Crystals, Symmetry and Tensors
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Lecture 8b
What is a group?

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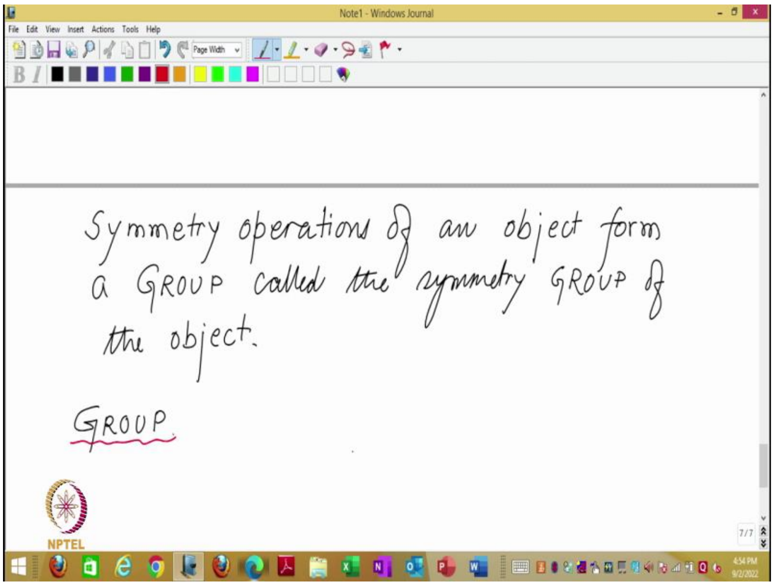

So, there is an important property of these symmetry operations which you can easily see and that is the origin of the name, point group, space group and so on that they form a group. Symmetry operations of an object form a group. So, they always occur in groups they do not occur separately. So, what exactly is a group? So, group is a mathematical concept, so we will define that.

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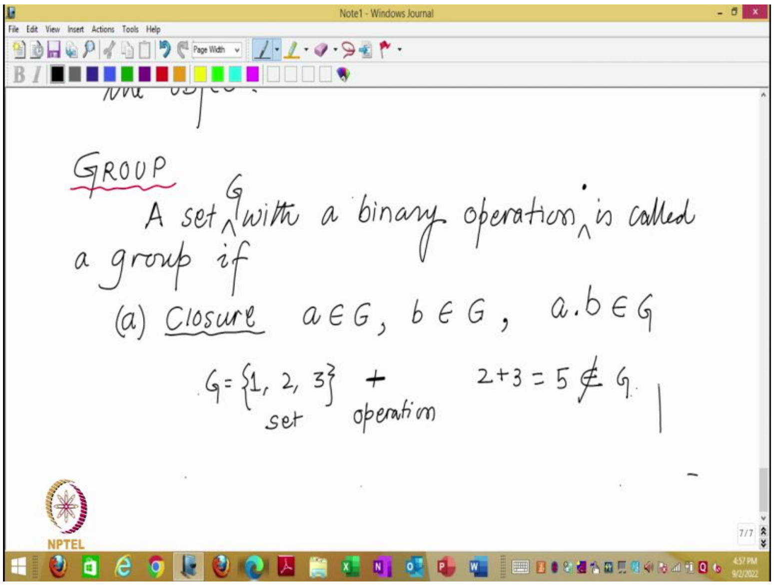

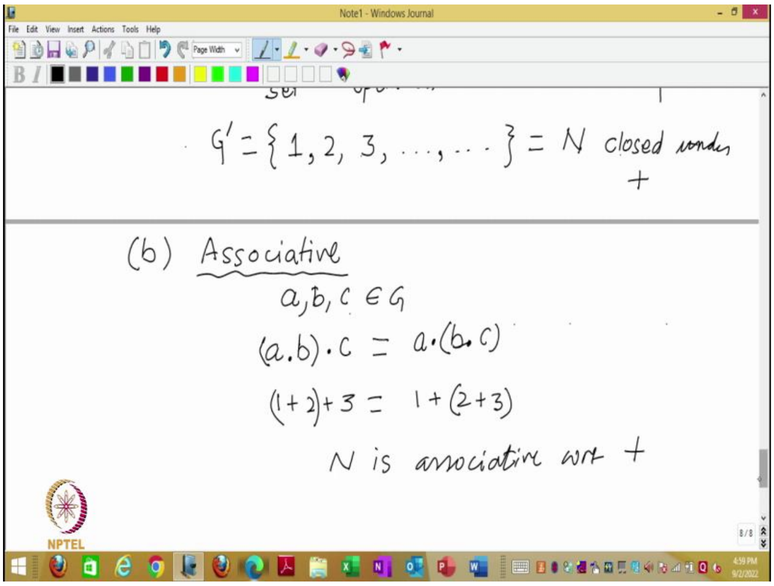

Let us define group. So, a set, for a group you need a set, a set of elements and you need a binary operation with a way of combining two elements. So, a set with a binary operation will called a group if it satisfies certain conditions. And what are those conditions? If, so let us see a set G with a binary operation, some binary operation.

