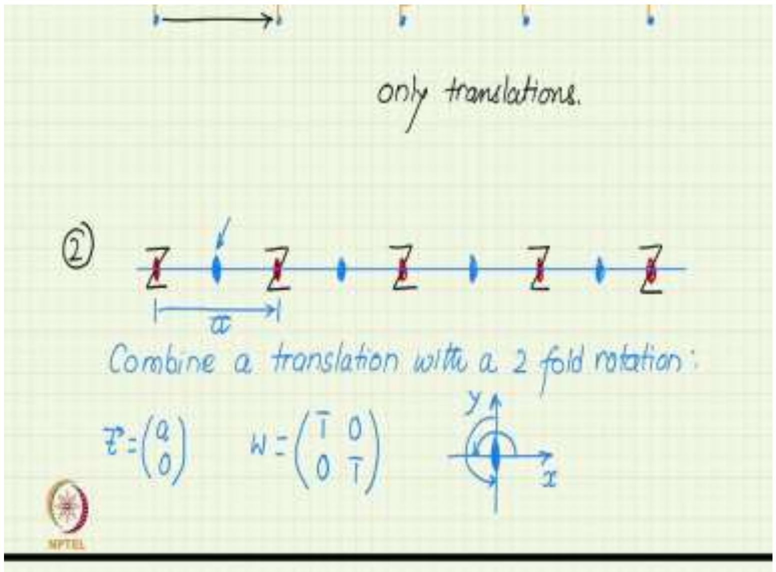
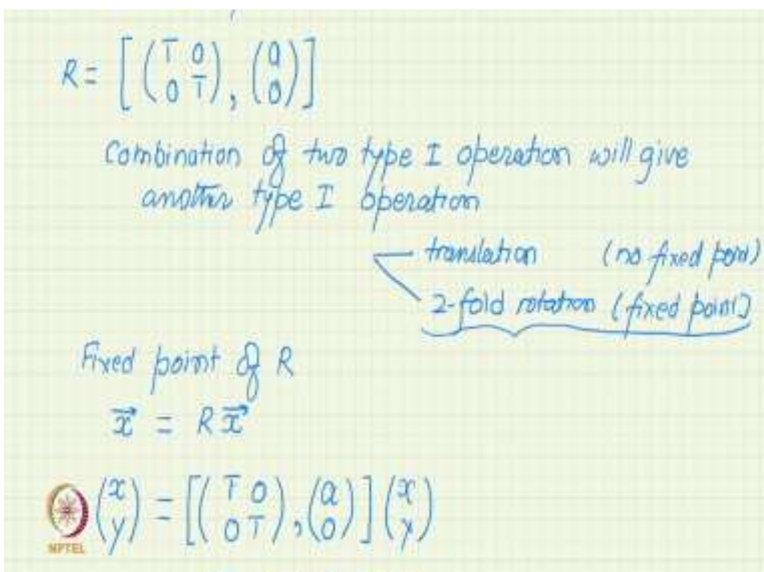
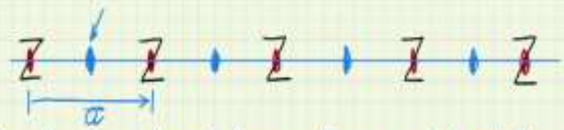




only translations.

② 

Combine a translation with a 2 fold rotation:

$\vec{t} = \begin{pmatrix} a \\ 0 \end{pmatrix}$ $\quad W = \begin{pmatrix} \bar{1} & 0 \\ 0 & \bar{1} \end{pmatrix}$

— translation (no fixed point)

2-fold rotation (fixed point)

