Crystals, Symmetry and Tensors
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Relation between Cube and Tetrahedron

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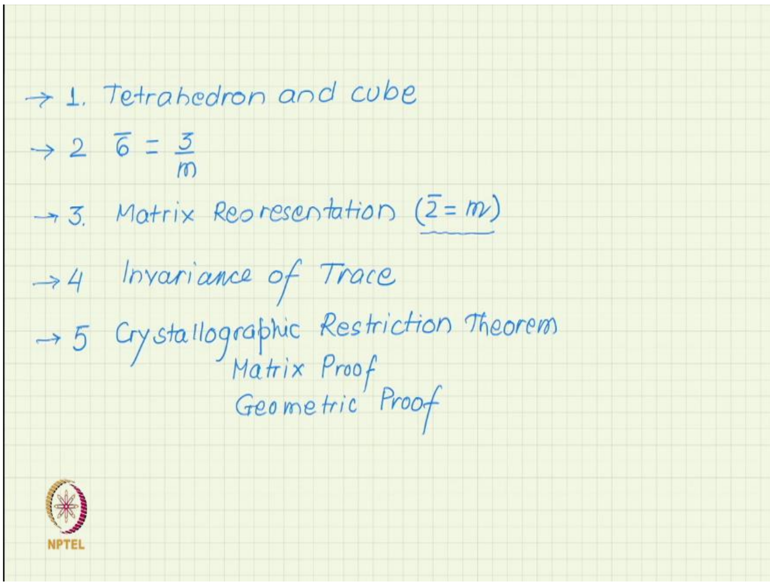

So, let us continue our discussion on symmetry. We will start with some a nice symmetry relation between tetrahedron and cube and then continue our discussion on roto-inversion axis that is what we were doing last time, matrix representation of symmetry operations. So, and last class we gave a geometric proof of the fact that 2 bar is equal to m that is a 2-fold roto-inversion axis is nothing but a mirror plane perpendicular to the axis passing through the inverse in the centre.

So, we will give a proof of that through matrix method. Then there is one homework, I had given you was invariance of trace. So, we will look at that problem and use that result to prove a very important theorem in crystallography which is the backbone you can say of entire crystallography is the crystallographic restriction or accuracy you can call it crystallographic restriction theorem. So, we will give you both the matrix proof and the geometric proof.

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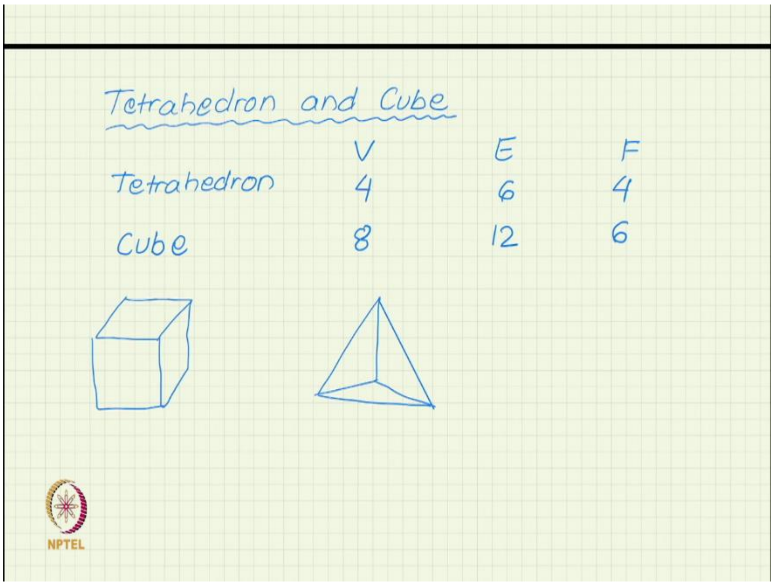

