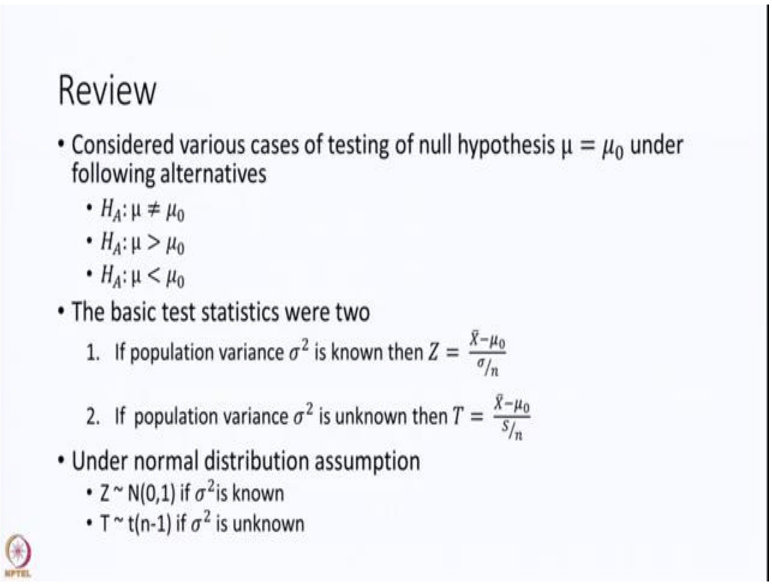
Dealing with Materials Data:
Collection, Analysis and Interpretation
Professor Hina A Gokhale
Department of Metallurgical Engineering and Materials Science
Indian Institute of Technology, Bombay
Lecture 73
Hypothesis Testing VI

Hello and welcome to dealing with materials data course, we have come a long way in learning about hypothesis testing, all the way we have worked with the case in which we have considered normal distribution as the population distribution.

(Refer Time Slide: 00:41)



So, if we review it quickly, what we have seen is we have tested the null hypothesis, if the population is normal which mean Mu and variance sigma square.

- Considered various cases of testing of null hypothesis $H_0\colon \mu = \mu_0$ under following alternatives
  - $H_A\colon \mu \neq \mu_0$
  - $H_A\colon \mu > \mu_0$
  - $H_A\colon \mu < \mu_0$

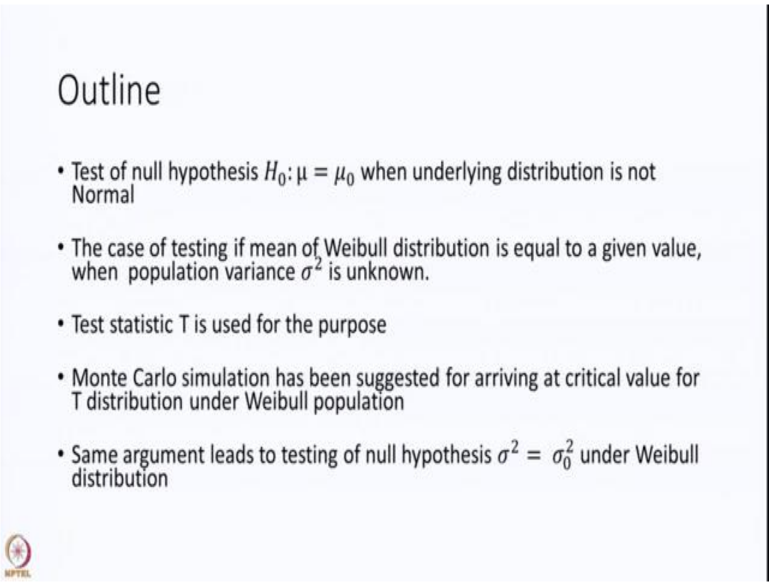
In both the cases, we found that if the population variance sigma square is known, then it reduces down to testing a hypothesis using the test statistics Z, which is a standard normal variate. It is a normal variate with mean 0 and variance 1. In case we assume that sigma square is unknown, then it reduces to a test statistic which is a students T distribution statistic.

And it is distributed a students T distribution with N minus 1 degrees of freedom. The advantage of the two is that none of the two statistics distribution under H0, when you assume that null hypothesis is true, does not it does not depend on any other parameter then the value n or in case of Z it does not depend even in the size of the sample.

Something I have not shown here, but we also went through a testing of hypothesis process for testing that the sample, the population variance is equal to a given value sigma 0 square. And we found that Chi square that is N minus 1 sample variance divided by the sigma naught square is the Chi square variate and that variate if we call W then or we have called it Y probably that is the variate which has a test statistic, it becomes a Chi square distribution with N minus 1 degree of freedom.