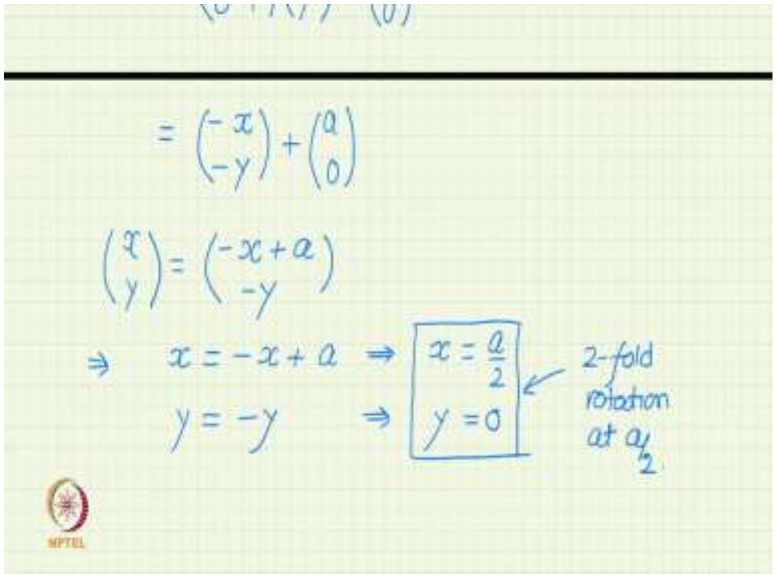
Fixed point of R

$$\vec{x} = R\vec{x}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left[\begin{pmatrix} \bar{1} & 0 \\ 0 & \bar{1} \end{pmatrix}, \begin{pmatrix} a \\ 0 \end{pmatrix}\right] \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} \bar{1} & 0 \\ 0 & \bar{1} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -x \\ -y \end{pmatrix} + \begin{pmatrix} a \\ 0 \end{pmatrix}$$



Now two vectors are equal when their components are equal. So x is equal to minus x plus a, which gives you x is equal to a by 2. And y is equal to minus y, which gives you y is equal to 0. So this is the fixed point. This is the fixed point of R. A by 2 0. That means although I am combining a translation of a, look we started with this red two-fold, which was the two-fold rotation and we combined it with the translation a, which was taking me to the next thread two-fold.

But the combination of the operation is actually generating a fixed point at a by 2. And what is that? That is a midpoint. And we have already decided that it is a two-fold rotation if it is a fixed point. Because it was type I, it was either translation or a two-fold rotation. Translation could not have left any point fixed, so we should not have got any solution. If it was a pure translation, we would not have got any solution. But we are getting a solution that a by 2 0 point is fixed. So, this is a two-fold rotation at a by 2. Okay?

So that is the reason why, although we decided that we will give two-fold motif, we will take a two-fold motif, which was having these red two-folds and translated it by a. But at the midpoints at a by 2, these blue two-folds got generated. So if this analysis would have given me some inconsistency, okay?

Sometimes it can give you an inconsistent equation which you cannot solve, okay? Means you will get x is equal to a, and x is equal to 2a, let us say. We will get some example like that. So then we will say that there is no fixed point and if there is no fixed point then it would have been translation.

Student: So, we will not get fixed point.

Professor Rajesh Prasad: Yeah, we will not get a fixed point. So here we assume now, but we were not guaranteed, we assume that x y is a fixed point for this R, and at the moment, R was, let us say a black box to me, I was not knowing that R is a two-fold rotation or if it could have turned out to be a reflection. But reflection we were ruling out because we know that we have written R by combining a rotation and a translation, so that cannot give me a reflection, things like that. So this was our second group.

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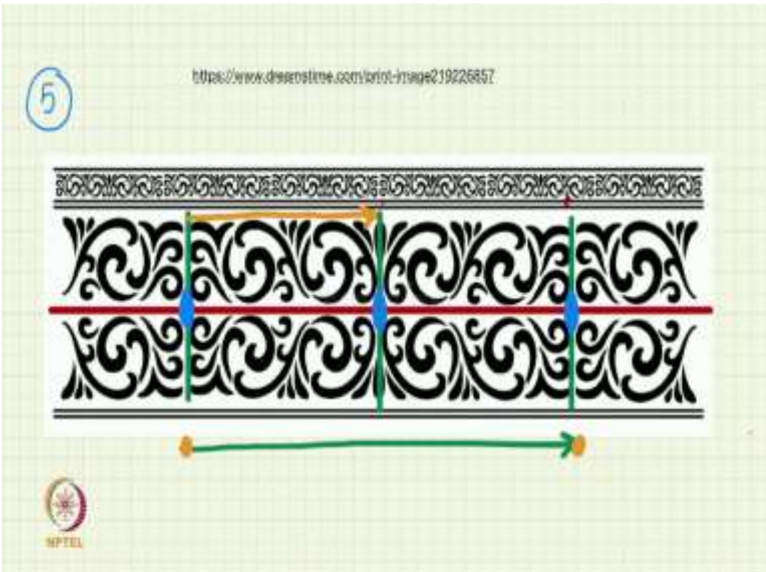