So, first property which a group should satisfy is the closure property that is if a, belongs to group and b belongs to group, so the combination of a and b by binary operation. So, let us say binary operation dot. So, then a dot b also belongs to the group, should not go out of the group. So, let us consider a set 1, 2, 3 and with the binary operation plus. Is it closed?

Student: No.

Professor Rajesh Prasad: Is not closed. 2 plus 3 is equal to 5 and it does not belong to the set. So, this is not a closed set under the binary operation of addition. So, if we want to have a closed set under binary operation of addition what will I have to do? So, that is the entire natural numbers we will have to take then only it will become closed.

Student: No.

Professor Rajesh Prasad: No?

Student: Identity.

Professor Rajesh Prasad: No, I am only looking at the property closer at the moment. So, it is closed under plus. The second property is associative. This says that if you have three objects then does not matter and you are combining these three objects. So, since you have a binary operation and you have to combine three or you do not have a ternary operation.

So, since you have a binary operation and you have to combine three objects, you have to decide whether you have to combine a and b first and then combine it with c or you combine b and c first and then combine it with a and it should not matter if it is associative, it is demanding that it should not matter. So, our plus naturally satisfies this. So, we do not worry about this because 1 plus 2 plus 3 the result does not depend upon which way do we do the addition. So, natural number N is associative with respect to plus.

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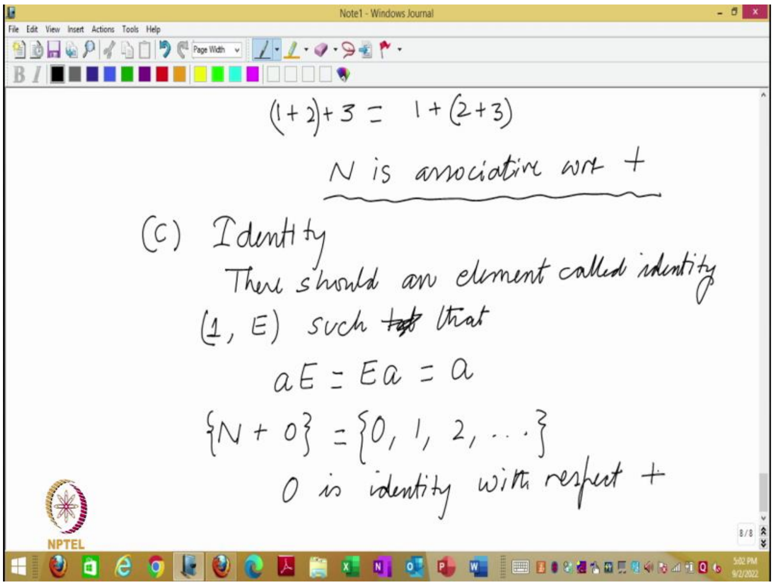