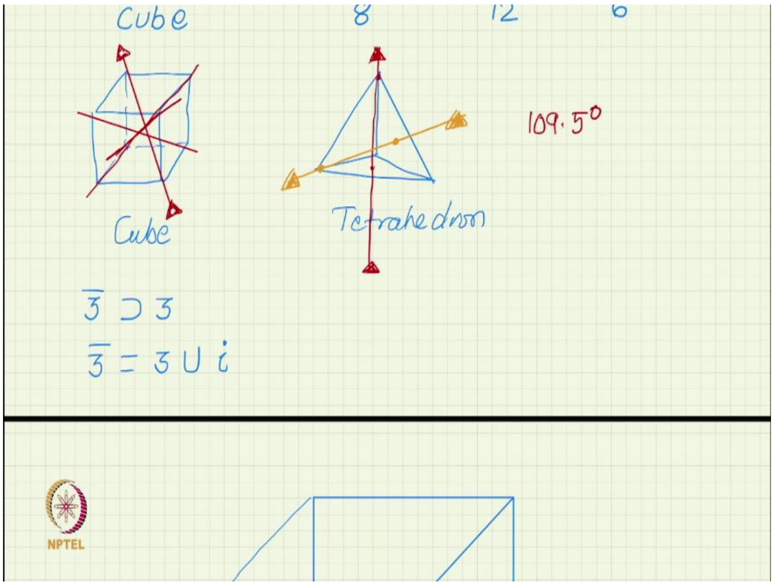

Tetrahedron and cube are related. So, tetrahedron has 4 vertices or 4 corners have 6 edges and 4 faces, the cube had 8 vertices, has 12 wedges and 6 faces. But, the two means, if you draw them independently appear to be related, but they have a very interesting relationship. So, what is the relation anyone, how is tetrahedron related to cube?

Student: (())(2:40) from 1 is to the centre point of the other surface and there will be also (())(2:48).

Professor: This axis you are saying from a vertex to centroid of the opposite face. So, what axis is that, 3-fold axis, very good and in the cube, in the cube where is a 3-fold axis. So, anybody diagonal of the cube is a 3-fold axis. But we saw last time, that in case of cube the 3-fold axis a little bit more than 3-fold. What was that?

3-fold roto inverse, but then 3-fold roto-inversion includes 3-fold, that 3 bar axis includes or is a super group of 3 axis. Or sometimes we say that the 3 bar is union of 3 and the inversion centre and that is why the symbol also includes a hole in the 3-fold triangle. Whereas tetrahedron has just a 3-fold. So, the 3-fold. So, it also has four 3-fold. That is interesting and cube also has four 3-fold because four faces are there.

So, each face centroid can be connected to the opposite vertex. So, you get four 3-fold one is shown here. So, for example, if you try to draw another one you can take the centre and this face centre and you get another 3-fold. If you do this with all the 4 vertices you will look four 3-fold in the cube case it is diagonal body diagonal and you will know that there are 4 body diagonals.

So, this is a is giving some indication of the relation between them that apparently they share the symmetry and the angle. Interestingly, if you find the angle between the 3-folds of tetrahedron and the 3-folds of the cube, they are exactly the same the angle is a well-known tetrahedral angle 109 point 5. So, in the cube also it is 109 point 5 in the tetrahedron also, it is 109 point 5. The angle between the body diagonals or angle between 3-folds.