So, it does not have any other parameter, only parameter is a known value depending on the sample size. And again, we tested the three alternatives and we found three critical regions in which we can reject the hypothesis when we assume that null hypothesis is true.

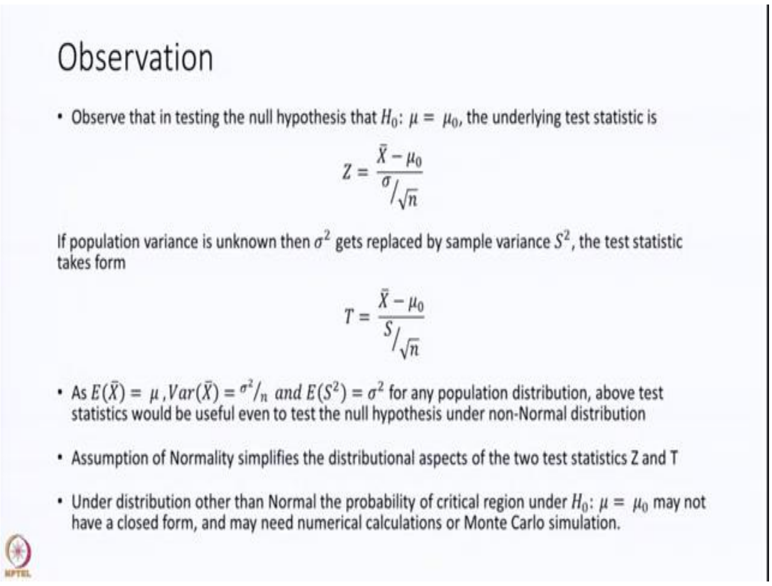
(Refer Time Slide: 03:32)



So, then comes what we want to do now, as I said before, we already made an assumption that the underlying population distribution is normal, but that may not be always the case. And as it was mentioned earlier also that in the material science and metallurgical engineering number of times the data do not follow a nice symmetric distribution like normal and we cannot really apply central limit theorem because the data size is not sufficiently large.

So, in this presentation today, in this session today, we would like to derive our test statistic to test the hypothesis, the same two hypotheses that the population mean is equal to a given value versus its three alternatives. And population variance is equal to given value versus its three alternative, but the underlying population distribution will not be assumed to be normal. In this case, as a case we have assumed it to be Weibull distribution.

(Refer Time Slide: 04:54)



We had mentioned it earlier, you just observe the test statistic that we found in testing the hypothesis that Mu is equal to Mu 0, when sigma square or the population variance is known is Z which depends only on X bar and rest of them are known parameters. If you say sigma square is unknown, then sigma square gets replaced by the sample variance.

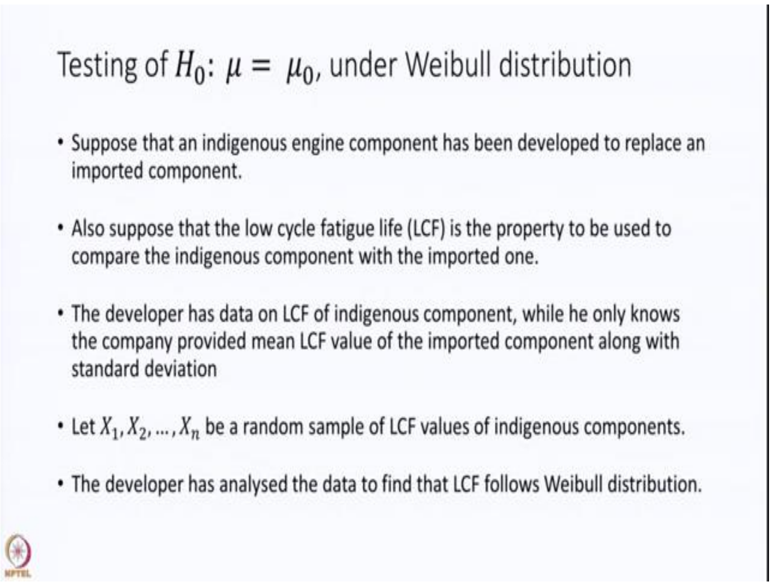
So, T statistic is depends on X bar and S and otherwise Mu and Mu 0 and N which are all the known quantity. This X bar and S are very special statistics, because expected value of X bar is always Mu and variants of X bar is always sigma square over n no matter what the underlying population distribution.

Also, if you look at it, the expected value of S square is also sigma square, independent of what distribution, underlying distribution is it need not be normal. So, this actually indicates that the same Z and T can well be used as test statistics to test the same hypothesis Mu is equal to Mu 0 when sigma square is known, and Mu is equal to Mu 0 when sigma square is unknown.