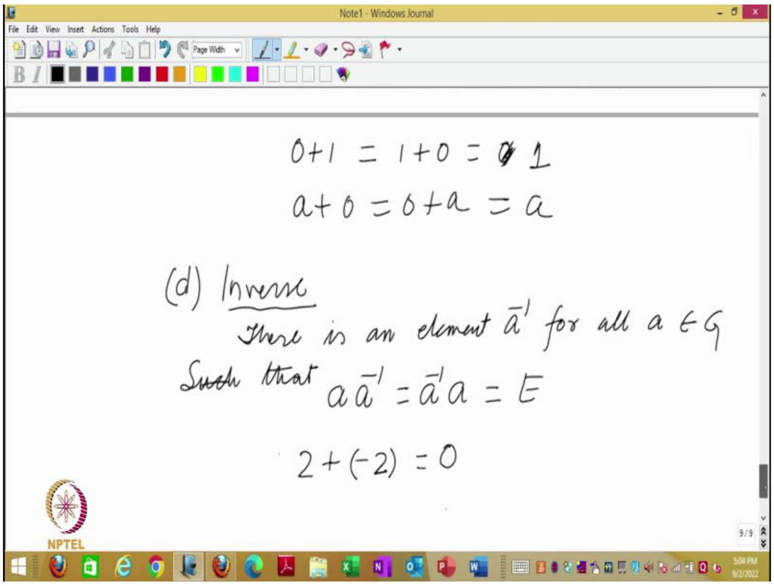

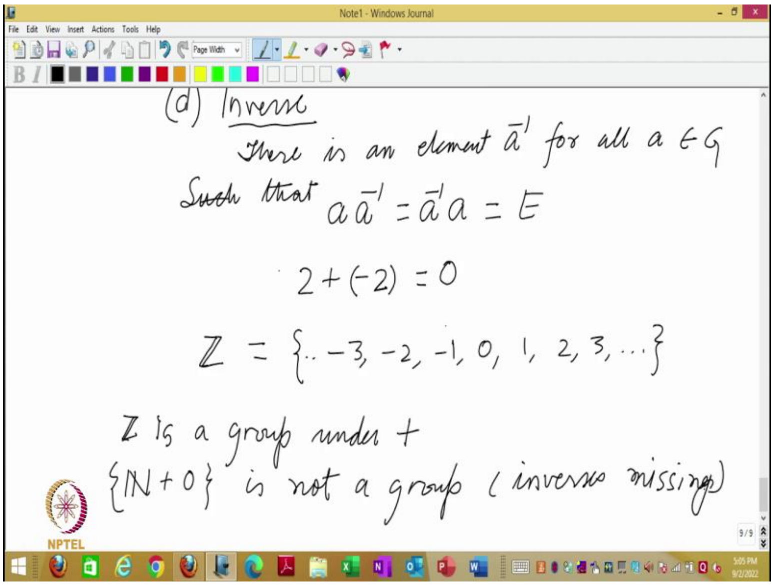

The third thing we require is an identity. There should be an especial element called identity. Now, identity is represented by various symbols often either by 1 or by E because the normal way of looking at the binary operation is to call the multiplication and in multiplication one acts like an identity. Sometimes we use the symbol E as an identity.

So, what is identity? a E should be equal to Ea should be equal to a. So, this element whichever way you combine with any given element either by pre multiplication or by post multiplication should not change your element. So, is the natural number is having the identity operation with plus?

Student: No.

Professor Rajesh Prasad: What you require for that?

Student: 0.

Professor Rajesh Prasad: 0. So, what is that called? N plus 0?

Student: Whole number.

Professor Rajesh Prasad: Whole number? Let us say N plus 0 whatever that is called. So, this will have 0, 1, 2 and everything up to infinity. So, 0 is identity with respect to the binary operation plus because it leaves, so, either I have 0 plus 1 or 1 plus 0 it will remain 1. In general, a plus 0 0 plus a is a.

So, now, we have identity also, but still it has not become a group because we have the final requirement for group is inverse. So, what is inverse? There is an element a inverse for all a belonging to G. Such that if you combine a and a inverse or a inverse and a you should get the identity. So, what is inverse with respect to addition?

Student: Negative.

Professor Rajesh Prasad: Negative number. So, inverse of 2 should be a number which well combined with 2 gives me the identity and identity addition is 0. So, 2 plus 2 minus 2 is 0, but we have not included that, we have not included the negative integers. So, I now have to add the negative numbers also. So, I will get the set of integers if I want a group, so N is not a group, N plus 0 is also not a group, but Z with all negative numbers included now it forms a group. So, we will say integers, set of integers is a group under plus N with 0 also is not a group because inverses are missing other three properties were there, inverse is missing.

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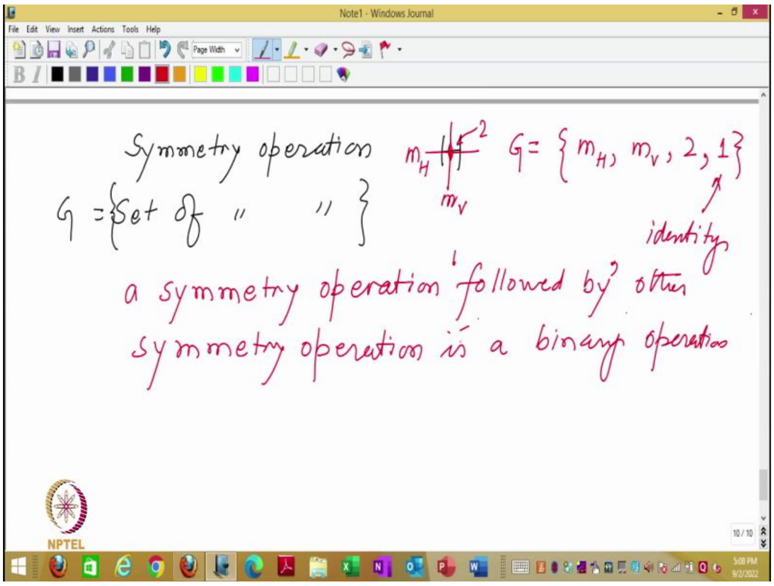

So, what has all this to do with symmetry? So, if you now look at symmetry, symmetry also, what will be a binary operation for symmetry, so symmetry is an operation, symmetry itself is an operation. So, for example, when we say reflection, that is an operation, rotation, that is an operation. So, our set is set of symmetry operations, our set G is set of symmetry operations.