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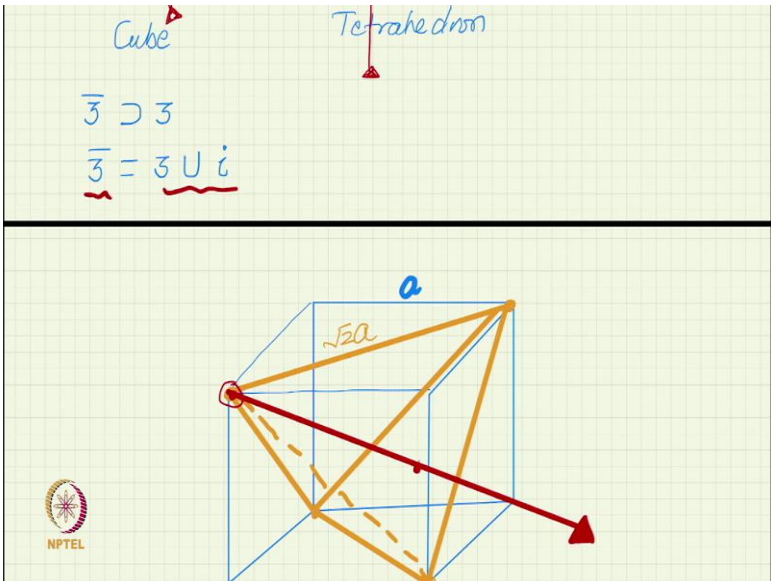

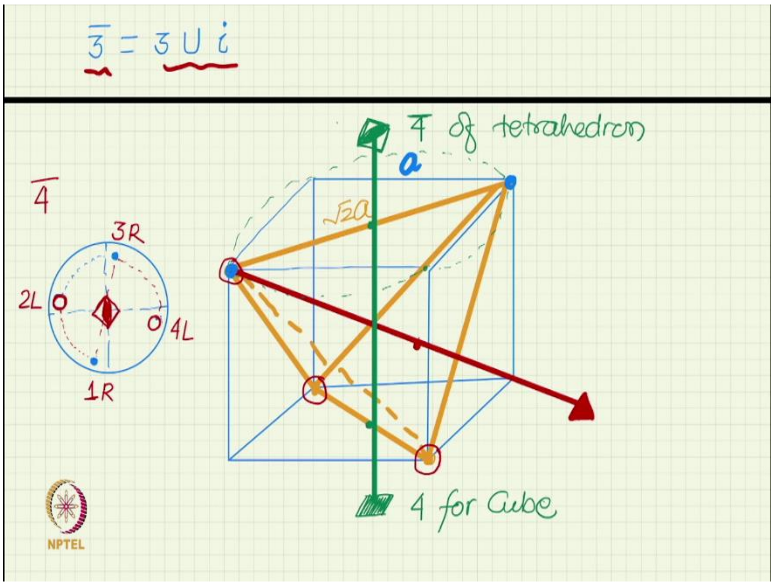

So, one way to explore that relationship is that cube has 8 corners, tetrahedron has 4 corners, if you select 4 corners properly, if you select 4 corners of the cube, you get a tetrahedron. So, let me show that to you. Suppose I start selecting I start from this corner, I select a diagonally opposite corner on this on a given face. So, on the top face, I take two corners which are diagonally opposite. I go to the bottom face and I select the opposite diagonal and ends of the opposite diagonal faces.

Now, I have 4 corners and these 4 corners define an exact tetrahedron, a regular tetrahedron. So, if you connect them. This is a tetrahedron you can see clearly that all edges are equal because all our face diagonal of the original cube. So, if the original cube had an edge length a all these are having an edge length root 2a which is one of the requirements of a regular tetrahedron also you can see that the faces since they are made up of a triangles and they are made up of sides which are equal.

So, all faces are equilateral triangle. All angles are 60 degrees exactly again as required for a regular tetrahedron. So, it is in this way a cube can be embedded sorry a tetrahedron can be embedded in a cube. And this is an interesting way to do any coordinate geometry of tetrahedron if you want to do, because the cube provides a nice coordinate system.

So, you can write for example, you can write the coordinates of all 4 vertices in a very simple way with respect to the axis of the cube. So, you can find all distances angle volume, whatever for tetrahedron using this cube as a reference, but what is interesting from our point of view at the moment, the symmetry point of view we already saw that the 3-fold axis were matching. So, 3-fold of the cube was parallel to the 3-fold of the tetrahedron and here you can see that because if you join any vertex. So, if I join this vertex to the opposite face centroid.