And we can follow the practice only thing is assumption of normality makes life easy because the distributional aspect of the test statistics Z and T become very obvious, one is a standard normal distribution and the other is a T distribution with n minus 1 degree of freedom, which may not be the case when you deviate from the assumption of normality.

But well in that case, with the advent of computers and so much of computing facility and software, there always should be possible to find a numeric solution through either the numerical analysis or through Monte Carlo simulation and that is exactly what we are going to demonstrate or rather show it here. So, let us consider a case that we want to test the hypothesis Mu is equal to Mu 0 under Weibull distribution.

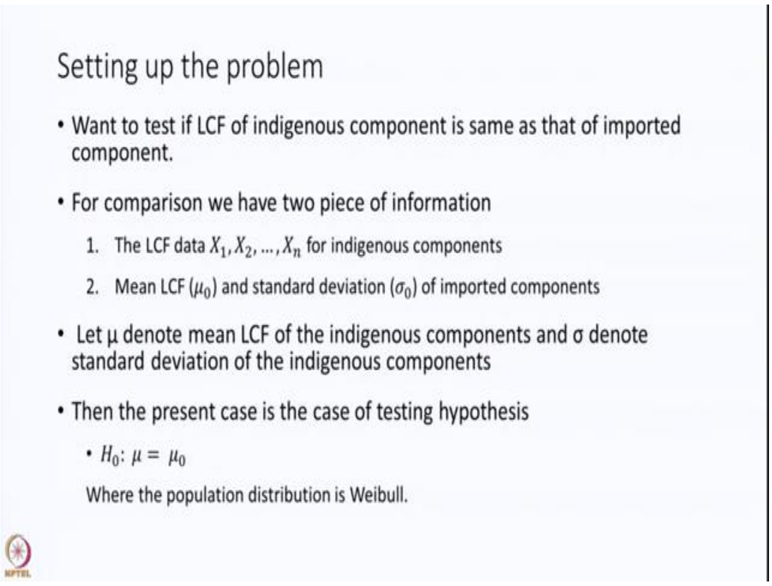
(Refer Time Slide: 07:33)



So, here I have given you a situation in which such a problem can arise. Suppose, there is an indigenous engine component and this component is now need to be replaced where we earlier the manufacturer was using it important component. Now, the developers of the indigenous component have to show that its performance or the performance criteria that the important component meets is are also made by the indigenous component. And suppose the criteria is matching the low cycle fatigue life LCF life is the property we would like to compare.

So, assume that we have a data, sample data from the indigenous component X1, X2, X3, XN and we would like to, from this data we would like to show that the mean LCF value and the standard deviation of LCF value from this sample is same as what you would get from the important component. Remember, it is one can show that LCF values can follow closely Weibull distribution.

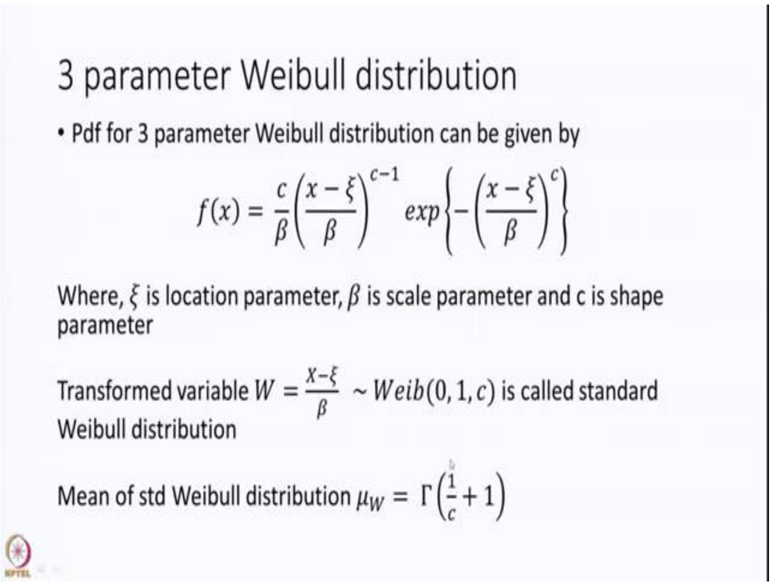
(Refer Time Slide: 09:08)



So, let us set up the problem, we want to test if LCF of indigenous component is same as that of an important component. So, for comparison we have two pieces of information, we have an LCF data X1, X2, X3, XN which is a random sample of size N. And we have mean value of important component Mu 0 and a standard deviation of important component sigma 0.

Let Mu denote the mean value of LCF of indigenous component and sigma denote the standard deviation of the indigenous component. Then, in the present case, we would like to test the hypothesis that Mu is equal to Mu 0, where the population distribution is Weibull.