So, for example, we saw that this H, which was highly symmetric, has a vertical mirror, horizontal mirror and a twofold, so it was having. So, what are its symmetry operations? So, let us call this, let us call this mV, let us call this mH and let us call this a 2. So, it has a symmetry operations mH, mV, 2 and identity, which we call 1.

But the symmetry operations are itself elements now they are elements of the set. So, elements of the set are operations, but for group we require a binary operation, so we require another operation which combines these two elements or any two elements in this. What will that operation be? 2 times 1 after the another.

So, followed by, I can combine when I say I will combine two operations. So, I will say one operation followed by another operation. Combine a horizontal mirror with a vertical mirror. So, first reflect horizontally then reflect vertically. So, with this definition now, that I collect all the symmetry operations of an object and I combine them by the operation of followed by? Does it form a group? So, let us explore that.

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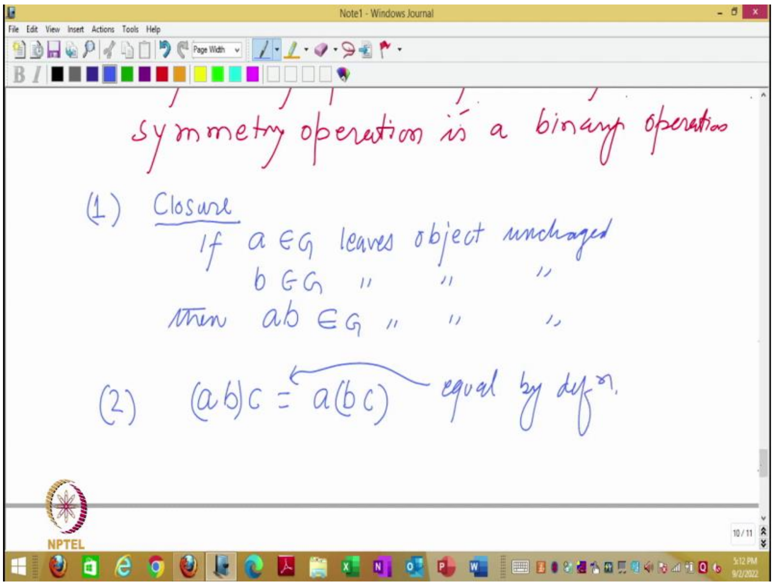

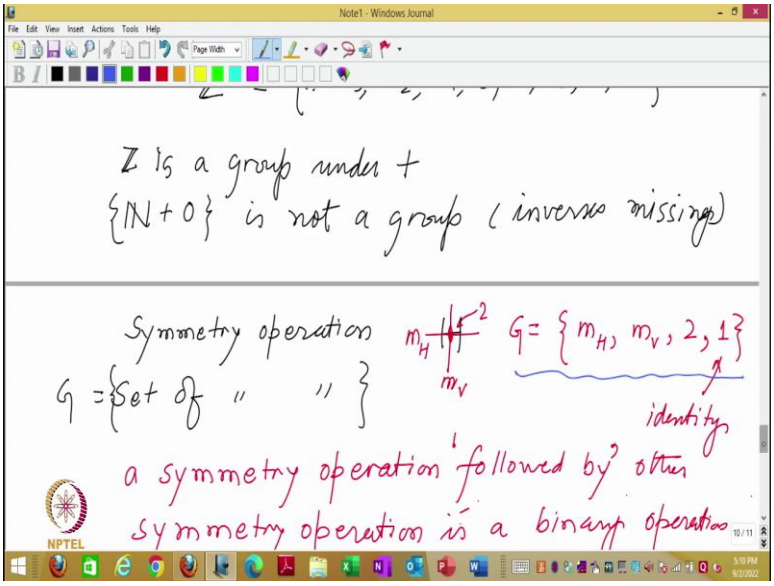

So, let us look at one by one. The first property was closure. What was the defining requirement for being a member of this group? What operations we collected in this group?

Student: (())(15:50).

Professor Rajesh Prasad: No, combination. Why is mH qualified to be part of this group? Because, mH left H unchanged.