So, this is the centroid of the opposite face. This is one of the corners and if I join it, you can see that on extension, that is also the body diagonal of the cube. So, it is a 3-fold for tetrahedron as well as 3-fold for the cube. Of course, in the cube case, it is 3-fold plus centre of inversion which makes it 3 bar, in the cube case it is 3 plus centre of inversion.

So, it is 3 bar in the tetrahedron case, it is just the 3-fold axis because, tetrahedron is not centrosymmetric and tetrahedron does not have a centre of inversion. That is one axis, but the other axis which we are interested in is what was the 4-fold axis of the cube. So, let us look at the 4-fold axis of the cube which was joining the two face centres two opposite face centres. So, this was the 4-fold axis of the cube.

What is it for the tetrahedron, what symmetry it is of the tetrahedron, cube it was 4-fold because, you can rotate any cube corner by 90 degree you get another corner and 90 degree and other corner. So, you can rotate by 90 degrees about this green axis and the cube will come into self-coincidence. So, for cube this was a 4-fold axis. For tetrahedron what it is tetrahedron, since this if this corner is there, the neighbouring corner is not there.

So, a 90-degree rotation will not bring it into self-coincidence you have to rotate all the way by 180-degree. So, what tetrahedron, this is a 2-fold axis, but if you look carefully, it is a little bit more than 2-fold is there a bit more than 2-fold remember what you what we did for a 4 bar axis. How did we define a 4 bar axis in the last class, rotate by 90 degree and invert.

So, that means if you started with a point about the equatorial plane rotated by 90 degree you came here, but you do not put a point there, you invert in the process of inversion from above the plane you go below the plane and you also change the handedness. So, blue becomes red change of handedness and a dot become circle which is going from about to below, if you continue this journey again.

So, if you rotate by 90-degree, so it travels below the plane here and then invert so it will login surface up. And again it will change handedness. So it will bring it into the original handedness and we will come there. If you again rotate and then invert, where you will get a point below then, again rotate and invert you come back to the original point. So, if this was your first point, and was a right-handed object, then this is the second point which is the left-handed object.

This is the third point, which is the right-handed object and this is a fourth point which is a left handed object and this axis became a 4-fold roto-inversion for which the symbol was we gave the graphical symbol as a square with a lens. And we gave that because we said that the 2-fold as you can see, that if you ignore inversions, then 2-fold also satisfies the symmetry, because blue point by 180-degree rotation goes to the blue point the red circle by 180-degree rotation goes to the red circle.

So, there is a 2-fold, but there is more than 2-fold because red and blue points are also related. The 2-fold is not recognising that symmetry 2-fold is only relating blue to blue and red to red. So, even if there was no relation between red and blue, there will be a 2-fold. But in this particular case, the red and blue are also related because you can go from this 1R two 2 L from the blue 1R to red 2L bye by rotation and inversion.

So, now, seeing this what do you conclude about this tetrahedron 2-fold, 2 blue points above, 2 red circles below just like in your stereogram. Can you see that. So, there are two points here 4 vertices for tetrahedron. So, 2 vertices on top 2 vertices below exactly at 90-degree turn 90-degree and shifted below. So, 2 points are here 2 points are there.

So, that is what is giving you the tetrahedral geometry, which is what is making this axis not just a 2-fold, but it is a 4-bar axis. So, we will draw a square rounded to recognise that. So, last time we gave you an example of for 4 bar, I gave an example, while I was leaving the class that tennis ball, but I later on when I thought that a better example from crystallography or geometry point of view is this tetrahedron.

So, what you usually think as a 2-fold axis of the tetrahedron is actually a 4 bar axis of tetrahedron. So, it is in a sense a realistic example of for 4 bar axis, which you are very familiar with because tetrahedron you are very familiar with and I hope all of you have a model of tetrahedron with you to go back and look at.