(Refer Time Slide: 10:17)



In reality, we have to assume three parameter Weibull distribution. In the case of special random variable, we have given a very brief introduction to three parameter Weibull distribution. So, we will revisit it here, the probability density function of three parameter Weibull distribution is given in this format,

$$f(x) = \frac{c}{\beta}\left(\frac{x-\xi}{\beta}\right)^{c-1} exp\left\{-\left(\frac{x-\xi}{\beta}\right)^{c}\right\}$$

where the ξ is called the location parameter β is called the scale parameter and C is called the shape parameter.

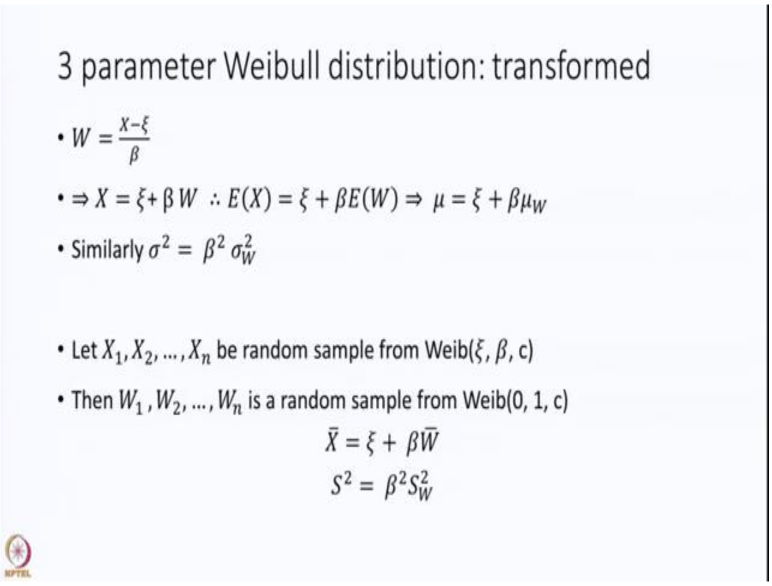
Now, let us take a transformed variable W, which is equal to

$W = \frac{X-\xi}{\beta} \sim Weib(0,\ 1,\ c)$ is called standard Weibull distribution

Weibull distribution with location parameter 0, scale parameter 1 and a shape parameter C. So, it means that this is this depends only now on one parameter which is called C, it is shape parameter.

This is also called a Standard Weibull distribution. Remember standard normal distribution does not depend on any parameter. Standard Weibull distribution depends on one parameter which is a shape parameter. The mean of standard Weibull distribution Mean of std Weibull distribution $\mu_W = \Gamma\left(\frac{1}{c} + 1\right)$

(Refer Time Slide: 11:34)



Now, if we transform the variable, the Weibull variable into

$$W = \frac{X - \xi}{\beta}$$

$$\Rightarrow X = \xi + \beta W \;\therefore E(X) = \xi + \beta E(W) \Rightarrow \mu = \xi + \beta\mu_W$$

Similarly $\sigma^2 = \beta^2 \sigma_W^2$

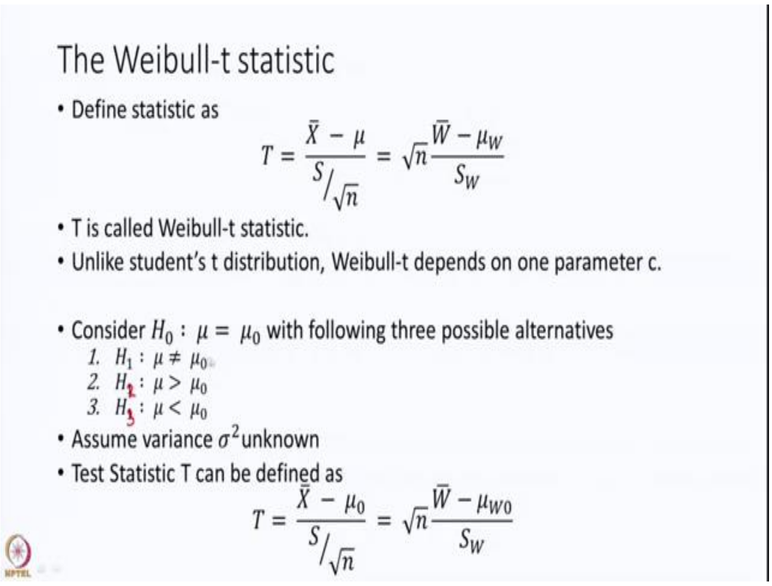
Let $X_1, X_2, \ldots, X_n$ be random sample from Weib(ξ, β, c)

Then $W_1, W_2, \ldots, W_n$ is a random sample from Weib(0, 1, c)

$$\bar{X} = \xi + \beta\bar{W}$$

$$S^2 = \beta^2 S_W^2$$

(Refer Time Slide: 12:50)



Let us define Weibull T statistic that is our t statistic that we already know

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} = \sqrt{n}\frac{\bar{W} - \mu_W}{S_W}$$

I call it T, but it would not follow the student's T distribution, whatever it may follow, I call it a Weibull t distribution and I call this a Weibull t statistic. T is called a Weibull t statistic and unlike T distribution Weibull t statistic also depends on one parameter, unknown parameter C, which is the shape of Weibull distribution.