Student: (())(16:03).

Professor Rajesh Prasad: So, if mH leaves H unchanged and mV leaves H unchanged. So, a combined operation of mH followed by mV?

Student: (())(16:16).

Professor Rajesh Prasad: Obviously, will leave it unchanged. And if that is also leaving it unchanged, so, that should also be a part of the group. So, means in my list of symmetry elements or symmetry operations of H, I cannot miss combination of mH and mV because mH is leaving H unchanged and V is leaving H unchanged. So, how cannot mH followed by mV be not part of that set that is also leaving it unchanged so that is also should be a symmetry operation. But I have not written mH combined mV.

So, closer is demanded that if a then ab, ab is actually a followed by b, first we apply b, nahi sorry, it is b, yeah, first we apply b and then a, so a followed by b yeah, a followed by b this should also belong to G because it leaves object unchanged by very definition. So, if a is a symmetry operation b is a symmetry operation the combined elements ab has to be a symmetry operation, you have no other way out, so the closure property has to be there.

Associativity, we have to check but usually it is always satisfied. So, in this case again what can happen, nothing else can happen, they have to be the same, ab leaves it unchanged and when you combine it with c that also leaves unchanged. So, the net result of the left-hand side ab applied first and then c is leaving the object unchanged, the right-hand side also leaves the object unchanged because c leaves it unchanged, you combine it with b that also leaves it unchanged and then you combine it with a that also leaves it unchanged. So, obviously the two are equal by very definition of symmetry. So, associativity is also satisfied.

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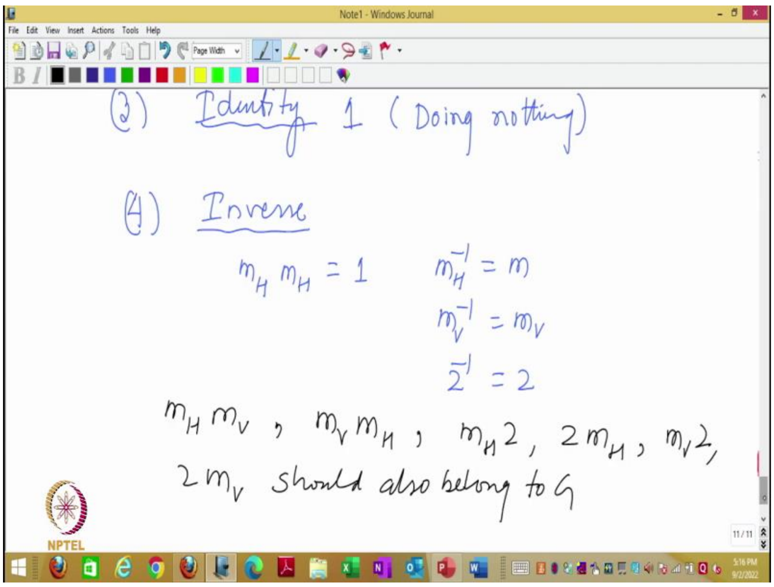