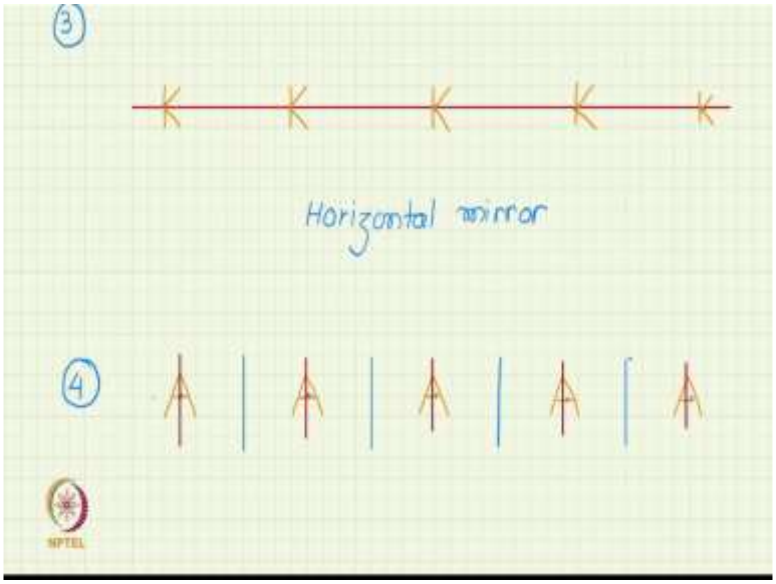

Now let us construct the third Frieze group. So what we did, our ingredients were two-fold rotation, horizontal mirror, vertical mirror and translation. So translation anyway is there and we begin with the translation. And now we have combined the two-fold translation with two-fold rotation. No symmetry the first group, two-fold rotation, the second group, so now let us play with mirrors. I have the mirror. So let us apply the horizontal mirror. So I say that in this pattern I will have a horizontal mirror. So I should select a motif which has horizontal mirror.

Student: B?

Professor Rajesh Prasad: B.

Student: A.

Professor Rajesh Prasad: A. A has a vertical mirror. A has a vertical mirror. So B is fine, C is fine, D, E, K. K, I like. Let us do it with K. Let me place K. This does not seem to create any more two-fold or any vertical mirrors or something, it remains a horizontal. Horizontal mirror. Now we have seen the pattern that what to do. What will you do in four?

Student: Vertical mirror.

Professor Rajesh Prasad: Apply vertical mirror. What motif you will select?

Student: A.

Professor Rajesh Prasad: A. The first letter. You have a vertical mirror. You have a vertical mirror, is it all you have generated? This is again the same thing, now I leave this as an exercise. You can do again a Seitz operator analysis of this. Combine a mirror with a translation, you will generate mirror at a by 2.

No way I can place a nice mirror symmetric A at the lattice point, and say that the midpoints will not have mirror. I do not have that freedom. So they will get mirror on their own. Buy one get one free. This is not a mirror, so let me not confuse by drawing that line. What can be the other one?

Student: Vertical and horizontal.

Professor Rajesh Prasad: Both vertical and horizontal, which we saw already in the peacock. No, not the peacock, the next pattern we had seen both horizontal and vertical. So this becomes our fifth Frieze group. This is all we can have if we limit ourselves to two-fold vertical mirror and horizontal mirror.