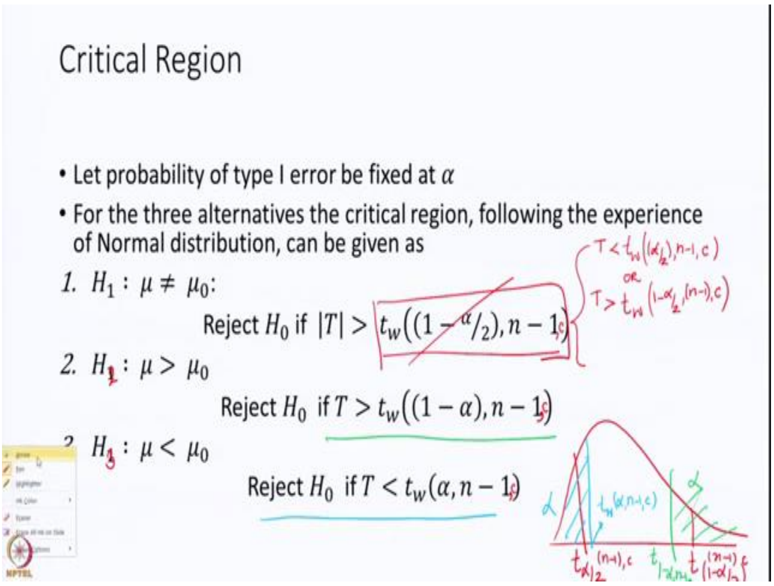
So, it depends on of course, the degrees of freedom should be n minus 1 and because we have estimated X bar. So, with the degrees of freedom will be n minus 1, but it will also have another parameter along with it which is C. Now, we consider the testing of hypothesis Mu is equal to Mu 0 against the three alternatives H1, sorry it should be H1, H2, H3 but there is a mistake here. Let us correct it so that the mistake does not continue.

- Consider $H_0 : \mu = \mu_0$ with following three possible alternatives

    1. $H_1 : \mu \neq \mu_0$
    2. $H_2 : \mu > \mu_0$
    3. $H_3 : \mu < \mu_0$

We straight away assume that sigma square is unknown. Then the T statistic as defined above can be a T statistic $T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} = \sqrt{n}\frac{\bar{W} - \mu_{W0}}{S_W}$ for *testing the hypothesis* $H_0 : \mu = \mu_0$.

(Refer Time Slide: 14:55)



Let us define what will be the critical region, you recall what we did in the past and we follow the same steps. So, if we fix the alpha and the type one error at alpha, then for the testing the null hypothesis against the three alternatives,

1. $H_1 : \mu \neq \mu_0$: Reject H_0 if $|T| > t_w\left(\left(1 - {}^{\alpha}/_{2}\right), n - 1\right)$

And of course, there will be a parameter I am sorry, of course, there will be a parameter C, which I have missed out. So, this is going to be your critical value, the alpha 1 minus alpha by

2 comes under the same argument, we are assuming that it is going to be symmetric. If it is not symmetric, then it has to have the two value I think it is, it would be more appropriate, if we write it that this can be written as

$$T < t_w\left((1-{}^{\alpha}/_{2}),\, n-1\right) \; or \; T > t_w\left((1-{}^{\alpha}/_{2}),\, n-1\right)$$

This has to be either this or that.

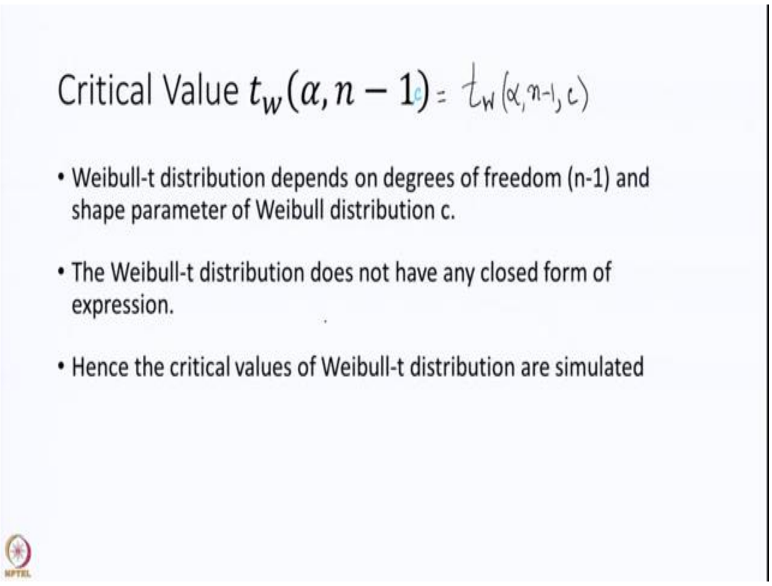
Then, if we consider the rejection with the alternate hypothesis