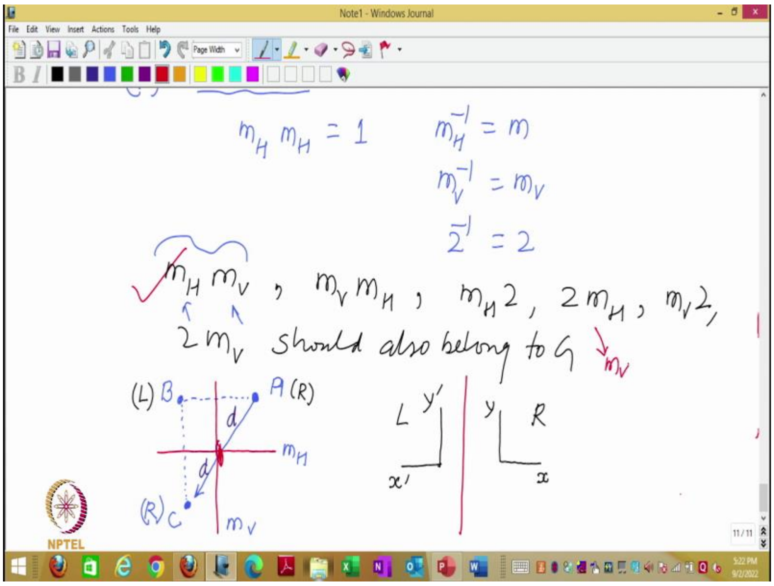

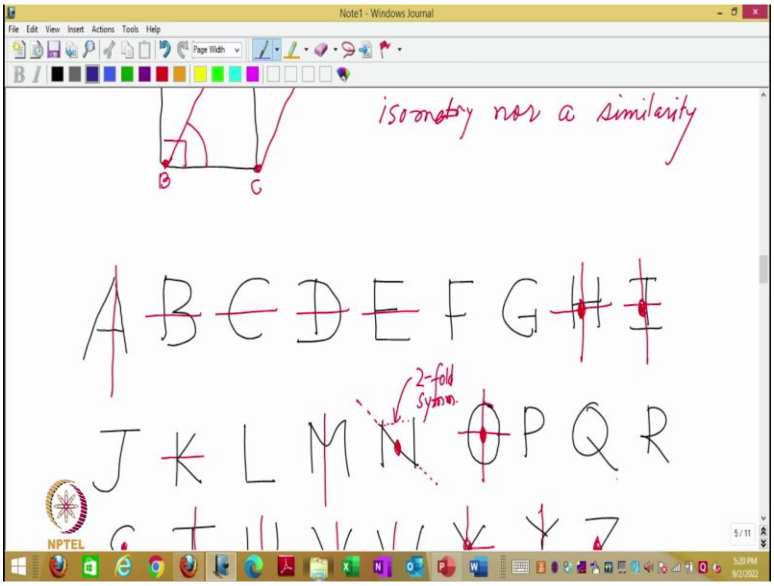

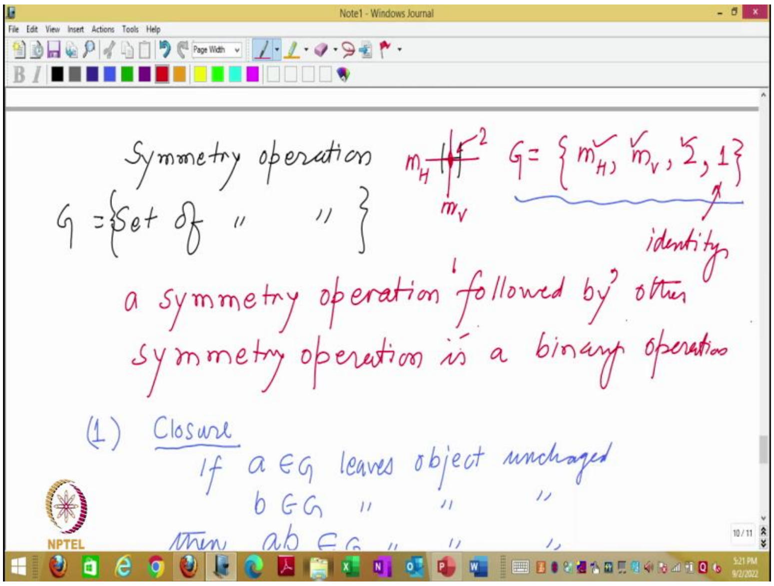

Professor Rajesh Prasad: Identity.

Student: 1.

Professor Rajesh Prasad: Doing nothing leaves it unchanged and we have called it 1, so identity is also there. And if you apply some symmetry operation, if I apply horizontal mirror and then do nothing what is the resultant, horizontal mirror. So, the doing nothing does qualify as an identity operation. What about inverse? So, if I apply a horizontal mirror I reflect and then I reflect again. What is the reflection of a reflection? The original object. So, reflection of reflection is the original object means you are doing nothing, so that is an identity.

So, mH is its own inverse, mH inverse is mH, mV inverse is mV and 2 inverse is 2. So, inverses are there. Now, but inverses are there, identity is there we have not yet an insured that whether the products are there. All combinations should also be there. So, now, since we are having these elements we need to ensure that mH mV, mV mH, mH2, 2mH, mV2, 2mV, these all should be part because they are also symmetry operation they are also leaving the object unchanged. Where are they? mH mV toh hai hi nahi.

Student: It is 2.

Professor Rajesh Prasad: mH mV is 2. Let us see how it is 2, let us try to prove that. Let us say we have an object and we have a two mirrors, we have a, this is mV, this is mH. Let us first apply mV. So, we are doing this operation first mV and then mH. So, first we apply mV, so, we reflect in the vertical mirror, I generate the point B. Now, actually although it is a point, we have to keep in mind that reflection changes the so-called chirality that is handedness.

Reflection will change a left-handed coordinate system to a right-handed coordinate system. In point it is not obvious, but suppose what will happen, suppose what will happen if I have a right-handed coordinate system x cross y is pointing up. I reflected in this red mirror, x prime cross y prime pointing down. So, if we call this right-handed coordinate system or a right-handed object, this is a left-handed object.

So, although we will draw points, we will imagine that actually the point is representing not only a location, but some sort of a right-handed object at that location. So, let us say that A is a right-handed object then what will become of B, B will be a left-handed object. Now, I apply another operation because I have to apply mV followed by, so I come here. Now, if a left-handed object is reflected then you get?

Student: Right-handed object.

Professor Rajesh Prasad: Again, a right-handed object. So, I get C which is again a right-handed object. So, the net result of mV followed by mH is A going to C directly. And by what operation A will go to see directly?

Student: 180-degree rotation.

Professor Rajesh Prasad: 180-degree rotation. And what does rotation do to the handedness?

Student: Nothing.

Professor Rajesh Prasad: It does not change. So, a right-handed object going 180-degree opposite with the same distance that you can prove geometrically that this distance d is same as this distance d. So, a right-handed object located here at arm's length is going 180 degree away to arm's length. Then what is the operation which is relating the 2? 180-degree rotation above the vertical axis.

That is why we were, so now that surprise is explained or that observation is explained if at all that was a surprise that whenever we were seeing two mirrors in any of the letters here, the two-fold was always there at the sitting at the center, because the combination of the two fold in two mirrors itself is a twofold. So, this is where that group properties coming. So, actually, mH and mV are included as 2 because we have included 2 in the group. So, the product of mH and mV is 2, you will find that mV mH is also 2. What will be mH2?

Student: mV.

Professor Rajesh Prasad: mH2 will be mV. Remember mH2 cannot be 2, why?

Student: Because using mH is we have changed the handedness.

Professor Rajesh Prasad: Yeah, using mH you are changing handedness, using 2 you are not changing the handedness, so the net result is a change of handedness. And a change of handedness cannot come from?

Student: 2.

Professor Rajesh Prasad: 2, so, mH and 2 will give you?

Student: mV.

Professor Rajesh Prasad: mV. So, everything is consistent. So, we get a nice group. So, that is the reason why we use the word group of symmetry, symmetry group.

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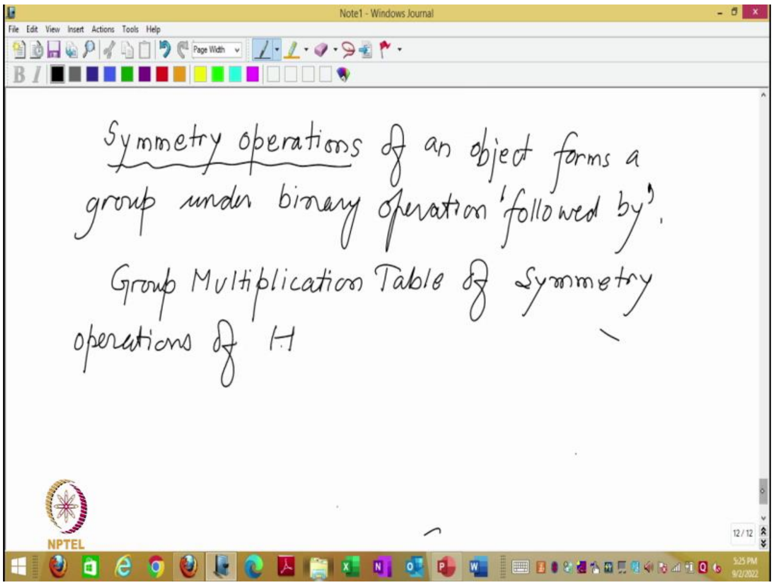