2. $H_2 : \mu > \mu_0$ Reject H_0 if $T > t_w((1-\alpha),\, n-1, c)$
3. $H_3 : \mu < \mu_0$ Reject H_0 if $T < t_w(\alpha,\, n-1, c)$

So, in other words like a Chi square distribution, we are assuming that this distribution will also be kind of asymmetric distribution. And then we are taking two values, this is t alpha by 2 and this is t 1 minus alpha by 2 and with the rest of the parameters, so that is n minus 1 and C, here also there is n minus 1 and C, this is the case with respect to this.

So, if you are looking for this situation, then you are going to look into this area to be alpha and therefore, this value is going to be t 1 minus alpha n minus 1 C. And if you are looking for the fourth case, if you are looking for this fourth case, then you will have to take some value here, where this value, this area is going to be alpha and therefore, this is going to be T Sub W alpha n minus 1 and C. I hope this is clear. So, these are going to be the critical region.

(Refer Time Slide: 19:42)

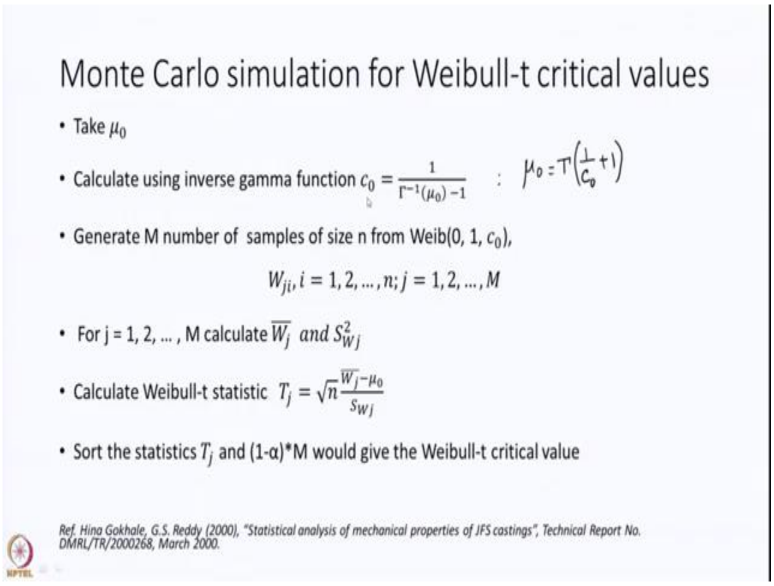


So, how do we find any critical value and again I have to add a C here. I have to add a C here because it is actually, so this should be written as

$$t_w(\alpha,\ n-1, c)$$

So, it is a Weibull t distribution depends on degrees of freedom n minus 1 and shape parameter of a Weibull distributions C. Therefore, Weibull t distribution does not have any closed form solution, the way we have defined it, it does not have a closed form solution hence the critical values of Weibull t are generally simulated, we will give you the here how to simulate it.

(Refer Time Slide: 20:45)



So, this Monte Carlo simulation can be carried out that.

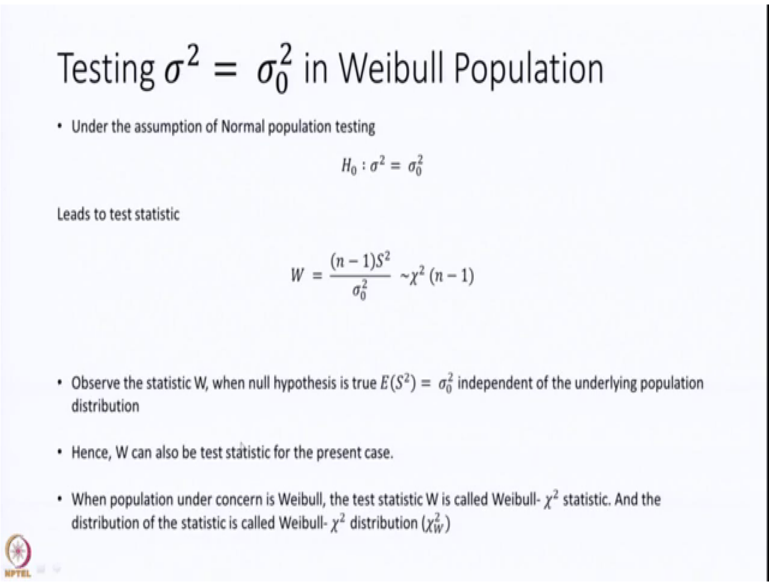
- Take μ_0
- Calculate using inverse gamma function $c_0 = \frac{1}{\Gamma^{-1}(\mu_0)-1}$
- Generate M number of samples of size n from Weib(0, 1, c_0),