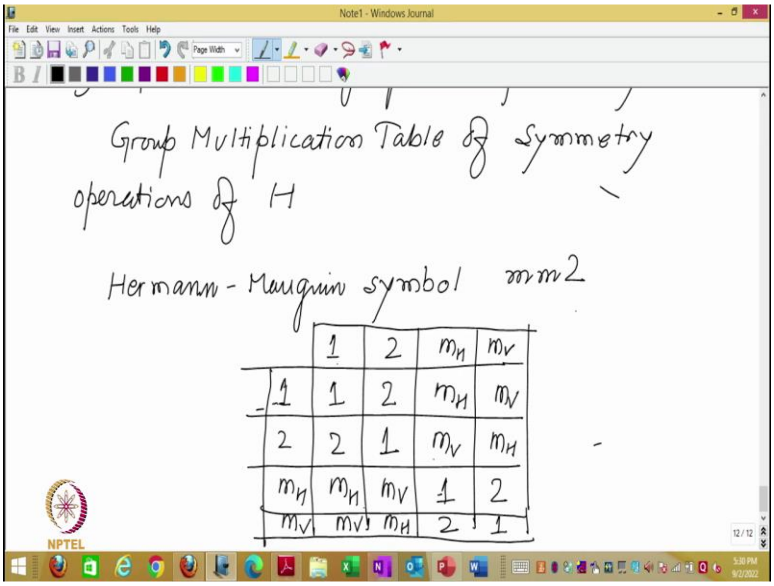

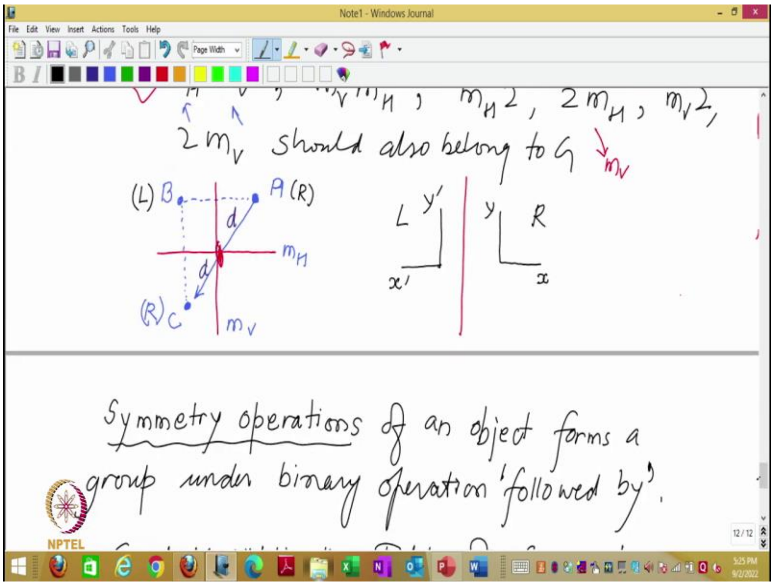

So, symmetry operations of an object forms a group under binary operation followed by. And we have seen that, one concrete example we have seen. We just finished the lecture with the, what is called a group multiplication table of symmetry operations of H. By the way since this is having to mirror planes and a twofold axis there is a so called Hermann-Mauguin symbol for it which is called mm2. So, one m represents one mirror another m represents another mirror and 2 represents the twofold axis at the intersection point.

So, let us create a group multiplication table for this. So, what do we mean by group multiplication table? We just have to write all the elements both horizontally and vertically. So, let us start with the identity 1, then 2 then mH and then mV. And I write 1, 2, mH, mV. All you have to do is to, in any box now, I will fill the row, the column element multiplied by the?

Student: Row element.

Professor Rajesh Prasad: Row element. So, 1 into 1 is?

Student: 1.

Professor Rajesh Prasad: 1 thankfully. 1 into 2?

Student: 2.

Professor Rajesh Prasad: Identity being combined with any operation. So, we will leave it unchanged. So, that row is easy for us to write. Similarly, if I do 2 into 1.

Student: 2.

Professor Rajesh Prasad: So, that will demand 2, so the column also is easy to write the first column. Now, the other this 3 by 3 9 elements you have to be careful. So, 2 into 2 gives you what?

Student: 1.

Professor Rajesh Prasad: 1, 2 into 2, 1. 2 into mH?

Student: mV.

Professor Rajesh Prasad: mV. 2 into mV?

Student: mH.

Professor Rajesh Prasad: mH. mH into 2?

Student: mV.

Professor Rajesh Prasad: mV. mH into mH?

Student: 1.

Professor Rajesh Prasad: 1. Reflection is its own inverse. This is a strange group interesting group, where each element is a self-inverse, do not think that this is a common property that elements are there inverses. If you are seeing only this group, we may conclude that, but this will not be true as we will look at other groups. So, in this case, each element is its own self inverse. mH into mV2, mV into 2?

Student: mH.

Professor Rajesh Prasad: mH. mV into mH? 2. So, you can see a beautiful table has been formed, which is justifying your group property. First of all, that you did not get in this table, means, we made a header of rows of all the elements and a header of column of all the elements and we are multiplying, on multiplication we did not generate any new element other than the top four which we had written, so that is closure.

For every row we are seeing and we are seeing that the row of one is leaving everything unchanged, so that is the identity. Similarly, column of 1 is leaving everything unchanged, so identity is there. Then we are seeing that in every row 1 appears once. So, and that is the inverse of that particular element. So, every element has an inverse. This kind of group multiplication table that way shows you the group structure also clearly and ensures you that the object is a group and you are not getting anything miss, it is not that the closure property is not satisfied.