$$W_{ji},\ i = 1,\ 2,\ \dots,\ n; j = 1,\ 2,\ \dots,\ M$$

- For j = 1, 2, … , M calculate $\overline{W_j}$ $and\ S_{Wj}^2$
- Calculate Weibull-t statistic $T_j = \sqrt{n}\frac{\overline{W_j}-\mu_0}{s_{Wj}}$
- Sort the statistics T_j and (1-α)*M would give the Weibull-t critical value

Once you have M capital M of T, j you sort them out from smallest to the largest. And if you take 1 minus alpha time M value of Tj that will be the alpha level critical value of Weibull t distribution depending on C0. So, what you really need to do is for different values of C0 you have to simulate this or as and when needed, you take what is your M 0, you convert it into your C0 and write a nice program, so that every time it generates this critical value and gives you, please remember when we do Monte Carlo simulation, it is very important to see the stability of it, make sure that the C that is given into it does not conflict with your Monte Carlo simulation process.

(Refer Time Slide: 23:55)



If suppose now we want to test sigma square is equal to sigma naught square okay.

Under the assumption of Normal population testing

$$H_0 : \sigma^2 = \sigma_0^2$$

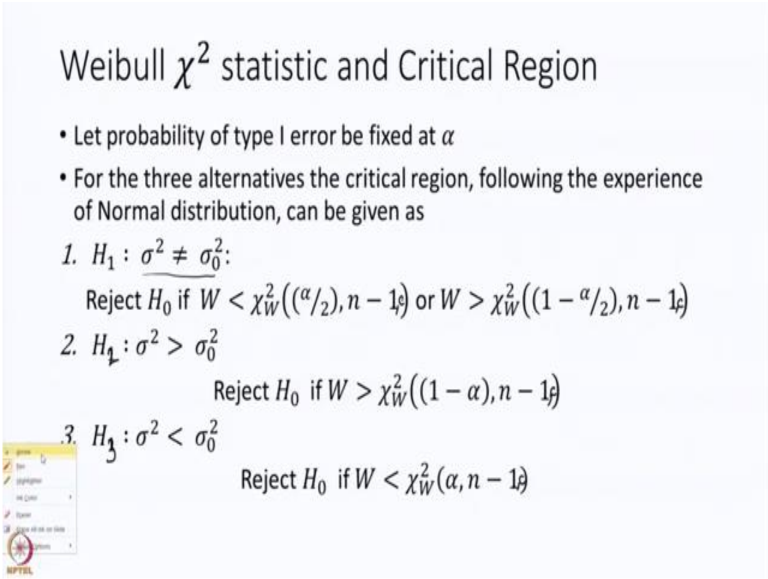
Leads to test statistic

$$W = \frac{(n-1)S^2}{\sigma_0^2} \sim \chi^2 (n-1)$$

Just as in the previous case of testing Mu is equal to Mu 0 we find rather we observe that expected value of S square is sigma naught square or a sigma square independent of what is underlying population.

Hence, W can also be a test statistic for the present case. And when population under concern is Weibull, such a statistic is called a Weibull Chi square statistics and the distribution is called Weibull Chi square distribution.

(Refer Time Slide: 24:50)



Again, you can find the critical region. Here also please remember the statistic will depend on C. So, let me write it down everywhere. So, we will have this with C, here also there will be a C. So, if you are taking null hypothesis, $H_0 : \sigma^2 = \sigma_0^2$

and your alternative

1. $H_1 : \sigma^2 \neq \sigma_0^2$: Reject H_0 if $W < \chi^2_W\left((\alpha/2), n-1, c\right)$ or

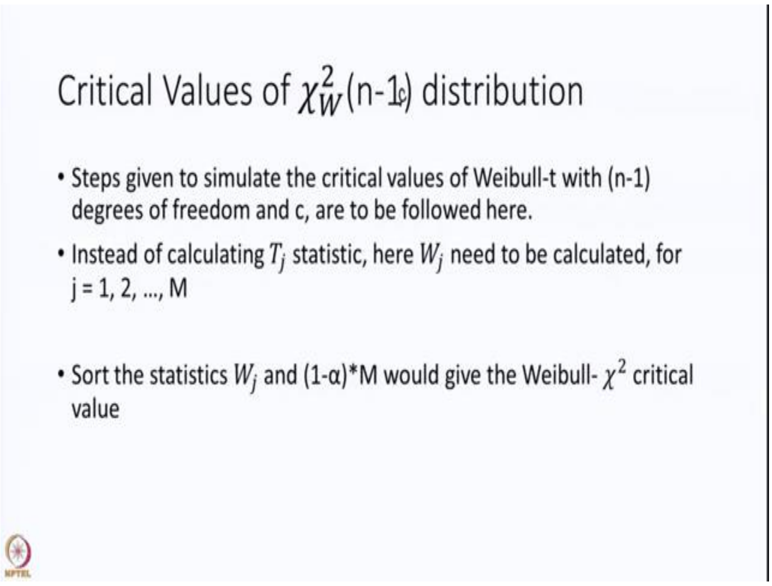
$$W > \chi^2_W\left((1-\alpha/2), n-1, c\right)$$

2. $H_2 : \sigma^2 > \sigma_0^2$ Reject H_0 if $W > \chi^2_W((1-\alpha), n-1, c)$

3. $H_3 : \sigma^2 < \sigma_0^2$ Reject H_0 if $W < \chi^2_W(\alpha, n-1, c)$

If your alternate is sigma square is greater than sigma 0 square, then we say that reject H0. If your Chi statistics W is greater than the Weibull Chi square with probability 1 minus alpha N minus 1 degrees of freedom and nuisance parameter C. And if sigma square is less than sigma 0 square is your alternative, then the critical value is Weibull Chi square at alpha probability n minus 1 degrees of freedom and nuisance parameter C.

(Refer Time Slide: 27:06)



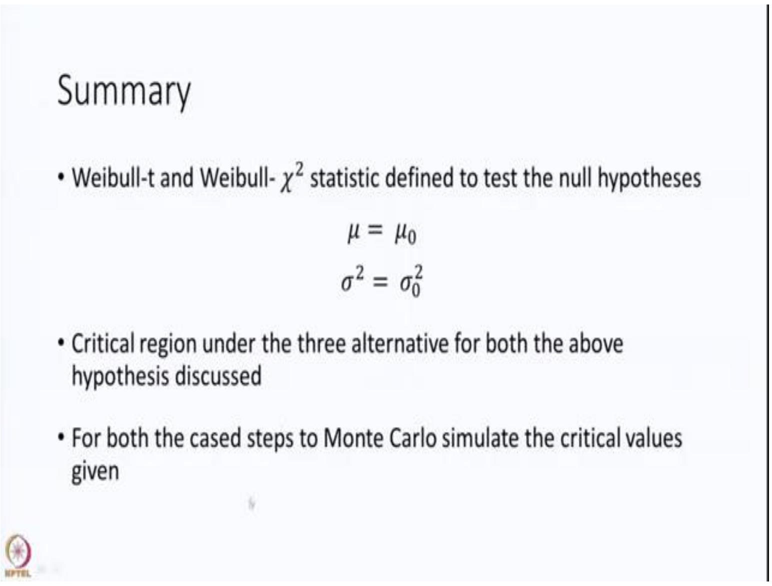
How do you find a critical values of Weibull Chi square distribution with n minus 1 degrees of freedom and nuisance parameters C, when we have to take the same steps as given to simulate the critical values instead of calculating test statistic, which is test statistic, which is Tj you calculate wj and sort wj from smallest to the largest and take 1 minus alpha times M th value which will give you Weibull Chi square critical value.

Please remember, here this value given is sigma naught square, so there will be a slight change in the beginning, he will take sigma naught square and from that you will calculate there is you have to find out the formula for sigma naught squared, which is given earlier and then you have to calculate the value of C naught and then do the simulation.

So, let us summarize it, we defined the testing of hypothesis process for testing Mu is equal to Mu 0 and sigma square is equal to sigma naught square under non normal distribution, in particular we took the case of Weibull. We did this because we found that the test statistics Z and T which we have chosen under normal population, their properties of choosing them for statistic is independent of what is the underlying population.

So, we decided that the same statistic can also be useful, we felt that it can also be useful to test the hypothesis under different population distribution. We did that with the Weibull distribution.

(Refer Time Slide: 29:14)



We found Weibull T and Weibull Chi square statistic, we found the critical region and we give the steps through Monte Carlo simulation to simulate the critical values. Please remember, there is nothing wholly about Weibull. This is just a case has been given to you, because when you said that distribution is not normal, there are too many possibilities come up. So, we have shown you one possibility, you can change it and see how the test procedure can develop as and when the need arises. Thank you.